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MACNEILLE COMPLETIONS OF HEYTING ALGEBRAS

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Abstract. In this note we provide a topological description of the MacNeille completion of a Heyting algebra similar to the description of the MacNeille completion of a Boolean algebra in terms of regular open sets of its Stone space. We also show that the only varieties of Heyting algebras that are closed under MacNeille completions are the trivial variety, the variety of all Boolean algebras, and the variety of all Heyting algebras.

1. Introduction

Based on Dedekind's construction of the reals by *cuts* of the rationals, MacNeille [8] gave a method to embed an arbitrary poset P into a complete lattice. This completion is known in the literature under many names including the MacNeille completion, the completion by cuts, the minimal completion, and the normal completion. It is briefly described as follows. For P a poset and $A \subseteq P$, let $L(A)$ be the collection of all lower bounds of A, $U(A)$ be the collection of all upper bounds of A, and call A a *normal ideal* of P if $A = LU(A)$. Then the collection P^* of all normal ideals of P is a complete lattice with $\bigwedge_I N_i = \bigcap_I N_i$ and $\bigvee_I N_i = LU(\bigcup_I N_i)$. Throughout, we call the lattice P^* the MacNeille completion of P.

There is natural embedding $\alpha : P \to P^*$ defined by setting $\alpha(p) = \downarrow p$ where $\downarrow p = \{x \in P | x \leq p\}$ is the principal ideal of P generated by p. The map α has many desirable order theoretic properties. Each element of P^* can be expressed

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as a join and as a meet of elements of the image of α, a fact often expressed by saying that the image of α is join and meet dense in P^*. Further, α preserves all existing joins and meets in P, a fact often expressed by saying that α is a *regular embedding*. Therefore, if P is a lattice, then α preserves all binary meets and joins, hence is a lattice embedding.

Unfortunately there are many algebraic properties that are not preserved by MacNeille completions. For instance, it is well known that the MacNeille completion of a distributive lattice (abbreviated: DL) need not be distributive [1, 6]. There are, however, interesting classes of lattices that are known to be closed under MacNeille completions. These include Boolean algebras (abbreviated: BAs) and Heyting algebras (abbreviated: HAs) [1].

In this paper we shall also consider co-Heyting algebras (abbreviated: co-HAs) and bi-Heyting algebras (abbreviated: bi-HAs). A co-HA is a lattice whose dual lattice is a HA and a bi-HA is a lattice that is both a HA and a co-HA. In the literature co-HAs are also known as dual Heyting algebras or Brouwerian algebras and bi-HAs are also known as double Heyting algebras or semi-Boolean algebras. As the MacNeille completion of the dual of a lattice L is isomorphic to the dual of the MacNeille completion of L, it follows that the classes of co-HAs and bi-HAs are closed under MacNeille completions.

There is an elegant topological characterization of the MacNeille completion of a Boolean algebra that plays a significant role in many areas of mathematics. Given a BA B, it is well known that B is isomorphic to the BA of all clopen sets of its Stone space X [13]. One can then recognize the MacNeille completion B^*, up to isomorphism, as the BA of all regular open, or equivalently, all regular closed sets of the Stone space X [1].

MacNeille completions play an important role in logic. As the variety of BAs is closed under MacNeille completions, it admits a regular completion. Rasiowa and Sikorski [11] used this fact to show completeness of classical predicate calculus with respect to its algebraic semantics. The closure of the varieties of HAs and bi-HAs under MacNeille completions was used in a similar way to establish completeness of intuitionistic and symmetric intuitionistic predicate calculi with respect to their algebraic semantics (see Rasiowa [10] and Rauszer [12]).

To determine the completeness of predicate intermediate logics with respect to their algebraic semantics one wishes to determine which varieties of Heyting algebras admit regular completions. The obvious starting point is to determine which varieties of HAs are closed under MacNeille completions. In this note we show the only varieties of HAs that are closed under MacNeille completions are the trivial variety, the variety of all BAs, and the variety of all HAs.

The paper is organized in the following fashion. In section 2 we give the basics of MacNeille completions of HAs. In section 3 we provide topological versions of MacNeille completions of DLs, HAs, co-HAs, and bi-HAs analogous to the result that the MacNeille completion of a BA is isomorphic to the regular open sets of its Stone space. Finally, in section 4 we characterize the varieties of HAs that are closed under MacNeille completions.

2. MacNeille completions of HAs

In this section we provide basic results about MacNeille completions of HAs. Most of the material in this section is well known. It is presented here both for the convenience of the reader and because some aspects of our approach differ from those usually found in the literature and are required in the sequel.

Lemma 2.1. *For A a HA and N an ideal of A, the following are equivalent:*

(1) *N is a normal ideal.*
(2) *N is closed under existing joins.*

Proof. (1) $\Rightarrow$ (2) Suppose N is a normal ideal, $S \subseteq N$ and $\bigvee S = a$. If $u \in U(N)$, then $a \leq u$, so $a \in LU(N)$. Since N is a normal ideal, $a \in N$.

(2) $\Rightarrow$ (1) Suppose N is an ideal that is closed under existing joins. We must show $N = LU(N)$. It is obvious that $N \subseteq LU(N)$. Conversely, suppose $a \in LU(N)$. Set $N_a = \{n \in N | n \leq a\}$. We first show $a = \bigvee N_a$. Let $b \in U(N_a)$. If $n \in N$, then $n \wedge a \in N_a$. So, $n \wedge a \leq b$, implying $n \leq a \to b$. Therefore, $a \to b \in U(N)$. As $a \in LU(N)$, it follows that $a \leq a \to b$, hence $a \leq b$. Thus, $a = \bigvee N_a$. Since N is closed under existing joins, it follows that $a \in N$. $\square$

Remark. In any lattice a normal filter is closed under existing meets. We give an example of a HA A and a filter F in A with F closed under existing meets, but F not normal. Let B be the finite and cofinite subsets of the natural numbers. Then define $A = \{(x, y) \in B \times \mathbf{2} | y = 1 \Rightarrow x \text{ cofinite}\}$ and $F = \{(x, y) \in A | x \supseteq \text{evens and } y = 1\}$.

