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Part I

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# 12 Varieties of algebras in fuzzy set theory

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**Abstract** Many algebras arise in the study of fuzzy set theory, including the unit interval with a negation, a t-norm, or both. We investigate equational properties of such algebras.

## 12.1 Introduction

Our purpose is to study equational properties of algebras that arise in fuzzy set theory. Each of the algebras we will consider is a bounded distributive lattice $(D, \wedge, \vee, 0, 1)$, perhaps with some additional operations. The difficulty in determining the equational properties of a given algebra depends greatly upon which, if any, additional operations are present.

Consider the situation for algebras having no further operations beyond the lattice operations $\wedge$ and $\vee$ and the bounds 0 and 1. Examples are the real unit interval $\mathbb{I}$ with $\wedge$ being min and $\vee$ being max, or the collection $\mathcal{F}(S)$ of fuzzy subsets of a set $S$ with $\wedge$ and $\vee$ defined componentwise from max and min on the unit interval. A fundamental theorem of Birkhoff states that a lattice equation holds in a non-trivial bounded distributive lattice if and only if it holds in the two element distributive lattice we denote by **2**. Thus, to determine whether a lattice equation holds in the real unit interval, or in the collection of fuzzy subsets of a nonempty set, it is necessary and sufficient to determine whether it holds in the two-element lattice **2**. This certainly provides a great simplification of the problem.

A similar situation arises with algebras having only lattice operations and an additional operation of negation, denoted $'$. Obvious examples are the unit interval with the negation $x' = 1 - x$, or the collection $\mathcal{F}(S)$ of fuzzy subsets of a set $S$ with operations defined componentwise from ones on the unit interval. However, many other negations are possible on the unit interval, and on $\mathcal{F}(S)$ as well, and these give rise to different algebras. Fortunately, a well-known result of Kalman [15] yields that an equation is valid in any one of these algebras described above if and only if it is valid in the three-element chain $\mathbf{3} = \{0, a, 1\}$ with negation $0' = 1$, $a' = a$ and $1' = 0$.

Consider algebras having an additional binary operation $\circ$ in addition to the usual lattice operations. Examples include the unit interval with $\circ$ being ordinary multiplication, the unit interval with $\circ$ being an arbitrary t-norm, or the collection $\mathcal{F}(S)$ of fuzzy subsets of a set with an operation $\circ$ defined componentwise through such an operation on the unit interval. Here matters are considerably more complicated as one can show there is no finite test algebra to play a role as above, even if one restricts attention to testing for validity of equations in the unit interval under multiplication. Still, there is much that can be said. For instance, we show that any equation holding in the algebra $(\mathbb{I}, \circ)$, where $\mathbb{I} = ([0,1], \wedge, \vee, 0, 1)$ and $\circ$ is ordinary multiplication, holds in any algebra $(\mathbb{I}, T)$ where $T$ is a continuous t-norm.

To continue on this path, we note that each of the algebras above is a bounded distributive lattice with a binary operation $\circ$ that is commutative, associative, and satisfies $x \circ (y \wedge z) = (x \circ y) \wedge (x \circ z)$, $x \circ (y \vee z) = (x \circ y) \vee (x \circ z)$ and $x \circ 1 = x$. We will call such an algebra a bounded distributive lattice ordered commutative monoid (abbreviated: *dl*-monoid).

We conjecture that any equation valid in the unit interval with ordinary multiplication is valid in all *dl*-monoids, and in particular is valid in any algebra $(\mathbb{I}, T)$ where $T$ is a t-norm. We have not proved this conjecture, but have verified it for equations involving at most two variables. Aside from its application to fuzzy set theory, this conjecture is likely of independent interest. It seems a natural companion to the well-known result [20] that any equation valid in the ordered group of real numbers under addition is valid in all lattice ordered abelian groups.

Finally, we consider algebras having lattice operations, a negation, and a binary operation as above. The unit interval with a negation and a t-norm is an example of such an algebra and is called a De Morgan system. A conorm can be obtained through the negation and t-norm, therefore its inclusion as a basic operation in a De Morgan system is not necessary.

There are several positive results about the equational theory of such De Morgan systems. For example, any De Morgan system whose t-norm is strict is isomorphic to one whose t-norm is ordinary multiplication. Thus, the equations valid in all strict De Morgan systems are exactly those valid in De Morgan systems based on ordinary multiplication. Analogous results hold for nilpotent De Morgan systems and the Łukasiewicz t-norm.

There are, however, a number of negative results showing the difficulties in developing the equational theory of De Morgan systems. There is no finite algebra that satisfies exactly those equations valid in all De Morgan systems. Worse still, the canonical example of the unit interval with the usual negation and ordinary multiplication cannot be used for this role either. In fact, given any two strict, non-isomorphic De Morgan systems, there are equations valid in one, but not the other.

This chapter is organized in the following manner. In the second section we give a brief review of some algebraic notions. In the third we define the basic lattices of interest and give a complete determination of their equational properties. In the fourth section we describe the situation for algebras with negation, and in the fifth we give several results about algebras with an additional binary operation. The sixth section develops the basic theory of *dl*-monoids. This paves the way for the seventh section where we verify that any equation in at most two variables valid in the unit interval with multiplication is valid in all *dl*-monoids. In the eighth and final section we consider algebras with a negation and a binary operation, especially De Morgan systems. Section 3, Section 4, portions of Section 5, and Section 8 represent surveys of existing results, many obtained by the second, fourth and fifth listed authors.

For background to this chapter the reader can consult [10] and [12] for connections between equational theories and fuzzy logic, [3] for general aspects of equational theories and universal algebra, and [16] and [19] for aspects of fuzzy sets.

## 12.2 Preliminaries

Given a set $A$, and a nonnegative integer $n$, we say a map $f: A^n \to A$ is an *n-ary* operation on $A$. Thus an $n$-ary operation takes as arguments $n$ values from $A$ and returns a single value from $A$. An *algebra* is a set equipped with a family of operations. An algebra may have any number of operations of any arities. A specification of the number of operations of an algebra and the arities of these operations is called the *type* of the algebra. For example, a bounded distributive lattice $(D, \wedge, \vee, 0, 1)$ is an algebra of type $(2,2,0,0)$ meaning that it has two binary operations $\wedge, \vee$ and two constants (operations taking zero arguments) 0, 1.

A *term* for a given type of algebra is an expression formed from a set of variables using the basic operations. An *equation* for algebras of a given type is a formal expression $s \approx t$ asserting the equality of two terms. For example $x \wedge (x \vee y) = x \vee (x \wedge y)$ is an equation for algebras having two binary operations $\wedge, \vee$. An algebra $\mathbb{A}$ is said to *satisfy an equation* $s \approx t$ if every possible substitution of elements of $A$ for variables in $s$ and $t$ produces an equality. We write $\mathbb{A} \models s \approx t$ to signify that $\mathbb{A}$ satisfies $s \approx t$.

For $\mathcal{K}$ a class of algebras and $\Sigma$ a set of identities we use $Eq(\mathcal{K})$ to denote the set of equations valid in each member of $\mathcal{K}$ and $mod(\Sigma)$ for the class of algebras satisfying each member of $\Sigma$. The notation $\mathcal{K} \models \Sigma$ means each member of $\mathcal{K}$ satisfies each equation in $\Sigma$.

12.2.1 DEFINITION
A class $\mathcal{K}$ of algebras is an *equational class*, or *variety*, if there is a set $\Sigma$ of identities such that $\mathcal{K} = mod(\Sigma)$.

Thus a variety is the class of all algebras satisfying some set of equations. Given any class $\mathcal{K}$ of algebras there is a smallest variety $\mathcal{V}(\mathcal{K})$ containing $\mathcal{K}$, namely $\mathcal{V}(\mathcal{K}) = mod(Eq(\mathcal{K}))$. In particular, there is a smallest variety $\mathcal{V}(\mathbb{A})$ containing a given algebra $\mathbb{A}$, and its members are those algebras that satisfy exactly the same equations as $\mathbb{A}$.