Lemma 2.2. *Let A be a HA and let M and N be normal ideals of A. Set*

$$K = \{k \in A | k \wedge m \in N \textit{ for all } m \in M\}.$$

Then

(1) *K is a normal ideal of A.*
(2) *K is the largest normal ideal whose intersection with M is contained in N.*

Proof. (1) By Lemma 2.1 it is enough to show that K is an ideal that is closed under existing joins. That K is an ideal follows directly from the facts that A is distributive and N is an ideal. Suppose $S \subseteq K$ and $a = \bigvee S$. We must show $a \in K$. Let $m \in M$. Then $a \wedge m = (\bigvee S) \wedge m = \bigvee\{s \wedge m | s \in S\}$. As $S \subseteq K$, $s \wedge m \in N$ for each $s \in S$. Therefore, $a \wedge m$ is the join of a subset of N. Since N is normal, $a \wedge m \in N$ by Lemma 2.1. Thus, $a \in K$. (2) We show more, that K is the largest downset whose intersection with M is contained in N. To see this, suppose D is a downset with $D \cap M \subseteq N$. Let $a \in D$. For each $m \in M$ we have that $a \wedge m \in D \cap M \subseteq N$. So, $a \in K$. That $K \cap M \subseteq N$ follows similarly. □

Theorem 2.3. *For A a HA define $\to^*$ on the MacNeille completion A^* of A by setting*

$$M \to^* N = \{k \in A | k \wedge m \in N \textit{ for all } m \in M\}.$$

Then A^* is a HA with implication $\to^*$ and $\alpha : A \to A^*$ is a HA-embedding.

Proof. By Lemma 2.2 $\to^*$ is well defined. Lemma 2.2 also gives that $M \to^* N$ is the relative pseudocomplement of M with respect to N, and it is well known that this implies A^* is a Heyting algebra. To show that α is a HA-embedding we must show $\alpha(a \to b) = \alpha(a) \to^* \alpha(b)$. But $\alpha(a) \to^* \alpha(b) = (\downarrow a) \to^* (\downarrow b) = \{k \in A | k \wedge m \in \downarrow b \text{ for all } m \in \downarrow a\} = \{k \in A | k \wedge a \leq b\} = \{k \in A | k \leq a \to b\} = \downarrow(a \to b) = \alpha(a \to b)$. □

A more algebraic description of the operation $\to^*$ is provided below.

Proposition 2.4. *For A a HA and M, N normal ideals of A,*

$$M \to^* N = \bigwedge\{\alpha(a \to b) | \alpha(a) \leq M \text{ and } N \leq \alpha(b)\}.$$

Proof. By definition $M \to^* N = \{c | c \wedge a \in N \text{ for all } a \in M\}$. As $b \in N$ iff $b \leq u$ for all $u \in U(N)$ we have $M \to^* N = \{c | c \wedge a \leq b \text{ for all } a \in M, b \in U(N)\} = \{c | c \leq a \to b \text{ for all } a \in M, b \in U(N)\} = \bigcap\{\downarrow(a \to b) | a \in M, b \in U(N)\}$. Since meets in the MacNeille completion are given by intersection, it then follows that $M \to^* N = \bigwedge\{\alpha(a \to b) | \alpha(a) \leq M, N \leq \alpha(b)\}$. □

Remark. It is well known that the MacNeille completion of a lattice A may also be realized as the lattice of normal filters of A partially ordered by $\supseteq$. The previous proposition states that for A a HA and x, y elements of the MacNeille completion, $x \to y$ is the meet of all elements of the form $a \to b$ where a, b are elements of A with a lying beneath x and b lying above y. Choosing to work with normal ideals of a HA is a natural choice as the infinite meet required to compute $x \to y$ corresponds to an intersection of normal ideals. One could work with normal filters of a HA, but then computing $x \to y$ would be somewhat problematic as the meet of a family of normal filters is obtained by taking UL of their union.

However, if one is considering MacNeille completions of co-HAs, working with normal filters is the natural choice. By a direct application of duality we obtain that a filter F of a co-HA is normal iff it is closed under all existing meets. Further, for x, y elements of the MacNeille completion we have $x \leftarrow y$ is formed by taking the join of all elements of the form $a \leftarrow b$ where a, b are elements of A with x lying beneath a and b lying beneath y. Working with normal filters is now preferable as the join of normal filters is given simply by their intersection, while the join of normal ideals is formed by taking LU of their union. Of course, when working with bi-HAs neither normal ideals nor normal filters is a perfectly convenient choice.

3. Topological characterization

In this section we provide a topological description of MacNeille completions in terms of Priestley spaces [9]. We briefly describe our notation and conventions. For A a DL let X be the set of all prime filters of A and for $a \in A$ set $\phi(a) = \{x \in X | a \in x\}$. The *Priestley space* of A is the set X, partially ordered by set inclusion, with the topology τ generated by the collection of all $\phi(a), -\phi(a)$. It is well known that τ is compact, Hausdorff, and has a basis of clopen sets. For $S \subseteq X$ define

$$\uparrow S = \{x \in X |\text{ there exists } y \in S \text{ with } y \leq x\},$$
$$\downarrow S = \{x \in X |\text{ there exists } y \in S \text{ with } x \leq y\}.$$

We call S an *upset* if $S = \uparrow S$ and we call S a *downset* if $S = \downarrow S$. The key fact is that ϕ is an isomorphism from A onto the lattice of clopen upsets of X. Throughout we use $\mathbf{I}$ and $\mathbf{C}$ for the interior and closure operators of τ.

Definition 1. For A a DL with Priestley space X and $S \subseteq X$ define

$\mathbf{J}S$ to be the largest open upset contained in S,

$\mathbf{D}S$ to be the smallest closed upset containing S.

Remark. The set of all open upsets of X is the set of open sets of a topology τ_1 on X, which is known as the spectral topology in the literature. By definition $\mathbf{J}$ is the interior operator of this topology. Similarly, the set of all closed upsets of X is the set of closed sets of a topology τ_2 on X. By definition $\mathbf{D}$ is the closure operator of this topology. The following lemma provides further descriptions of $\mathbf{J}, \mathbf{D}$ and shows $\{\phi(a)|a \in A\}$ is a basis of open sets for τ_1 and a basis of closed sets for τ_2.