To reiterate, our primary purpose is to give methods to determine which equations will hold in a given algebra $\mathbb{A}$ or class of algebras $\mathcal{K}$ arising in fuzzy set theory. Our technique will be to find an algebra $\mathbb{B}$ whose equational theory is easily determined such that $\mathbb{B}$ generates the same variety as $\mathbb{A}$ or as $\mathcal{K}$. Our primary tools are the algebraic techniques described below.

An algebra $\mathbb{B}$ is a *subalgebra* of an algebra $\mathbb{A}$ if the underlying set of $\mathbb{B}$ is a subset of that of $\mathbb{A}$ and the operations of $\mathbb{B}$ are the restrictions of those of $\mathbb{A}$. A map $f: \mathbb{A} \to \mathbb{B}$ is a *homomorphism* if it is compatible with the basic operations. If the map $f$ is onto, we say $\mathbb{B}$ is a *homomorphic image* of $\mathbb{A}$. For a family $\mathbb{A}_i$ $(i \in I)$ of algebras of the same type, the *product* $\prod_{i \in I} \mathbb{A}_i$ is the algebra whose underlying set is the Cartesian product of the underlying sets of the $\mathbb{A}_i$ and whose operations are defined componentwise. If all the algebras $\mathbb{A}_i$ equal some algebra $\mathbb{A}$ we call the product of the $\mathbb{A}_i$ the *power* $\mathbb{A}^I$ of $\mathbb{A}$. An algebra $\mathbb{B}$ is a *subdirect product* of the family $\mathbb{A}_i$ if $\mathbb{B}$ is a subalgebra of $\prod_{i \in I} \mathbb{A}_i$ and for each $i \in I$ the natural homomorphism from $\mathbb{B}$ to $\mathbb{A}_i$ is onto. Of basic importance is the following theorem of Birkhoff.

12.2.2 THEOREM
*The variety $\mathcal{V}(\mathcal{K})$ generated by $\mathcal{K}$ is the smallest class of algebras containing $\mathcal{K}$ and closed under taking homomorphic images, subalgebras and products.*

One final result, again due to Birkhoff, will be used. This result says that an equation will hold in all members of a variety $\mathcal{V}$ if and only if it holds in certain very special algebras in $\mathcal{V}$ called subdirectly irreducibles. In order to define these, we first briefly review the notion of a congruence.

Given an algebra $\mathbb{A}$, an equivalence relation $\theta$ on the underlying set of $\mathbb{A}$ is called a *congruence* of $\mathbb{A}$ if it is compatible with the operations of $\mathbb{A}$. Specifically this means that for each $n$-ary operation $f$:

$$\text{If } a_i\,\theta\,b_i \text{ for } i = 1,\ldots,n, \text{ then } f(a_1,\ldots,a_n)\,\theta\,f(b_1,\ldots,b_n).$$

Clearly the identical relation $\Delta$ which relates each element only to itself is a congruence on $\mathbb{A}$, as is the universal relation $\nabla$ which relates any two elements of $\mathbb{A}$. An algebra $\mathbb{A}$ is said to be *subdirectly irreducible* if there is a smallest congruence which is not equal to the identity $\Delta$. This is equivalent to requiring that there be elements $a \neq b$ such that $(a,b)$ belongs to every congruence other than the identical relation. The significance of subdirectly irreducibles is conveyed by the following theorem, also due to Birkhoff.

12.2.3 THEOREM
*An equation holds in a variety $\mathcal{V}$ if and only if it holds in every subdirectly irreducible algebra in $\mathcal{V}$.*

The key point is that in many varieties, including the ones of interest here, the subdirectly irreducibles are much better behaved than arbitrary members of the variety. Thus, determining the subdirectly irreducibles can provide a tractable method to determining equational properties.

## 12.3 The basic lattices

The real unit interval $[0,1]$ forms a bounded distributive lattice under its usual ordering with the operations of $\wedge$ and $\vee$ given by min and max. We then define the following.

$$\mathbb{I} = ([0,1],\wedge,\vee,0,1) \text{ is the bounded unit interval.}$$

Just as the real interval $\mathbb{I}$ plays a basic role in the theory of fuzzy sets, a lattice constructed from the collection of closed subintervals of $[0,1]$ plays a basic role in the theory of interval-valued fuzzy sets. This lattice is most easily described by noting that there is a bijection between the non-empty closed subintervals of $[0,1]$ and the set of ordered pairs $(a,b)$ with $0 \leq a \leq b \leq 1$. We then define the following.

$$\mathbb{I}^{[2]} \text{ is } \{(a,b) \mid 0 \leq a \leq b \leq 1\} \text{ with } \wedge,\vee,0,1 \text{ defined componentwise.}$$

A *fuzzy subset* of a set $S$ is a function $f\colon S \to [0,1]$. The collection of all fuzzy subsets of $S$ is therefore the set of all maps from $S$ to $[0,1]$, which is the power $[0,1]^S$. This collection of fuzzy subsets of $S$ can naturally be considered a lattice by defining operations componentwise from those of $\mathbb{I}$. We define the following.

$$\text{The bounded lattice } \mathcal{F}(S) \text{ of fuzzy subsets of a set } S \text{ is the power } \mathbb{I}^S.$$

An *interval-valued fuzzy subset* of a set $S$ is defined to be a mapping from $S$ to the set of all non-empty closed subintervals of $[0,1]$, or equivalently, a mapping from $S$ to $\{(a,b) \mid 0 \leq a \leq b \leq 1\}$. The set of all interval-valued fuzzy subsets of $S$ can naturally be considered a bounded lattice by defining lattice operations componentwise through those of $\mathbb{I}^{[2]}$. We then define the following.

The bounded lattice $\mathcal{IF}(S)$ of interval-valued fuzzy subsets of $S$ is $(\mathbb{I}^{[2]})^S$.

Birkhoff showed that any bounded distributive lattice that has more than one element generates the variety of all bounded distributive lattices. As each of the lattices above is distributive we have the following.

12.3.1 COROLLARY
*A bounded lattice equation is valid in* $\mathbb{I}$, $\mathbb{I}^{[2]}$, $\mathcal{F}(S)$ *or* $\mathcal{IF}(S)$ *if and only if it is valid in the two-element lattice* **2**.

This result is of practical use. It gives a simple and effective method for determining whether a bounded lattice equation is valid in one of the lattices listed above—one simply checks whether the equation is valid in the two-element lattice **2**. Further, equipping these lattices with additional operations, such as a negation or a t-norm, will in no way affect this result for equations involving only the bounded lattice operations $\wedge, \vee, 0, 1$.

## 12.4 Lattices with a negation

We next consider bounded distributive lattices with an additional unary operation.

12.4.1 DEFINITION
A *negation* on a lattice is a unary operation $'$ that satisfies

1. $(x \wedge y)' = x' \vee y'$,
2. $(x \vee y)' = x' \wedge y'$,
3. $x'' = x$.

A bounded distributive lattice with a negation is a *De Morgan algebra*.

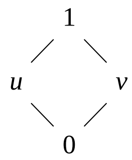
Of basic importance in the study of fuzzy sets is the negation $'$ on the lattice $\mathbb{I}$ defined by $x' = 1 - x$. We call this the *usual negation* on $\mathbb{I}$. There are other negations on $\mathbb{I}$, such as the negation defined by $x' = \sqrt{1 - x^2}$. However, any negation on $\mathbb{I}$ produces an algebra isomorphic to $\mathbb{I}$ with the usual negation [1, 8].

Similarly, there is a negation on the lattice $\mathbb{I}^{[2]}$ of particular importance in the study of interval-valued fuzzy sets. This negation, which is called the *usual negation* on $\mathbb{I}^{[2]}$, is defined by $(a,b)' = (1-b, 1-a)$. One again has the result that any negation on $\mathbb{I}^{[2]}$ produces an algebra isomorphic to $\mathbb{I}^{[2]}$ with the usual negation [9].

Finally, by the *usual negations* on the lattices $\mathcal{F}(S)$ and $\mathcal{IF}(S)$, we mean the negations defined componentwise through the usual negations on $\mathbb{I}$ and $\mathbb{I}^{[2]}$ respectively. We note that there are negations on $\mathcal{F}(S)$ and on $\mathcal{IF}(S)$ producing algebras that are not isomorphic to $\mathcal{F}(S)$ or $\mathcal{IF}(S)$ with the usual negations.

12.4.2 DEFINITION
Define two finite De Morgan algebras as follows.

1. **3** is the lattice $0 < a < 1$ with negation $0' = 1, a' = a, 1' = 0$.
2. $\mathbb{D}$ is the lattice below with negation $0' = 1$, $u' = u$, $v' = v$, $1' = 0$.




These algebras are important in the study of equational properties of De Morgan algebras. Kalman [15] showed that there are exactly four varieties of De Morgan algebras; the variety of all De Morgan algebras, the variety of De Morgan algebras satisfying $x \wedge x' \leq y \vee y'$, which is known as the variety of *Kleene algebras*, the variety of Boolean algebras, and the trivial variety of one-element algebras. Further, $\mathbb{D}$ generates the variety of all De Morgan algebras and **3** generates the variety of all Kleene algebras. This yields the following which can also be found in [2, 10].

12.4.3 COROLLARY
*An equation is valid in $\mathbb{I}$ or $\mathcal{F}(S)$ with the usual negations if and only if it is valid in* **3***; and an equation is valid in $\mathbb{I}^{[2]}$ or $\mathcal{IF}(S)$ with the usual negations if and only if it is valid in $\mathbb{D}$.*

Again, we have a simple and effective method for determining whether an equation is valid in one of the algebras listed above—one simply checks whether the equation is valid in the three- or four-element lattice with negation. See [13] for further discussion, including descriptions of normal forms and truth tables in these settings.

## 12.5 The unit interval with a t-norm

We recall several basic definitions which can be found in [17].

12.5.1 DEFINITION
A *t-norm* is a binary operation on the unit interval that is commutative, associative and satisfies

1. $T(x, y \wedge z) = T(x, y) \wedge T(x, z)$,
2. $T(x, y \vee z) = T(x, y) \vee T(x, z)$,
3. $T(x, 1) = 1$.

For a t-norm $T$ and element $x \in \mathbb{I}$, define recursively $x^n$ by setting $x^0 = 1$ and $x^{n+1} = T(x, x^n)$. We now define several classes of t-norms of particular importance in our study.

12.5.2 DEFINITION
Let $T$ be a t-norm. We say

1. $T$ is *continuous* if it is continuous under the usual topology on $\mathbb{I}$.
2. $T$ is *strict* if it is continuous and $x > 0, y < z \Longrightarrow T(x, y) < T(x, z)$.
3. $T$ is *nilpotent* if it is continuous and $x \neq 1 \Longrightarrow x^n = 0$ for some $n$.
4. $T$ is *idempotent* if it satisfies $T(x, x) = x$, or equivalently $x^2 = x$.

One can easily show that any t-norm satisfies $T(x,y) \leq x \wedge y$. It then follows from this that there is exactly one t-norm that is idempotent. While there are many different strict and nilpotent t-norms, we shall see there are canonical examples of each.

12.5.3 DEFINITION

1. The *product* t-norm $T_{\mathbf{P}}$ is defined by $T_{\mathbf{P}}(x,y) = xy$.
2. The *Łukasiewicz* t-norm $T_{\mathbf{L}}$ is defined by $T_{\mathbf{L}}(x,y) = (x+y-1) \vee 0$.
3. The *minimum* t-norm $T_{\mathbf{M}}$ is defined by $T_{\mathbf{M}}(x,y) = x \wedge y$.

Note that the product t-norm $T_{\mathbf{P}}$ is strict, the Łukasiewicz t-norm $T_{\mathbf{L}}$ is nilpotent, and the minimum t-norm $T_{\mathbf{M}}$ is idempotent. These examples are canonical in the following sense.

12.5.4 THEOREM
*Let $T$ be a t-norm.*

1. *If $T$ is strict, then the algebra $(\mathbb{I}, T)$ is isomorphic to $(\mathbb{I}, T_{\mathbf{P}})$.*
2. *If $T$ is nilpotent, then the algebra $(\mathbb{I}, T)$ is isomorphic to $(\mathbb{I}, T_{\mathbf{L}})$.*
3. *If $T$ is idempotent, then the algebra $(\mathbb{I}, T)$ is equal to $(\mathbb{I}, T_{\mathbf{M}})$.*

Thus, if $T$ is a strict t-norm, the algebras $(\mathbb{I}, T)$ and $(\mathbb{I}, T_p)$ generate the same variety, and if $T$ is nilpotent then $(\mathbb{I}, T)$ and $(\mathbb{I}, T_{\mathbf{L}})$ generate the same variety. In [8], it is shown that each of the algebras $(\mathbb{I}, T_{\mathbf{P}})$ and $(\mathbb{I}, T_{\mathbf{L}})$ can be obtained from the other using homomorphic images, subalgebras and products. Thus these algebras generate the same variety. We therefore have the following.

12.5.5 THEOREM ( [8])
*If $T$ is a t-norm that is either strict or nilpotent, then*

$$\mathcal{V}(\mathbb{I}, T_{\mathbf{P}}) = \mathcal{V}(\mathbb{I}, T) = \mathcal{V}(\mathbb{I}, T_{\mathbf{L}}).$$

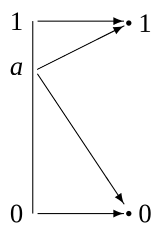
The situation for the idempotent t-norm $T_{\mathbf{M}}$ is particularly simple.

12.5.6 PROPOSITION
*For $(\mathbf{2}, \min)$, the lattice $\mathbf{2}$ with extra binary operation* min*,*

$$\mathcal{V}(\mathbb{I}, T_{\mathbf{M}}) = \mathcal{V}(\mathbf{2}, \min) \subset \mathcal{V}(\mathbb{I}, T_{\mathbf{P}}).$$

*Proof.* For $a \in (0,1]$, let $\varphi_a \colon (\mathbb{I}, T_{\mathbf{M}}) \to (\mathbf{2}, \min)$ be the map defined by

$$\varphi_a(x) = \begin{cases} 1 & \text{if } x \geq a, \\ 0 & \text{if } x < a. \end{cases}$$



Each $\varphi_a$ is a homomorphism, and this family of maps separates points. So the product homomorphism embeds $(\mathbb{I}, T_M)$ into $\prod_{a\in(0,1]}(\mathbf{2}, \min)$. This shows that $(\mathbb{I}, T_\mathbf{M})$ is in $\mathcal{V}(\mathbf{2}, \min)$, and therefore that $\mathcal{V}(\mathbb{I}, T_\mathbf{M}) \subseteq \mathcal{V}(\mathbf{2}, \min)$. The containments $\mathcal{V}(\mathbf{2}, \min) \subseteq \mathcal{V}(\mathbb{I}, T_\mathbf{M})$ and $\mathcal{V}(\mathbf{2}, \min) \subseteq \mathcal{V}(\mathbb{I}, T_\mathbf{P})$ follow as $(\mathbf{2}, \min)$ is a subalgebra of $(\mathbb{I}, T_\mathbf{M})$ and $(\mathbb{I}, T_\mathbf{P})$. The equation $s^2 \approx s$ which holds in $(\mathbb{I}, T_\mathbf{M})$ does not hold in $(\mathbb{I}, T_\mathbf{P})$ so the inclusion is proper. □

The following lemma, which is similar to Ling's [18] characterization of continuous t-norms as ordinal sums of strict, nilpotent and idempotent t-norms, is key to determining equational properties of continuous t-norms.