Lemma 3.1. *Let A be a DL with Priestley space X. Then for $S \subseteq X$*

(1) $\mathbf{J}S = -\downarrow -\mathbf{I}S$.
(2) $\mathbf{J}S = \bigcup\{\phi(a)|\phi(a) \subseteq S\}$.
(3) $\mathbf{D}S = \uparrow\mathbf{C}S$.
(4) $\mathbf{D}S = \bigcap\{\phi(a)|S \subseteq \phi(a)\}$.

Proof. For (1) and (2) we show

$$-\downarrow -\mathbf{I}S \subseteq \bigcup\{\phi(a)|\phi(a) \subseteq S\} \subseteq \mathbf{J}S \subseteq -\downarrow -\mathbf{I}S.$$

For the first containment suppose $x \in -\downarrow -\mathbf{I}S$. Then $x \notin \downarrow -\mathbf{I}S$ so for each $y \in -\mathbf{I}S$ we have $x \not\leq y$, hence the prime filter x is not contained in the prime filter y. So, for each $y \in -\mathbf{I}S$ there is $a_y \in x$ with $a_y \notin y$. Then $\bigcap\{\phi(a_y)|y \in -\mathbf{I}S\} \cap (-\mathbf{I}S) = \emptyset$. By a standard compactness argument there is $a \in A$ with $x \in \phi(a)$ and $\phi(a) \subseteq \mathbf{I}S$. This shows the first containment. The second follows as each $\phi(a)$ is an open upset, and the third as $\mathbf{J}S$ is an open upset and $-\downarrow -\mathbf{I}S$ is the largest upset contained in $\mathbf{I}S$.

For (3) and (4) we show

$$\uparrow\mathbf{C}S \subseteq \mathbf{D}S \subseteq \bigcap\{\phi(a)|S \subseteq \phi(a)\} \subseteq \uparrow\mathbf{C}S.$$

The first containment follows as $\mathbf{D}S$ is both closed and an upset. The second as each $\phi(a)$ is a closed upset. For the third, suppose $x \notin \uparrow\mathbf{C}S$. Then for each $y \in \mathbf{C}S$ we have $y \not\leq x$, so for each $y \in \mathbf{C}S$ there is $a_y \in A$ with $a_y \in y$ and $a_y \notin x$. Then $\bigcap\{-\phi(a_y)|y \in \mathbf{C}S\} \cap \mathbf{C}S = \emptyset$. By a standard compactness argument there is $a \in A$ with $x \notin \phi(a)$ and $\mathbf{C}S \subseteq \phi(a)$. The third containment follows. □

Definition 2. For A a DL with Priestley space X define

$\mathcal{I}$ to be the ideal lattice of A partially ordered by $\subseteq$,

$\mathcal{F}$ to be the filter lattice of A partially ordered by $\supseteq$,

$\mathcal{O}$ to be the lattice of open upsets of X partially ordered by $\subseteq$,

$\mathcal{C}$ to be the lattice of closed upsets of X partially ordered by $\subseteq$.

We then define maps $\Gamma : \mathcal{I} \to \mathcal{O}$ and $\Delta : \mathcal{F} \to \mathcal{C}$ by setting

$$\Gamma I = \bigcup\{\phi(a)|a \in I\},$$
$$\Delta F = \bigcap\{\phi(a)|a \in F\}.$$

Proposition 3.2. *For A a DL with Priestley space X, each of $\mathcal{I}, \mathcal{F}, \mathcal{O}, \mathcal{C}$ is a complete lattice and Γ, Δ are lattice isomorphisms.*

Proof. As each of $\mathcal{I}, \mathcal{F}, \mathcal{O}, \mathcal{C}$ is closed under either unions or intersections, each is a complete lattice. For each $a \in A$, $\phi(a)$ is an open upset, so Γ is well defined. For I, J ideals of A, obviously $I \subseteq J$ implies $\Gamma I \subseteq \Gamma J$, and as each $\phi(a)$ is compact $\Gamma I \subseteq \Gamma J$ implies $I \subseteq J$. To see Γ is onto, suppose S is an open upset. Then $S = \mathbf{J}S$, and for $I = \{a | \phi(a) \subseteq S\}$, Lemma 3.1 (2) gives $S = \mathbf{J}S = \Gamma I$. This shows Γ is an isomorphism. Next, Δ is well defined since each $\phi(a)$ is a closed upset. For F, G filters of A, surely $F \supseteq G$ implies $\Delta F \subseteq \Delta G$, and as each $-\phi(a)$ is compact, $\Delta F \subseteq \Delta G$ implies $F \supseteq G$. To see Δ is onto, suppose S is a closed upset. Then $S = \mathbf{D}S$, and for $F = \{a | S \subseteq \phi(a)\}$, Lemma 3.1 (4) gives $S = \mathbf{D}S = \Delta F$. So Δ is an isomorphism. □

For A a DL with dual space X consider the following diagram.

$$\begin{tikzcd}
\mathcal{I} \arrow[r, shift left, "U"] \arrow[d, "\Gamma"'] & \mathcal{F} \arrow[l, shift left, "L"] \arrow[d, "\Delta"] \\
\mathcal{O} \arrow[r, shift left, "\mathbf{D}"] & \mathcal{C} \arrow[l, shift left, "\mathbf{J}"]
\end{tikzcd}$$

Here U, L are the maps taking upper bounds and lower bounds respectively.

Lemma 3.3. *For A a DL with Priestley space X*

(1) $\Delta U = \mathbf{D}\Gamma$.
(2) $\Gamma L = \mathbf{J}\Delta$.

Proof. (1) For I an ideal of A note $\bigcup\{\phi(a) | a \in I\} \subseteq \phi(b)$ iff $b \in U(I)$. The result follows by Lemma 3.1 (4). For F a filter of A note $\phi(b) \subseteq \bigcap\{\phi(a) | a \in F\}$ iff $b \in L(F)$. The result follows by Lemma 3.1 (2). □

Definition 3. For A a DL with Priestley space X set

$$\begin{aligned}\mathcal{NI} &= \{I \in \mathcal{I} | I = LUI\},\\ \mathcal{NF} &= \{F \in \mathcal{F} | F = ULF\},\\ \mathcal{RO} &= \{S \in \mathcal{O} | S = \mathbf{JD}S\},\\ \mathcal{RC} &= \{S \in \mathcal{C} | S = \mathbf{DJ}S\}.\end{aligned}$$

We partially order $\mathcal{NI}, \mathcal{RO}$ and $\mathcal{RC}$ by set inclusion, $\mathcal{NF}$ by reverse set inclusion.