12.5.7 LEMMA
*If $T$ is a continuous t-norm, the algebra $(\mathbb{I}, T)$ is isomorphic to a subdirect product of algebras $(\mathbb{I}, T_u)$ where each $T_u$ is either a strict t-norm, a nilpotent t-norm, or the idempotent t-norm.*

*Proof.* Let $T$ be a continuous t-norm. Let $Z = \{x \in [0,1] \mid T(x,x) = x\}$. Since $T$ is continuous, $Z$ is a closed subset of $[0,1]$, so

$$[0,1] - Z = \bigcup_{s\in A} X_s$$

where $\{X_s\}_{s\in A}$ is a finite or countably infinite collection of pairwise disjoint open intervals. Also,

$$[0,1] - \overline{\bigcup_{s\in A} X_s} = \bigcup_{t\in B} Y_t$$

for some finite or countably infinite collection of disjoint open intervals $Y_t$. For $s \in S$, the definition of the open interval $X_s$ provides that its closure is equal to $[a,b]$ for some $a,b \in Z$. Then for any $x,y \in [a,b]$ we have that $a = T(a,a) \le T(x,y) \le T(b,b) = b$, so $T$ restricts to a binary operation on the interval $[a,b]$ we denote $T_s$. Define $0_s = a$, $1_s = b$ and set

$$\mathbb{A}_s = (\overline{X_s}, \wedge, \vee, T_s, 0_s, 1_s).$$

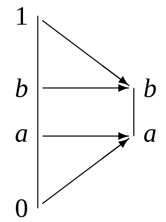
The operation $T_s$ is commutative, associative, and distributes over both joins and meets as it inherits these properties from $T$. If $x \in [a,b]$, then $T(x,b) \le x \le b = T(b,b)$, thus, as $T$ is continuous, $x = T(y,b)$ for some $x \le y \le b$. This gives $T(x,b) = T(T(y,b),b) = T(y,T(b,b)) = T(y,b) = x$, showing that $1_s = b$ is a unit for $T_s$. Note also that $T_s$ is continuous as it is the restriction of a continuous operation $T$, and the definition of $X_s$ provides that $T_s$ has no idempotents other than the endpoints $a,b$.

From the above remarks it follows that $\mathbb{A}_s$ is isomorphic to an algebra $(\mathbb{I}, \hat{T}_s)$ where $\hat{T}_s$ is a continuous t-norm with no nontrivial idempotents, and consequently either a strict or nilpotent t-norm (see [17] for a proof). For $t \in T$ we have $Y_t$ is an open interval contained in the closed set $Z$. So the closure of $Y_t$ is a closed interval $[a,b]$ contained in $Z$. From the definition of $Z$ we have $T(x,x) = x$ for each $x \in [a,b]$, and it follows that $T$ restricts to an idempotent operation $T_t$ on $[a,b]$. Define $0_t = a$, $1_t = b$ and set

$$\mathbb{B}_t = (\overline{Y_t}, \wedge, \vee, T_t, 0_t, 1_t).$$

Again, $T_t$ is commutative, associative, and distributes over joins and meets as it inherits these properties from $T$. Further, for $x \in [a,b]$ we have $x = T(x,x) \le T(x,b) \le x$, hence $b = 1_t$ is a unit for $T_t$. Then as $T_t$ is idempotent, it follows that $\mathbb{B}_t$ is isomorphic to $\mathbb{I}$ with the idempotent t-norm. For $[a,b] = \overline{X_s}$ or $\overline{Y_t}$ define $\varphi_{ab}\colon (\mathbb{I}, T) \to \mathbb{A}_s$ or $\mathbb{B}_t$ by

$$\varphi_{ab}(x) = \begin{cases} b & \text{if } x \geq b, \\ x & \text{if } a \leq x \leq b, \\ a & \text{if } x \leq a. \end{cases}$$




Each $\varphi_{ab}$ is order preserving and therefore, as our algebras are chains, preserves $\wedge, \vee$. Also, by definition, each $\varphi_{ab}$ preserves bounds. To see that $\varphi_{ab}$ is a homomorphism it remains to show that $\varphi_{ab}(T(x,y)) = T_s(\varphi_{ab}(x), \varphi_{ab}(y))$ if $[a,b] = \overline{X_s}$, or a similar statement involving $T_t$ if $[a,b] = \overline{Y_t}$. This follows from the definition of $T_s$ and $T_t$ if $x,y \in [a,b]$. If $x \leq a$ or $y \leq a$ then $T(x,y) \leq a$ and the result follows. If $x \geq b$ and $y \geq b$ then as $T(b,b) = b$ we have $T(x,y) \geq b$ and the result follows. Finally, if $x \leq b$ and $y \geq b$ we have $x = T(x,b) \leq T(x,y) \leq x$, and the result follows.

Since each $\varphi_{ab}$ is a homomorphism, the product of these maps gives a homomorphism from $(\mathbb{I}, T)$ to $\prod_{s \in S} \mathbb{A}_s \times \prod_{t \in T} \mathbb{A}_t$. To show that this map is an embedding, we need to show that the family of maps $\varphi_{ab}$ separates points. Let $x < y \in [0,1]$. There is no problem if either $x$ or $y$ lies in one of the intervals $X_s$. Suppose $x,y \in Z$. If one of the intervals $X_s = [a,b]$ lies between $x$ and $y$, then $\varphi_{ab}(x) = a \neq b = \varphi_{ab}(y)$. If this is not the case, every element between $x$ and $y$ is in $Z$ so that $x,y \in \overline{Y_t} = [a,b]$ for some $t$, and $\varphi_{ab}(x) = x \neq y = \varphi_{ab}(y)$. Each map $\varphi_{ab}$ is onto $\mathbb{A}_s$ or $\mathbb{B}_t$ as the case may be. Thus $(\mathbb{I}, T)$ is a subdirect product of the algebras $\mathbb{A}_s$ and $\mathbb{B}_t$, and each of these is isomorphic to the unit interval with a strict, nilpotent, or the idempotent t-norm. □

12.5.8 THEOREM
*If $T$ is a continuous t-norm that is not idempotent, then*

$$\mathcal{V}(\mathbb{I}, T) = \mathcal{V}(\mathbb{I}, T_{\mathbf{P}}).$$

*Proof.* By Lemma 12.5.7, $(\mathbb{I}, T)$ can be embedded as a subdirect product of algebras $(\mathbb{I}, T_u)$ where each $T_u$ is either a strict, nilpotent, or the idempotent t-norm. By Theorem 12.5.5 and Proposition 12.5.6 each of the algebras $(\mathbb{I}, T_u)$ belongs to $\mathcal{V}(\mathbb{I}, T_{\mathbf{P}})$. This implies that $(\mathbb{I}, T)$ also belongs to $\mathcal{V}(\mathbb{I}, T_{\mathbf{P}})$, hence $\mathcal{V}(\mathbb{I}, T) \subseteq \mathcal{V}(\mathbb{I}, T_{\mathbf{P}})$. As $(\mathbb{I}, T)$ is embedded as a subdirect product of the algebras $(\mathbb{I}, T_u)$, the projections from $(\mathbb{I}, T)$ to the factors $(\mathbb{I}, T_u)$ are all onto mappings. Since $T$ is not idempotent, it cannot be the case that all of the $T_u$ are idempotent. Therefore there is a strict or nilpotent $T_u$ with $(\mathbb{I}, T_u)$ a homomorphic image of $(\mathbb{I}, T)$. It follows from Theorem 12.5.5 that $\mathcal{V}(\mathbb{I}, T_{\mathbf{P}}) \subseteq \mathcal{V}(\mathbb{I}, T)$. □

We summarize our results in terms of equational properties.