A Galois connection [2] between complete lattices P, Q is a pair of order inverting maps $f : P \to Q$ and $g : Q \to P$ with $p \leq gf(p)$ and $q \leq fg(q)$ for all $p \in P$, $q \in Q$. The lattice $\mathcal{G}P$ of Galois closed elements of P is the set $\{p \in P | p = gf(p)\}$ with the partial ordering inherited from P, and the lattice $\mathcal{G}Q$ of Galois closed elements of Q is the set $\{q \in Q | q = fg(q)\}$ with the partial ordering inherited from Q. Both $\mathcal{G}P$ and $\mathcal{G}Q$ are complete lattices, but not sublattices of P and Q. Meets in $\mathcal{G}P$ and $\mathcal{G}Q$ agree with those in P and Q, but the join of a family $\{p_i\}_I$ in $\mathcal{G}P$ is $gf(\bigvee_I p_i)$ where $\bigvee_I p_i$ is the join in P, and the join of a family $\{q_i\}_I$ in $\mathcal{G}Q$ is $fg(\bigvee_I q_i)$ where $\bigvee_I q_i$ is the join in Q. Finally, the maps f, g restrict to mutually inverse dual isomorphisms between $\mathcal{G}P$ and $\mathcal{G}Q$.

Lemma 3.4. *Let A be a DL with Priestley space X.*

(1) *L, U is a Galois connection between $\mathcal{I}$ and $\mathcal{F}^d$.*
(2) *$\mathcal{NI} = \mathcal{GI}$ and $\mathcal{NF} = (\mathcal{G}(\mathcal{F}^d))^d$.*
(3) *L, U restrict to mutually inverse isomorphisms between $\mathcal{NI}$ and $\mathcal{NF}$.*
(4) *$\mathbf{D}, \mathbf{J}$ is a Galois connection between $\mathcal{O}$ and $\mathcal{C}^d$.*
(5) *$\mathcal{RO} = \mathcal{GO}$ and $\mathcal{RC} = (\mathcal{G}(\mathcal{C}^d))^d$.*
(6) *$\mathbf{D}, \mathbf{J}$ restrict to mutually inverse isomorphisms between $\mathcal{RO}$ and $\mathcal{RC}$.*
(7) *Γ restricts to an isomorphism between $\mathcal{NI}$ and $\mathcal{RO}$.*
(8) *Δ restricts to an isomorphism between $\mathcal{NF}$ and $\mathcal{RC}$.*

Proof. Statements (1), (2) and (3) are well known. By Proposition 3.2, $\Gamma : \mathcal{I} \to \mathcal{O}$ and $\Delta : \mathcal{F}^d \to \mathcal{C}^d$ are isomorphisms and Lemma 3.3 yields that $\Gamma LU = \mathbf{JD}\Gamma$ and $\Delta UL = \mathbf{DJ}\Delta$. A standard argument then gives (4). The definitions of $\mathcal{RO}$ and $\mathcal{RC}$ show that the underlying set of $\mathcal{RO}$ agrees with the underlying set of $\mathcal{GO}$ and the underlying set of $\mathcal{RC}$ agrees with that of $\mathcal{G}(\mathcal{C}^d)$. The orderings on $\mathcal{RO}, \mathcal{GO}$ and $\mathcal{RC}$ are $\subseteq$ while that of $\mathcal{G}(\mathcal{C}^d)$ is $\supseteq$, hence (5) follows. Statement (6) follows from general considerations of Galois connections. Using the fact that $\Gamma LU = \mathbf{JD}\Gamma$ it follows that $I \in \mathcal{NI}$ iff $\Gamma I \in \mathcal{RO}$, and (7) follows. Statement (8) follows similarly as $\Delta UL = \mathbf{DJ}\Delta$. $\square$

Theorem 3.5. *Let A be a DL with Priestley space X.*

(1) *$\mathcal{RO}$ and $\mathcal{RC}$ are isomorphic to the MacNeille completion of A.*
(2) *In $\mathcal{RO}$, $\bigwedge S_i = \mathbf{J}\bigcap S_i$ and $\bigvee S_i = \mathbf{JD}\bigcup S_i$.*
(3) *In $\mathcal{RC}$, $\bigwedge S_i = \mathbf{DJ}\bigcap S_i$ and $\bigvee S_i = \mathbf{D}\bigcup S_i$.*

Proof. In the previous proposition we have shown $\mathcal{NI}$ is isomorphic to $\mathcal{RO}$ and that $\mathcal{RO}$ is isomorphic to $\mathcal{RC}$. This establishes (1). As $\mathcal{RO} = \mathcal{GO}$ it follows that meets in $\mathcal{RO}$ agree with meets in $\mathcal{O}$ and joins in $\mathcal{RO}$ are found by taking the join in $\mathcal{O}$ and applying $\mathbf{JD}$. By Definition 1 meets in $\mathcal{O}$ are formed by taking intersection and applying $\mathbf{J}$, and joins in $\mathcal{O}$ are given by union. This establishes (2). As $\mathcal{RC} = (\mathcal{G}(\mathcal{C}^d))^d$, joins in $\mathcal{RC}$ agree with meets in $\mathcal{C}^d$, and hence with joins in $\mathcal{C}$. Meets in $\mathcal{RC}$ are formed by taking joins in $\mathcal{C}^d$ and applying $\mathbf{DJ}$, hence by taking meets in $\mathcal{C}$ and applying $\mathbf{DJ}$. By Definition 1, joins in $\mathcal{C}$ are formed by taking union and applying $\mathbf{D}$, and meets in $\mathcal{C}$ are given by intersection. This establishes (3). $\square$

Remark. The MacNeille completion of a BA A is isomorphic to the lattice of regular open sets of the Stone space X of A. But a set S is regular open iff it is a fixed point of the composition of the interior and closure operators on X. By the previous theorem and Remark 3 the MacNeille completion of a DL A is realized as the subsets S of the Priestley space of A which are fixed points of the composition of an interior and closure operator, but it is the interior operator of the topology τ_1 and the closure operator of topology τ_2 that are used. Of course, if A is a BA, then τ_1 and τ_2 agree and are the Stone topology on X.