12.5.9 THEOREM
*If $T$ is any continuous t-norm other than the idempotent t-norm, then an equation is valid in the unit interval $\mathbb{I}$ with t-norm $T$ if and only if it is valid in the interval $\mathbb{I}$ with the product t-norm. Further, an equation is valid in the interval $\mathbb{I}$ with the idempotent t-norm if and only if it is valid in the finite algebra $(\mathbf{2}, \min)$.*

To conclude this section we note that there are algebras $(\mathbb{I},T)$ in $\mathcal{V}(\mathbb{I},T_{\mathbf{P}})$ where $T$ is not continuous. The drastic t-norm $T_D$ defined in [17] is one such example. Actually, more can be shown. Using $\mathbb{A}$ to denote the subalgebra of $(\mathbb{I},T_{\mathbf{L}})$ with underlying set $\{0,\frac{1}{2},1\}$ we have

$$\mathcal{V}(\mathbb{I},T_D)=\mathcal{V}(\mathbb{A})\subset\mathcal{V}(\mathbb{I},T_{\mathbf{P}}).$$

The proof is similar to that of Proposition 12.5.6.

## 12.6 Distributive lattice ordered commutative monoids

To study further properties of t-norms, it is convenient to work in a more general setting.

12.6.1 Definition

A *bounded distributive lattice ordered commutative monoid* (abbreviated: *dl*-monoid) is a bounded distributive lattice with an additional commutative, associative binary operation $\circ$ that satisfies

1. $x\circ(y\wedge z)=(x\circ y)\wedge(x\circ z)$,
2. $x\circ(y\vee z)=(x\circ y)\vee(x\circ z)$,
3. $x\circ 1=x$.

The class of all *dl*-monoids is a variety we denote by $\mathcal{M}$.

Note that $(I,T)$ is an example of a *dl*-monoid for any t-norm $T$. We next describe a particular *dl*-monoid that plays an important role.

12.6.2 Definition

The infinite cyclic algebra $\mathbb{Z}$ is the *dl*-monoid consisting of the chain $0<\cdots<z_2<z_1<z_0=1$ with binary operation $\circ$ given by

$$x\circ y=\begin{cases}z_{m+n} & \text{if } x=z_m \text{ and } y=z_n,\\ 0 & \text{if either } x \text{ or } y \text{ is } 0.\end{cases}$$

Note, if $\mathbb{A}$ is a *dl*-monoid and $a\in A$, then there is a homomorphism $\varphi\colon\mathbb{Z}\to\mathbb{A}$ mapping the generator $z_1$ of $\mathbb{Z}$ to $a$. This yields the following.

12.6.3 Theorem

*The infinite cyclic algebra $\mathbb{Z}$ is the free dl-monoid on one generator. Thus, an equation in one variable is valid in $\mathbb{Z}$ if and only if it is valid in every dl-monoid.*

The following is key to studying equational properties of *dl*-monoids.

12.6.4 Proposition

*Two dl-monoids satisfy the same equations in the variables $x_1,\ldots,x_k$ if and only if they satisfy the same inequalities*

$$s_1\wedge\cdots\wedge s_m\le t_1\vee\cdots\vee t_n$$

*where each $s_i$ and each $t_j$ is a product of the variables $x_1,\ldots,x_k$.*

*Proof.* Note first that the equation $s \approx t$ holds if and only if each of the inequalities $s \leq t$ and $t \leq s$ holds, and an inequality $s \leq t$ holds if and only if the equation $s \wedge t \approx t$ holds. Thus two *dl*-monoids satisfy the same equations if and only if they satisfy the same inequalities. Suppose we wish to see whether an inequality $s \leq t$ holds. Because the monoid operation $\circ$ distributes over the lattice operations $\wedge, \vee$ in any *dl*-monoid, we can assume that $s$ is a disjunction $s = s_1 \vee s_2 \vee \cdots \vee s_n$ of terms $s_i$, where each $s_i$ is a conjunction $s_i = s_{i1} \wedge s_{i2} \wedge \cdots \wedge s_{in_i}$ of terms $s_{ih}$ with each $s_{ih}$ a product of variables. Similarly, we can assume $t$ is a conjunction $t = t_1 \wedge t_2 \wedge \cdots \wedge t_m$ of terms $t_j$, where each $t_j$ is a disjunction $t_j = t_{j1} \vee t_{j2} \vee \cdots \vee t_{jm_j}$ of terms $t_{jk}$ with each $t_{jk}$ a product of variables. Now the inequality

$$s_1 \vee s_2 \vee \cdots \vee s_n \leq t_1 \wedge t_2 \wedge \cdots \wedge t_m$$

holds if and only if each of the inequalities $s_i \leq t_j$, $i = 1, \ldots, n$, $j = 1, \ldots, m$ holds, and these inequalities are of the form asserted in the statement of the result. □

12.6.5 THEOREM
*An equation is valid in the interval $\mathbb{I}$ with product t-norm $T_{\mathbf{P}}$ if and only if it is valid in the infinite cyclic algebra $\mathbb{Z}$.*

*Proof.* By the previous result, it suffices to show $(\mathbb{I}, T_{\mathbf{P}})$ and $\mathbb{Z}$ satisfy the same inequalities $s \leq t$ where $s = s_1 \wedge s_2 \wedge \cdots \wedge s_m$, $t = t_1 \vee t_2 \vee \cdots \vee t_n$ with

$$s_i = a_1^{p_{i1}} \cdots a_k^{p_{ik}} \text{ and } t_j = a_1^{q_{j1}} \cdots a_k^{q_{jk}}.$$

Here we are assuming the inequality involves $k$ variables $a_1, \ldots, a_k$ and each variable occurs in each product (maybe with exponent 0). If $b \in (0,1)$, then the subalgebra of $(\mathbb{I}, T_{\mathbf{P}})$ generated by $b$ is isomorphic to $\mathbb{Z}$, so any equation valid in $(\mathbb{I}, T_{\mathbf{P}})$ is valid in $\mathbb{Z}$. To show the converse, we assume the inequality $s \leq t$ fails in $(\mathbb{I}, T_{\mathbf{P}})$ and show that it also fails in $\mathbb{Z}$. By continuity, we know that this inequality failing in the real unit interval $[0,1]$ means that it fails in $(0,1)$. Thus it fails for some choice of $a_1, \ldots, a_k$ in $(0,1)$. Given $a \in (0,1)$, we can write $a_i = a^{\lambda_i}$ where $\lambda_i$ belongs to $(0, \infty)$. Thus the function

$$f(\lambda_1, \ldots, \lambda_k) = \bigwedge_{i=1}^{m} a^{\lambda_1 p_{i1} + \cdots + \lambda_k p_{ik}} - \bigvee_{i=1}^{n} a^{\lambda_1 q_{j1} + \cdots + \lambda_k q_{jk}}$$

has at least one positive value. By continuity, we can find a positive value with $\lambda_1, \ldots, \lambda_k$ rational, say $\lambda_1 = \frac{u_1}{v}, \ldots, \lambda_k = \frac{u_k}{v}$. Then, we have that $a_1 = a^{\lambda_1}, \ldots, a_k = a^{\lambda_k}$ provides an instance where the original inequality $s \leq t$ fails. Let $b = a^{\frac{1}{v}}$. Then $a_1 = b^{u_1}, \ldots, a_k = b^{u_k}$. It then follows that the values $a_1, \ldots, a_k$ producing a failure of $s \leq t$ lie in the subalgebra of $(\mathbb{I}, T_{\mathbf{P}})$ generated by $b$, and this subalgebra is isomorphic to $\mathbb{Z}$. □

Combining this result with those of the previous section, we have that any equation valid in the infinite cyclic algebra $\mathbb{Z}$ is valid in $(\mathbb{I}, T)$ for any continuous t-norm $T$. This result has other applications as well.