Lemma 3.6. *Let A be a DL with Priestley space X, let $a, b \in A$ and $S \subseteq X$. If A is a HA then*

(1) *$\phi(a \to b) = \mathbf{J}(-\phi(a) \cup \phi(b))$.*
(2) *S clopen $\Rightarrow {\downarrow}S$ is clopen.*
(3) *S an upset $\Rightarrow \mathbf{C}S = \mathbf{D}S$.*

If A is a co-HA then

(1) *$\phi(a \leftarrow b) = \mathbf{D}(-\phi(a) \cap \phi(b))$.*
(2) *S clopen $\Rightarrow {\uparrow}S$ is clopen.*
(3) *S an upset $\Rightarrow \mathbf{I}S = \mathbf{J}S$.*

Proof. We show only the statements for HAs, the statements for co-HAs are proved similarly. (1) By definition $\phi(a \to b)$ is the largest clopen upset whose intersection with $\phi(a)$ is contained in $\phi(b)$, hence $\phi(a \to b)$ is the largest clopen upset contained in $-\phi(a) \cup \phi(b)$. Therefore $\phi(a \to b) \subseteq \mathbf{J}(-\phi(a) \cup \phi(b))$. For the other containment suppose $x \in \mathbf{J}(-\phi(a) \cup \phi(b))$. Then by Lemma 3.1(1) $x \in -\downarrow-\mathbf{I}(-\phi(a)\cup\phi(b))$, and as both $\phi(a), \phi(b)$ are clopen, $x \notin\downarrow(\phi(a)\cap-\phi(b))$. So for each $y \in \phi(a)\cap-\phi(b)$ we have $x \not\leq y$, hence for each such y there is $c_y \in A$ with $x \in \phi(c_y)$ and $y \notin \phi(c_y)$. Therefore $\bigcap\{\phi(c_y)|y \in \phi(a)\cap-\phi(b)\}\cap\phi(a)\cap-\phi(b) = \emptyset$, and compactness yields some $c \in A$ with $x \in \phi(c)$ and $\phi(c) \cap \phi(a) \cap -\phi(b) = \emptyset$. So $\phi(c) \cap \phi(a) \subseteq \phi(b)$ giving $c \leq a \to b$. Thus $x \in \phi(c) \subseteq \phi(a \to b)$. (2) As every clopen set is a finite union of sets of the form $\phi(a)\cap-\phi(b)$, we may assume $S = \phi(a) \cap -\phi(b)$. By (1) $\phi(a \to b) = -\downarrow(\phi(a) \cap -\phi(b))$, and as $\phi(a \to b)$ is clopen, it follows that $\downarrow(\phi(a) \cap -\phi(b))$ is clopen. (3) By definition, $\mathbf{C}S \subseteq \mathbf{D}S$. If $x \notin \mathbf{C}S$, then as the clopen sets are a base for the topology, there is a clopen set K with $x \in K$ and $K \cap S = \emptyset$. As S is an upset, $\downarrow K \cap S = \emptyset$ and by (2) $\downarrow K$ is clopen. So $-\downarrow K$ is a clopen upset, hence equal to $\phi(a)$ for some $a \in A$. Then $x \notin \phi(a)$ and $S \subseteq \phi(a)$. It then follows from Lemma 3.1(4) that $x \notin \mathbf{D}S$. □

Remark. Esakia has shown that condition (2) for HAs and condition (2) for co-HAs characterize the dual spaces of HAs and co-HAs among the dual spaces of DLs. He has used this to build a duality theory for HAs and certain Priestley spaces [3], a duality theory for co-HAs and certain Priestley spaces [4], and a duality theory for bi-HAs and certain Priestley spaces [4] (for bi-HAs see also Rauszer [12]).

Lemma 3.7. *Let A be a DL with Priestley space X and let $a \in A$.*

(1) *If A is a HA and $S \in \mathcal{RC}$, then $S \wedge \phi(a) = S \cap \phi(a)$.*
(2) *If A is a co-HA and $S \in \mathcal{RO}$, then $S \vee \phi(a) = S \cup \phi(a)$.*

Here $\wedge$ is taken in the lattice $\mathcal{RC}$ and $\vee$ is taken in the lattice $\mathcal{RO}$.

Proof. We prove the first statement, the second is similar. By Theorem 3.5 and Lemma 3.6 $S\wedge\phi(a) = \mathbf{CJ}(S\cap\phi(a))$. Since $\mathbf{J}(S\cap\phi(a)) \subseteq S\cap\phi(a)$ and this latter set is closed, $\mathbf{CJ}(S \cap \phi(a)) \subseteq S \cap \phi(a)$. For the other inclusion it suffices to show $S \subseteq -\phi(a) \cup \mathbf{CJ}(S \cap \phi(a))$. As $S \in \mathcal{RC}$ we have $S = \mathbf{CJ}S \subseteq \mathbf{C}(-\phi(a) \cup \mathbf{J}S)$ $= \mathbf{C}(-\phi(a) \cup (\mathbf{J}S \cap \phi(a))$. As $\phi(a)$ is a clopen upset, $\phi(a) = \mathbf{J}\phi(a)$. Hence $S \subseteq \mathbf{C}(-\phi(a)\cup(\mathbf{J}S\cap\mathbf{J}\phi(a))) = \mathbf{C}(-\phi(a)\cup\mathbf{J}(S\cap\phi(a))) = \mathbf{C}(-\phi(a))\cup\mathbf{CJ}(S\cap\phi(a))$. As $\phi(a)$ is clopen, $S \subseteq -\phi(a) \cup \mathbf{CJ}(S \cap \phi(a))$ as required. □

Remark. Algebraically the first part of the above lemma states that for N a normal filter of a HA A and $a \in A$, the filter generated by $\{a\} \cup N$ is normal. This follows from observing that $b \in UL(\{a\} \cup N)$ implies $a \to b \in UL(N)$. The algebraic counterpart to the second part of the lemma states that for N a normal ideal of a co-HA A and $a \in A$, the ideal generated by $\{a\} \cup N$ is normal.