12.6.6 THEOREM
*For $\mathcal{V}$ a variety of dl-monoids, these are equivalent.*

1. *Each finitely generated algebra in $\mathcal{V}$ is finite.*

2. *The infinite cyclic algebra* $\mathbb{Z}$ *is not in* $\mathcal{V}$.
3. *The algebra* $(\mathbb{I}, T_{\mathbf{P}})$ *is not in* $\mathcal{V}$.
4. *The free algebra in* $\mathcal{V}$ *on one generator is finite.*

*Proof.* (1) implies (2) follows as $\mathbb{Z}$ is finitely generated, in fact generated by a single element we denote by $z$, and is infinite. Theorem 12.6.5 yields (2) is equivalent to (3). To establish (2) implies (4), recall $\mathbb{Z}$ is the free *dl*-monoid on one generator. So, if we use $\mathbb{F}$ for the free algebra in $\mathcal{V}$ on one generator, there is a homomorphism $\varphi$ mapping $\mathbb{Z}$ onto $\mathbb{F}$. As $\mathbb{Z}$ does not belong to $\mathcal{V}$, the map $\varphi$ is not one-one, so there are powers $m < m+k$ with $\varphi(z^m) = \varphi(z^{m+k})$. Since $\varphi$ is a lattice homomorphism, it follows that $\varphi(z^m) = \varphi(z^{m+1})$, and therefore that $\mathbb{F} = \{0, \varphi(z)^m, \varphi(z)^{m-1}, \ldots, \varphi(z), 1\}$. To show (4) implies (1), suppose $\mathbb{F}$ has $k$ elements. Then for $g$ the generator of $\mathbb{F}$ we have $g^k = g^{k+1}$. As $\mathbb{F}$ is free in $\mathcal{V}$, this equation holds in every algebra in $\mathcal{V}$. Suppose $\mathbb{A} \in \mathcal{V}$ is finitely generated. Then only finitely many elements occur as powers of the generators, and as $\circ$ is commutative, only finitely many elements can be obtained as products of powers of the generators. As $\circ$ distributes over $\wedge, \vee$, $\mathbb{A}$ is generated as a distributive lattice by the elements which are products of powers of the generators, and hence is finite. □

We next produce a sequence of results which characterize the subdirectly irreducible *dl*-monoids as certain chains, and establish that every variety of *dl*-monoids is generated by its finite subdirectly irreducibles. Results of Fuchs [7], in a slightly more general setting, already showed each subdirectly irreducible *dl*-monoid is a chain, but did not yield our other results. We remind the reader an algebra is subdirectly irreducible if it has a least nontrivial congruence.

12.6.7 PROPOSITION
*A subdirectly irreducible dl-monoid* $(\mathbb{L}, \circ)$ *has a least nonzero element a. Further,* $(0, a)$ *belongs to each nontrivial congruence on* $(\mathbb{L}, \circ)$*, and the least nontrivial congruence on* $(\mathbb{L}, \circ)$ *is* $\Delta \cup \{(0,a),(a,0)\}$.

*Proof.* Since $(\mathbb{L}, \circ)$ is subdirectly irreducible, there is a pair $(a,b)$ with $a \neq b$ that belongs to every nontrivial congruence of $(\mathbb{L}, \circ)$. Any element $c \in L$ induces a congruence defined by $x \equiv_c y$ if and only if $x \vee c = y \vee c$. Then if $b \neq 0$, $a \equiv_b b$ implies $a \vee b = b \vee b = b$ so $a \leq b$. Also if $a \neq 0$, $a \equiv_a b$ implies that $b \leq a$ so $a = b$. Thus exactly one member of the pair $(a,b)$ is 0, and we take $b = 0$. Then for any $0 \neq c \in L$, we have $0 \equiv_c a$, implying $c = 0 \vee c = a \vee c$ , whence $a \leq c$. Thus $a$ is the least nonzero element of $L$, and it follows that $\equiv_a = \Delta \cup \{(0,a),(a,0)\}$ is the least nontrivial congruence. □

Recall that a nonempty subset $I$ of a lattice $\mathbb{L}$ is called an *ideal* of $\mathbb{L}$ if $x, y \in I$ imply $x \vee y \in I$, and $x \in I$ and $y \leq x$ imply $y \in I$. An ideal is called *prime* if $x \wedge y \in I$ implies $x \in I$ or $y \in I$.

12.6.8 DEFINITION
For $(\mathbb{L}, \circ)$ a *dl*-monoid, $x \in L$ and $I$ an ideal of $\mathbb{L}$ define

$$(I : x) = \{y \in L \mid x \circ y \in I\}.$$

For $I = \{0\}$ we call this the *annihilator* of $x$ and write $(\{0\}, x) = (0 : x)$.

We note that $x \leq y$ implies $(I:x) \supseteq (I:y)$, $(I:x \vee y) = (I:x) \cap (I:y)$, and if $I$ is a prime ideal, then $(I:x \wedge y) = (I:x) \cup (I:y)$.

12.6.9 PROPOSITION
*For* $(\mathbb{L}, \circ)$ *a dl-monoid and* $I$ *a prime ideal of* $\mathbb{L}$ *set*

$$x \equiv y \text{ if and only if } (I:x) = (I:y).$$

*Then* $\equiv$ *is a congruence on* $(\mathbb{L}, \circ)$.

*Proof.* Let $x \equiv y$, and suppose $w(x \vee z) \in I$. Then $w(x \vee z) = wx \vee wz \in I$ implies $wx \in I$ and $wz \in I$, whence $wy \in I$ and $wz \in I$, and thus $wy \vee wz = w(y \vee z) \in I$. It follows by symmetry that

$$(I:(x \vee z)) = (I:(y \vee z)).$$

Suppose $w(x \wedge z) \in I$. Then $w(x \wedge z) = wx \wedge wz \in I$ implies either $wx \in I$ or $wz \in I$, whence either $wy \in I$ or $wz \in I$ and thus $w(y \wedge z) \in I$. It follows by symmetry that

$$(I:(x \wedge z)) = (I:(y \wedge z)).$$

Finally, suppose $w(px) \in I$. Then $(wp)x \in I$, hence $(wp)y = w(py) \in I$. It now follows by symmetry that

$$(I:px) = (I:py).$$

Thus $\equiv$ is a congruence. □

12.6.10 THEOREM
*A dl-monoid* $(\mathbb{L}, \circ)$ *is subdirectly irreducible if and only if it has a least nonzero element and the annihilator ideals*

$$\{(0:x) \mid x \in L\}$$

*are distinct.*

*Proof.* If $(\mathbb{L}, \circ)$ is subdirectly irreducible, then by Proposition 12.6.7, it has a least nonzero element $a$, and the pair $(0,a)$ belongs to every nontrivial congruence. The set $\{0\}$ is an ideal, and since $a$ lies below every nonzero element, $\{0\}$ is a prime ideal. Thus the relation defined by

$$x \equiv y \text{ if and only if } (0:x) = (0:y)$$

is a congruence. But $(0:0) = L \neq (0:a)$ implies $(0,a)$ does not belong to this congruence. Thus $\equiv = \Delta$, that is, the relation induced by the annihilators must be the trivial one. It follows that the annihilators of different elements are distinct. Now suppose the annihilators are distinct and $\mathbb{L}$ has a least nonzero element $a$. If $\equiv$ is any congruence, and $x \equiv y$ with $x \neq y$, then $(0:x) \neq (0:y)$ so there is an element $w$ such that $wx = 0$ and $wy \neq 0$ (or the other way around). But then $0 = wx \wedge a \equiv wy \wedge a = a$, so every nontrivial congruence contains the pair $(0,a)$. Thus $(\mathbb{L}, \circ)$ is subdirectly irreducible. □

12.6.11 PROPOSITION
*If* $(\mathbb{L}, \circ)$ *is subdirectly irreducible with least element* $a$, *then*

$$(0:a) = L - \{1\}.$$

*Proof.* Suppose $x \neq 1$. Then $(0:x) \neq (0:1) = \{0\}$ implies there is a nonzero $b \in L$ such that $xb = 0$. But this implies that $xa = 0$, hence $x \in (0:a)$. □