Theorem 3.8. *Let A be a DL with Priestley space X.*

(1) *$\mathcal{RO}$ and $\mathcal{RC}$ are isomorphic to the MacNeille completion of A.*
(2) *In $\mathcal{RO}$, $\bigwedge S_i = \mathbf{J}\bigcap S_i$ and $\bigvee S_i = \mathbf{JD}\bigcup S_i$.*
(3) *In $\mathcal{RC}$, $\bigwedge S_i = \mathbf{DJ}\bigcap S_i$ and $\bigvee S_i = \mathbf{D}\bigcup S_i$.*

If A is a HA then

(1) *$\mathcal{RO} = \{S \in \mathcal{O} | S = \mathbf{JC}S\}$ and $\mathcal{RC} = \{S \in \mathcal{C} | S = \mathbf{CJ}S\}$.*
(2) *In $\mathcal{RO}$, $\bigwedge S_i = \mathbf{J}\bigcap S_i$, $\bigvee S_i = \mathbf{JC}\bigcup S_i$, and $S \to T = \mathbf{J}(-S \cup T)$.*
(3) *In $\mathcal{RC}$, $\bigwedge S_i = \mathbf{CJ}\bigcap S_i$, $\bigvee S_i = \mathbf{C}\bigcup S_i$, and $S \to T = \mathbf{CJ}(-S \cup T)$.*

If A is a co-HA then

(1) *$\mathcal{RO} = \{S \in \mathcal{O} | S = \mathbf{ID}S\}$ and $\mathcal{RC} = \{S \in \mathcal{C} | S = \mathbf{DI}S\}$.*
(2) *In $\mathcal{RO}$, $\bigwedge S_i = \mathbf{I}\bigcap S_i$, $\bigvee S_i = \mathbf{ID}\bigcup S_i$, and $S \leftarrow T = \mathbf{ID}(-S \cap T)$.*
(3) *In $\mathcal{RC}$, $\bigwedge S_i = \mathbf{DI}\bigcap S_i$, $\bigvee S_i = \mathbf{D}\bigcup S_i$, and $S \leftarrow T = \mathbf{D}(-S \cap T)$.*

If A is a bi-HA then

(1) *$\mathcal{RO} = \{S \in \mathcal{O} | S = \mathbf{IC}S\}$ and $\mathcal{RC} = \{S \in \mathcal{C} | S = \mathbf{CI}S\}$.*
(2) *In $\mathcal{RO}$, $\bigwedge S_i = \mathbf{I}\bigcap S_i$, $\bigvee S_i = \mathbf{IC}\bigcup S_i$, $S \to T = \mathbf{I}(-S \cup T)$, and $S \leftarrow T = \mathbf{IC}(-S \cap T)$.*
(3) *In $\mathcal{RC}$, $\bigwedge S_i = \mathbf{CI}\bigcap S_i$, $\bigvee S_i = \mathbf{C}\bigcup S_i$, $S \to T = \mathbf{CI}(-S \cup T)$, and $S \leftarrow T = \mathbf{C}(-S \cap T)$.*

Proof. Each of the statements about DLs is established in Theorem 3.5. For HAs Lemma 3.6 gives $\mathbf{D} = \mathbf{C}$ for upsets, and for co-HAs Lemma 3.6 gives $\mathbf{J} = \mathbf{I}$ for upsets, so for bi-HAs both $\mathbf{D} = \mathbf{C}$ and $\mathbf{J} = \mathbf{I}$ for upsets. This establishes all the statements about meets and joins.

Suppose A a HA and $S, T \in \mathcal{RO}$. We show $S \to T = \mathbf{J}(-S \cup T)$. Note that finite meets in $\mathcal{RO}$ are given by intersection. So for $U \in \mathcal{RO}$, $U \subseteq S \to T$ iff $S \cap U \subseteq T$ iff $U \subseteq -S \cup T$. In particular, $S \to T \subseteq -S \cup T$ and as $S \to T$ is an open upset, $S \to T \subseteq \mathbf{J}(-S \cup T)$. Each element in the MacNeille completion of A is the join of the elements of A it dominates. Therefore $S \to T = \mathbf{JC}\bigcup\{\phi(a) | \phi(a) \subseteq -S \cup T\}$, and by Lemma 3.1 $S \to T = \mathbf{JCJ}(-S \cup T)$. But $\mathbf{JC}$ is a closure operator on $\mathcal{O}$, so $\mathbf{J}(-S \cup T) \subseteq \mathbf{JCJ}(-S \cup T) = S \to T \subseteq \mathbf{J}(-S \cup T)$.

Suppose A is a HA and $S, T \in \mathcal{RC}$. We show $S \to T = \mathbf{CJ}(-S \cup T)$. Note that finite meets in $\mathcal{RC}$ are not generally given by intersection, but Lemma 3.7 does give $S \wedge \phi(a) = S \cap \phi(a)$ for $a \in A$. It follows that $\phi(a) \subseteq S \to T$ iff $S \cap \phi(a) \subseteq T$ iff $\phi(a) \subseteq -S \cup T$. Using the fact that each element of the MacNeille completion is the join of the elements it dominates, $S \to T = \mathbf{C}\bigcup\{\phi(a) | \phi(a) \subseteq -S \cup T\}$. So by Lemma 3.1 $S \to T = \mathbf{CJ}(-S \cup T)$.

The corresponding statements for co-HAs are proved similarly. Finally, for A a bi-HA, the descriptions of $\rightarrow$ and $\leftarrow$ in $\mathcal{RO}$ and $\mathcal{RC}$ follow directly from the above results as $\mathbf{J} = \mathbf{I}$ and $\mathbf{D} = \mathbf{C}$ for upsets in a bi-HA. $\square$

Remark. A DL A is complete iff each normal ideal of A is principal. Using the fact that Γ restricts to an isomorphism from $\mathcal{NI}$ to $\mathcal{RO}$, it follows that A is complete iff each member of $\mathcal{RO}$ is clopen, which is equivalent to $\mathbf{D}S$ being clopen for each open upset S. Priestley calls such spaces extremally order disconnected [9]. For A a HA, Lemma 3.6 shows A is complete iff $\mathbf{C}S$ is clopen for each open upset S. For A a bi-HA, Lemma 3.6 shows A is complete iff each regular open upset of the dual space of A is clopen, generalizing the well-known result for Boolean algebras.

4. Varieties of HAs

In this section we prove the only varieties of HAs that are closed under MacNeille completions are the trivial variety, the variety of all BAs, and the variety of all HAs. Throughout we use the following notations: $\mathbf{1}, \mathbf{2}, \mathbf{3}$ are the one, two, and three-element chains considered as HAs, $\mathbf{N}$ is the set of natural numbers and $X = \mathbf{N} \cup \{\infty\}$. For lattices A, B we use $A \oplus B$ for the ordinal sum of A and B, which is the lattice formed by placing the lattice B on top of the lattice A.