12.6.12 THEOREM
*Every subdirectly irreducible dl-monoid is a chain.*

*Proof.* Suppose $(\mathbb{L}, \circ)$ is subdirectly irreducible with least nonzero element $a$, and let $c, d \in L$. If $c$ and $d$ are not comparable, then we have

$$c \wedge d < c < c \vee d.$$

It is easy to see that

$$(0 : c \wedge d) \supseteq (0 : c) \supseteq (0 : c \vee d),$$

and by Theorem 12.6.10, both inclusions are proper. Thus there is $p \in (0 : c \wedge d)$ such that $p \notin (0:c)$, and $q \in (0:c)$ such that $q \notin (0 : c \vee d)$. This means

$$pc \neq 0, pd = 0, qc = 0, qd \neq 0.$$

But this means that

$$(c \vee d)(p \wedge q) = (c \vee d)p \wedge (c \vee d)q \geq cp \wedge dq \geq a \neq 0,$$
$$(c \vee d)(p \wedge q) = c(p \wedge q) \vee d(p \wedge q) \leq cq \vee dp = 0.$$

Thus there is no such pair $p, q$. It follows that every pair of elements is comparable, i.e., $\mathbb{L}$ is a chain. □

As each variety of *dl*-monoids is generated by subdirectly irreducibles, we have shown each variety of *dl*-monoids is generated by its members which are chains. We will show each such variety is generated by its finite subdirectly irreducibles, in particular by its finite chains. We require a lemma.

12.6.13 LEMMA
*A finitely generated dl-monoid $\mathbb{C}$ whose underlying lattice is a chain contains no infinite strictly increasing chains.*

*Proof.* Suppose $\mathbb{C}$ is is generated by $\{g_1, g_2, \ldots, g_n\}$ and $0 < x_1 < x_2 < \cdots$ is a strictly increasing chain. Since the underlying lattice of $\mathbb{C}$ is a chain, each nonzero element of $\mathbb{C}$ may be written as a product of powers of the generators. In particular each $x_i = g_1^{k_{i1}} g_2^{k_{i2}} \cdots g_n^{k_{in}}$ with each $k_{ij}$ a nonnegative integer. If $k_{1m} \leq k_{im}$ for all $m = 1, 2, \ldots, n$ then $x_1 \geq x_i$. So for each $i > 1$, there is at least one $m$ such that $k_{1m} > k_{im} \geq 0$. As the sequence of $x_i$'s is infinite and there are only finitely many different natural numbers below the $k_{1m}$'s, there must be some pair of natural numbers $(i_1, m_1)$ for which $\{i \mid k_{im_1} = k_{i_1 m_1}\}$ is infinite. Take the subsequence of all $x_i$'s with $k_{im_1} = k_{i_1 m_1}$. Again, for each $i > i_1$, there is at least one $m$ such that $k_{im} > k_{i_1 m} \geq 0$, and $m \neq m_1$. Thus there is some $(i_2, m_2)$ for which $\{i \mid k_{im_1} = k_{i_1 m_1}$ and $k_{im_2} = k_{i_2 m_2}\}$ is infinite. Again, take the subsequence of all $x_i$'s with both $k_{im_1} = k_{i_1 m_1}$ and $k_{im_2} = k_{i_2 m_2}$. Continuing in this fashion, we get an $(i_n, m_n)$ for which $\{i \mid k_{im_1} = k_{i_1 m_1}, k_{im_2} = k_{i_2 m_2}, \ldots, k_{im_n} = k_{i_n m_n}\}$ is infinite. This then says that $\{i \mid x_i = g_1^{k_{i_1 m_1}} g_2^{k_{i_2 m_2}} \cdots g_n^{k_{i_n m_n}}\}$ is infinite. But our assumption implies that no two $x_i$'s were equal, which is a contradiction. The lemma follows. □

12.6.14 PROPOSITION
*Each finitely generated subdirectly irreducible dl-monoid is a finite algebra whose underlying lattice is a chain.*

*Proof.* Suppose $(\mathbb{L},\circ)$ is a finitely generated subdirectly irreducible *dl*-monoid. By Theorem 12.6.12, $\mathbb{L}$ is a chain, so by Lemma 12.6.13 there are no infinite strictly increasing chains in $\mathbb{L}$. We show also that there are no infinite strictly decreasing chains in $\mathbb{L}$. Suppose $x_1 > x_2 > x_3 > \cdots$ is a strictly decreasing chain in $\mathbb{L}$. If $x_i > x_{i+1}$ then clearly $(0 : x_i) \subseteq (0 : x_{i+1})$, and Theorem 12.6.10 provides that the annihilators $(0 : x_n)$ are strictly increasing. Choosing $y_n \in (0 : x_{n+1}) - (0 : x_n)$ we have $y_1 < y_2 < y_3 < \cdots$ is a strictly increasing chain in $\mathbb{L}$, so this chain is finite. Thus the chain $x_1 > x_2 > x_3 > \cdots$ is also finite. As $\mathbb{L}$ is a chain containing no infinite strictly increasing or infinite strictly decreasing chains, $\mathbb{L}$ is finite. □

12.6.15 THEOREM
*Every variety of dl-monoids is generated by its finite subdirectly irreducible members, all of which are chains.*

*Proof.* It is well known that varieties are closed under direct limits, that every algebra is the direct limit of its finitely generated subalgebras, and that every finitely generated algebra is a subdirect product of finitely generated subdirectly irreducibles. Thus, every variety is generated by its finitely generated subdirectly irreducible algebras, and by Proposition 12.6.14 these are finite chains. □

12.6.16 DEFINITION
For $(\mathbb{L},\circ)$ a finite *dl*-monoid, define the residual $\eta$ on $L$ by

$$\eta(x) = \bigvee \{y \in L \mid y \circ x = 0\}.$$

Combining Theorem 12.6.10 and Proposition 12.6.14 yields the following.

12.6.17 COROLLARY
*A finite dl-monoid is subdirectly irreducible if and only if its underlying lattice is a chain and its residual* $\eta$ *is a negation in the sense of Definition* 12.4.1.

There is difficulty in extending this result to the infinite setting as one needs completeness to ensure the residual $\eta$ is defined. Suppose we assume $\mathbb{A}$ is a *dl*-monoid whose underlying lattice is complete and that satisfies the infinite distributive law $x \circ (\bigvee y_i) = \bigvee (x \circ y_i)$. Then $\mathbb{A}$ is subdirectly irreducible if and only if it is a chain with least nonzero element whose residual $\eta$ is a negation.

Determining equational properties of *dl*-monoids has been reduced to the setting of finite chains whose residual $\eta$ is a negation, but the problem is still far from trivial. With a computer, one can check that there are dozens of such chains with, say, 10 elements. It is not clear whether an effective procedure to determine all such *n*-element chains can be found. And if one is given a particular such chain, there can be difficulties in working with it. Here, a first glimpse of trouble occurs already with quite small chains.

12.6.18 EXAMPLE
The four element chain with operation $\circ$ given by

$\bullet$ 1
|
$\bullet$ $e = e \circ e$
|
$\bullet$ $a$
|
$\bullet$ $0 = e \circ a = a \circ a$

is a subdirectly irreducible *dl*-monoid.

The 4-element *dl*-monoid above can be shown to belong to $\mathcal{V}(\mathbb{I}, T_{\mathbf{P}})$, but this is not a trivial task. It is a homomorphic image of a subalgebra of an ultrapower of $(\mathbb{I}, T_{\mathbf{P}})$, but not a homomorphic image of a subalgebra of $(\mathbb{I}, T_{\mathbf{P}})$ (due to the idempotent $e$).

## 12.7 Equations in two variables

In Section 12.5 we showed that any equation valid in the unit interval $\mathbb{I}$ with the product t-norm $T_{\mathbf{P}}$ is valid in $\mathbb{I}$ with any continuous t-norm. We suspect more is true—that any equation satisfied by $(\mathbb{I}, T_{\mathbf{P}})$ is satisfied by $(\mathbb{I}, T)$ for any t-norm $T$. This would be one of the consequences of the following conjecture.