Definition 4. Define recursively HAs L_n as follows:

$$\begin{aligned} L_0 &= \mathbf{3} \\ L_{n+1} &= (\underbrace{L_n \times \cdots \times L_n}_{n+2 \text{ times}}) \oplus \mathbf{1} \end{aligned}$$

So $x \in L_{n+1}$ implies that $x = 1$ or $x = (a_0, \ldots, a_{n+1})$ for some $a_0, \ldots, a_{n+1} \in L_n$.

Theorem 4.1. *The variety generated by $\{L_n | n \geq 0\}$ is the variety of all HAs.*

Proof. Define recursively

$$\begin{aligned} J_0 &= \mathbf{2} \\ J_{n+1} &= (\underbrace{J_n \times \cdots \times J_n}_{n+2 \text{ times}}) \oplus \mathbf{1} \end{aligned}$$

Then by induction we have $J_n \leq L_n$. So the variety generated by $\{J_n | n \geq 0\}$ is contained in the variety generated by $\{L_n | n \geq 0\}$. But Jaskowski [7] has shown that the J_n's generate the variety of all HAs. $\square$

Notation. Each L_n has a unique coatom, which we denote c.

Definition 5. For each $n \geq 0$ define

$$S_n = \{f \in L_n^X | f(\infty) \in \{0, c, 1\}, f(\infty) = 0 \Rightarrow f =_{a.e.} 0, f(\infty) \neq 0 \Rightarrow f =_{a.e.} 1\}.$$

Here $f =_{a.e.} 0$ means $f(x) = 0$ except for finitely many values of x.

Proposition 4.2. *For each $n \geq 0$, $S_n \leq L_n^X$.*

Proof. Note first that $\{0, c, 1\}$ is a subalgebra of L_n isomorphic to **3**. Therefore if $f, g \in S_n$, then $f(\infty)$ and $g(\infty)$ belong to $\{0, c, 1\}$, hence $(f \wedge g)(\infty), (f \vee g)(\infty)$ and $(f \to g)(\infty)$ belong to $\{0, c, 1\}$. Note also that the constant function 0 and the constant function 1 belong to S_n. We next show S_n is closed under $\wedge, \vee$. Suppose $f, g \in S_n$. If $(f \wedge g)(\infty) = 0$, then either $f(\infty) = 0$ or $g(\infty) = 0$. So either $f =_{a.e.} 0$ or $g =_{a.e.} 0$, hence $f \wedge g =_{a.e.} 0$. If $(f \wedge g)(\infty) \neq 0$, then $f(\infty) \neq 0$ and $g(\infty) \neq 0$, so $f =_{a.e.} 1$ and $g =_{a.e.} 1$, hence $f \wedge g =_{a.e.} 1$. So S_n is closed under $\wedge$. Showing S_n is closed under $\vee$ is similar. To show S_n is closed under $\to$ suppose $f, g \in S_n$. If $(f \to g)(\infty) = 0$, then $f(\infty) \neq 0$ and $g(\infty) = 0$. So $f =_{a.e.} 1$ and $g =_{a.e.} 0$, hence $f \to g =_{a.e.} 0$. If $(f \to g)(\infty) \neq 0$, then as $\{0, c, 1\}$ is isomorphic to **3** it follows that either $f(\infty) = 0$ or $g(\infty) \neq 0$. If $f(\infty) = 0$ then $f =_{a.e.} 0$ so $f \to g =_{a.e.} 1$, and if $g(\infty) \neq 0$ then $g =_{a.e.} 1$ so $f \to g =_{a.e.} 1$. ☐

Definition 6. For each $n \geq 0$ define $\alpha_n : L_{n+1} \to \mathcal{P}(S_n)$ by setting

$$\begin{aligned} \alpha_n(1) &= S_n \\ \alpha_n((a_0, \ldots, a_{n+1})) &= \{f | f(\infty) \neq 1 \text{ and } m \equiv k \bmod (n+2) \Rightarrow f(m) \leq a_k\}. \end{aligned}$$

In a sequence of lemmas we will show α_n is a HA-embedding of L_{n+1} into the MacNeille completion S_n^* of S_n.

Lemma 4.3. *For any $n \geq 0$, α_n is a set mapping from L_{n+1} to S_n^*.*

Proof. We must show $\alpha_n(x)$ is a normal ideal of S_n for each $x \in L_{n+1}$. Clearly $\alpha_n(1) = S_n$ is a normal ideal of S_n. Suppose $x = (a_0, \ldots, a_{n+1})$. For $m \in \mathbf{N}$, say with $m \equiv k \bmod (n+2)$, define a function $v_m : X \to L_n$ by setting

$$v_m(y) = \begin{cases} a_k & \text{if } y = m \\ c & \text{if } y = \infty \\ 1 & \text{otherwise} \end{cases}$$

Then $v_m \in S_n$ and v_m is an upper bound of $\alpha_n(x)$. But if $f \in S_n$ and $f \leq v_m$ for each $m \geq 0$, then $f \in \alpha_n(x)$. Hence $\alpha_n(x) = LU(\alpha_n(x))$. ☐

Lemma 4.4. *For each $n \geq 0$, α_n preserves finite meets.*

Proof. If $x, y \in L_{n+1}$ and either x or y equals 1, then as $\alpha_n(1) = S_n$ it follows that $\alpha_n(x \wedge y) = \alpha_n(x) \wedge \alpha_n(y)$. Otherwise, $x = (a_0, \ldots, a_{n+1})$, $y = (b_0, \ldots, b_{n+1})$ and $x \wedge y = (c_0, \ldots, c_{n+1})$ where $c_i = a_i \wedge b_i$. Suppose $f \in S_n$ and $m \equiv k \mod (n+2)$. Then as $f(m) \leq c_k$ iff $f(m) \leq a_k$ and $f(m) \leq b_k$ it follows that $f \in \alpha_n(x \wedge y)$ iff $f \in \alpha_n(x) \cap \alpha_n(y)$. The result follows as meets in S_n^* are given by intersections. $\square$

Lemma 4.5. *For each $n \geq 0$, α_n preserves finite joins.*

Proof. If $x, y \in L_{n+1}$ and either x or y equals 1, then as $\alpha_n(1) = S_n$ it follows that $\alpha_n(x \vee y) = \alpha_n(x) \vee \alpha_n(y)$. Otherwise, $x = (a_0, \ldots, a_{n+1})$ and $y = (b_0, \ldots, b_{n+1})$. Suppose $f \in \alpha_n(x \vee y)$ and $u \in U(\alpha_n(x) \cup \alpha_n(y))$. For a natural number $m \in \mathbf{N}$ with $m \equiv k \mod (n+2)$ consider the functions $p_m, q_m : X \to L_n$ given by

$$p_m(y) = \begin{cases} a_k & \text{if } y = m \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad q_m(y) = \begin{cases} b_k & \text{if } y = m \\ 0 & \text{otherwise} \end{cases}$$

Then $p_m \in \alpha_n(x)$ and $q_m \in \alpha_n(y)$, hence $u(m) \geq a_k \vee b_k \geq f(m)$. We have shown $f(m) \leq u(m)$ for all $m \in \mathbf{N}$. If $f(\infty) = c$, then $f =_{a.e.} 1$, hence $u =_{a.e.} 1$, and as $u \in S_n$ we have $u(\infty)$ is either c or 1. Thus $f(\infty) \leq u(\infty)$ and therefore $f \leq u$. This shows $\alpha_n(x \vee y) \subseteq LU(\alpha_n(x) \cup \alpha_n(y)) = \alpha_n(x) \vee \alpha_n(y)$. The other inclusion follows as α_n is order preserving. $\square$

Lemma 4.6. *For each $n \geq 0$, α_n preserves implication $\to$.*

Proof. If $x, y \in L_{n+1}$ and either x or y equals 1, then as $\alpha_n(1) = S_n$ it follows that $\alpha_n(x \to y) = \alpha_n(x) \to \alpha_n(y)$. Also if $x \leq y$, then as α_n is order preserving $\alpha_n(x \to y) = \alpha_n(x) \to \alpha_n(y)$. Otherwise $x = (a_0, \ldots, a_{n+1})$ and $y = (b_0, \ldots, b_{n+1})$ and there is some $0 \leq j \leq n+1$ with $a_j \not\leq b_j$. It follows that $x \to y = (c_0, \ldots, c_{n+1})$ where $c_i = a_i \to b_i$. Suppose $f \in \alpha_n(x) \to \alpha_n(y)$. For $m \in \mathbf{N}$ with $m \equiv k \mod (n+2)$ define $r_m : X \to L_n$ by setting

$$r_m(y) = \begin{cases} a_k & \text{if } y = m \\ 0 & \text{otherwise} \end{cases}$$

Then $r_m \in \alpha_n(x)$. So $f \wedge r_m \in (\alpha_n(x) \to \alpha_n(y)) \cap \alpha_n(x) \subseteq \alpha_n(y)$. This gives $f(m) \wedge r_m(m) \leq b_k$, and it follows that $f(m) \wedge a_k \leq b_k$, hence $f(m) \leq a_k \to b_k$. We have shown $m \equiv k \mod (n+2)$ implies $f(m) \leq c_k$. As $c_j < 1$, we have $f(m) < 1$ for infinitely many m, and as $f \in S_n$ it follows that $f(\infty) = 0$. Therefore $f \in \alpha_n(x \to y)$. This shows $\alpha_n(x) \to \alpha_n(y) \subseteq \alpha_n(x \to y)$. The other inclusion follows from general principles as α_n is a lattice homomorphism between HAs. $\square$

Lemma 4.7. *For each $n \geq 0$, α_n is an embedding.*

Proof. As α_n is a HA homomorphism, it is sufficient to show $x \neq 1$ implies $\alpha_n(x) \neq S_n$. But this follows directly from the definition of α_n as $x \neq 1$ and $f \in \alpha_n(x)$ imply $f(\infty) \neq 1$. □

Proposition 4.8. *For each $n \geq 0$, $\alpha_n : L_{n+1} \to S_n^*$ is a HA-embedding.*

Proof. This is the content of the previous five lemmas. □

Theorem 4.9. *The closure of $\{\mathbf{3}\}$ under $\mathbf{H}, \mathbf{S}, \mathbf{P}$ and MacNeille completions is the variety of all HAs.*

Proof. Let K be the smallest class of algebras containing $\mathbf{3}$ and closed under $\mathbf{H}, \mathbf{S}, \mathbf{P}$ and MacNeille completions. By induction on n we show $L_n \in K$. For $n = 0$ this is trivial as $L_0 = \mathbf{3}$. Assume $L_n \in K$. We have shown in Proposition 4.8 that L_{n+1} is isomorphic to a subalgebra of the MacNeille completion of S_n, and we have shown in Proposition 4.2 that S_n is a subalgebra of a power L_n^X of L_n. Therefore $L_{n+1} \in K$. It follows that $\{L_n | n \in \mathbf{N}\} \subseteq K$, so by Theorem 4.1 K contains the class of all HAs. But the class of all HAs is closed under $\mathbf{H}, \mathbf{S}, \mathbf{P}$ and MacNeille completions, hence K equals the class of all HAs. □

Corollary 4.10. *The only varieties of HAs that are closed under MacNeille completions are the trivial variety, the variety of all BAs, and the variety of all HAs.*

Proof. Any variety of HAs different from the trivial variety and the variety of all BAs contains the HA $\mathbf{3}$. □

Remark. By duality, the only varieties of co-HAs that are closed under MacNeille completions are the trivial variety, the variety of all BAs, and the variety of all co-HAs. There are, however, varieties of bi-HAs that are closed under MacNeille completions. For example, the variety of bi-HAs generated by $\mathbf{3}$ is closed under MacNeille completions. A proof of this fact can be obtained from the work of Givant and Venema [5] or by using the topological considerations of Section 3.

5. Concluding remarks

It remains an open question whether there are varieties of HAs other than the trivial variety, the variety of all BAs, and the variety of all HAs that admit a regular completion. In particular, it is unknown whether the variety of HAs generated by $\mathbf{3}$ admits a regular completion.

Using the familiar connection between closure algebras and HAs, the results of this paper have direct application to MacNeille completions of closure algebras. More generally, it may also be possible to extend the techniques and results of this paper to MacNeille completions of modal algebras.

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