12.7.1 CONJECTURE
*The equations satisfied by the interval $\mathbb{I}$ with the product t-norm $T_{\mathbf{P}}$ are exactly the equations that are satisfied by all dl-monoids.*

If true, this conjecture would imply $\mathcal{V}(\mathbb{I}, T_{\mathbf{P}})$ is the variety of all *dl*-monoids, and would provide a finite set of equations that define $\mathcal{V}(\mathbb{I}, T_{\mathbf{P}})$, namely the equations used to define *dl*-monoids. We have not proved this result, but can prove the version of it restricted to equations having at most two variables. Thus, we will prove the following.

12.7.2 THEOREM
*Any equation involving at most two variables that is satisfied by the interval $\mathbb{I}$ with product t-norm $T_{\mathbf{P}}$ is satisfied by all dl-monoids, and in particular, is satisfied by $\mathbb{I}$ with any t-norm $T$.*

Our tools will be Proposition 12.6.4 and Theorem 12.6.12. These reduce the problem to showing an inequality of the form $s_1 \wedge \cdots \wedge s_m \leq t_1 \vee \cdots \vee t_n$, with each $s_i$ and $t_j$ a product of the variables, is valid in $(\mathbb{I}, T_{\mathbf{P}})$ if and only if it is valid in every *dl*-monoid whose lattice reduct is a chain.

We use $\mathcal{C}$ to denote the class of all *dl*-monoids whose underlying lattices are chains, and $\mathcal{C} \models s \leq t$ to mean that the inequality $s \leq t$ is valid in all members of $\mathcal{C}$. We write $x \circ y$ and $T_{\mathbf{P}}(x, y)$ simply as $xy$, and use the notation $x^n$ as described after Definition 12.5.1.

12.7.3 LEMMA
*For $x, y, x_1, y_1$ nonnegative integers, the following are equivalent.*

1. $(\mathbb{I}, T_{\mathbf{P}}) \models a^x b^y \leq a^{x+x_1} \vee b^{y+y_1}$.
2. $xy \geq x_1 y_1$.
3. $\mathcal{C} \models a^x b^y \leq a^{x+x_1} \vee b^{y+y_1}$.

*Proof.* Assume that $(\mathbb{I}, T_{\mathbf{P}}) \models a^x b^y \leq a^{x+x_1} \vee b^{y+y_1}$. If $x_1 = 0$ or $y_1 = 0$ then trivially $xy \geq x_1 y_1$. So assume $x_1 > 0$ and $y_1 > 0$. Let $b \in (0,1)$ and $a = b^z$, where $z = (y+y_1)/(x+x_1)$. Then

$$b^{zx+y} = a^x b^y \leq a^{x+x_1} \vee b^{y+y_1} = b^{zx+zx_1} \vee b^{y+y_1} = b^{zx+zx_1}$$

and $b^{zx+y} \leq b^{zx+zx_1}$ implies

$$zx + y \geq zx + zx_1 = y + y_1,$$

from which it follows that $y \geq zx_1$ and $zx \geq y_1$. Thus $zxy \geq zx_1 y_1$, or since $z \neq 0$, $xy \geq x_1 y_1$. Thus (1) implies (2). To show that $xy \geq x_1 y_1$ implies $\mathcal{C} \models a^x b^y \leq a^{x+x_1} \vee b^{y+y_1}$, we will induct on $xy$. If $xy = 0$ then $x_1 y_1 = 0$, so either $x_1 = 0$ or $y_1 = 0$ and $\mathcal{C} \models a^x b^y \leq a^{x+x_1} \vee b^{y+y_1}$ follows. Assume $xy > 0$. As $xy \geq x_1 y_1$, either $x \geq x_1$ or $y \geq y_1$. We will assume without loss of generality that $x \geq x_1$ and consider the two cases $y \geq y_1$ and $y \leq y_1$. First suppose $y \geq y_1$. One readily verifies that $a^x b^y \leq a^{2x} \vee b^{2y}$ is valid in any chain by considering the alternatives $a^x \leq b^y$ and $b^y \leq a^x$. As $x \geq x_1$ and $y \geq y_1$ we then have $2x \geq x + x_1$ and $2y \geq y + y_1$. It follows that $\mathcal{C} \models a^x b^y \leq a^{x+x_1} \vee b^{y+y_1}$. Now suppose $y \leq y_1$. Note that subtracting $x_1 y$ from both sides of the inequality $xy \geq x_1 y_1$ yields

$$(x - x_1) y \geq x_1 (y_1 - y).$$

Our assumptions $x \geq x_1$ and $y \leq y_1$ imply that each of the terms in the above inequality is nonnegative. Therefore by the inductive hypothesis

$$\mathcal{C} \models a^{x-x_1} b^y \leq a^x \vee b^{y_1}.$$

Let $\mathbb{C} \in \mathcal{C}$ and $a, b \in \mathbb{C}$. If $b^y \leq a^{x_1}$ then $a^x b^y \leq a^{x+x_1}$. And if $a^{x_1} \leq b^y$ then

$$\begin{aligned} a^x b^y &= a^{x_1} a^{x-x_1} b^y \leq a^{x_1} (a^x \vee b^{y_1}) \\ &= a^{x_1} a^x \vee a^{x_1} b^{y_1} \leq a^{x_1} a^x \vee b^y b^{y_1} = a^{x+x_1} \vee b^{y+y_1}. \end{aligned}$$

This shows (2) implies (3). It is a trivial observation that (3) implies (1), because $(\mathbb{I}, T_{\mathbf{P}}) \in \mathcal{C}$. □

12.7.4 LEMMA
*For $x, y, x_1, y_1$ nonnegative integers, the following are equivalent.*

1. $(\mathbb{I}, T_{\mathbf{P}}) \models a^{x+x_1} \wedge b^{y+y_1} \leq a^x b^y$.
2. $xy \leq x_1 y_1$.
3. $\mathcal{C} \models a^{x+x_1} \wedge b^{y+y_1} \leq a^x b^y$.

*Proof.* The proof is similar to that of the previous lemma, we only sketch the outline. For (1) implies (2), let $b \in (0,1)$ and set $a = b^z$ where $z$ satisfies $(x + x_1) z = y + y_1$. For (2) implies (3), we show that $xy \leq x_1 y_1$ implies $\mathcal{C} \models a^{x+x_1} \wedge b^{y+y_1} \leq a^x b^y$ by inducting on $x_1 y_1$. The case $x_1 y_1 = 0$ implies either $x = 0$ or $y = 0$ leading to a trivial case such as $\mathcal{C} \models a^{x_1} \wedge b^{y+y_1} \leq b^y$. For $x_1 y_1 > 0$ we may assume without loss of generality that $x_1 \geq x$.

If also $y_1 \geq y$ then from the observation $\mathcal{C} \models a^{2x} \wedge b^{2y} \leq a^x b^y$ our result follows. We are left to consider $x_1 \geq x$ and $y_1 \leq y$. Noting $xy \leq x_1 y_1$ implies $x(y - y_1) \leq (x_1 - x)y_1$ the inductive hypothesis gives $\mathcal{C} \models a^{x_1} \wedge b^y \leq a^x b^{y-y_1}$. Our result then follows by considering the two cases $a^x \leq b^{y_1}$ and $b^{y_1} \leq a^x$. And, of course, (3) implies (1) is again trivial. □

For the next step, we establish an inequality with a more general right hand side. This is obtained easily by eliminating trivial cases and then reducing to the previous case.

12.7.5 LEMMA
*For nonnegative integers $x, y, p_1, q_1, p_2, q_2$, these are equivalent.*

1. $(\mathbb{I}, T_{\mathbf{P}}) \models a^x b^y \leq a^{p_1} b^{q_1} \vee a^{p_2} b^{q_2}$.
2. $\mathcal{C} \models a^x b^y \leq a^{p_1} b^{q_1} \vee a^{p_2} b^{q_2}$.

*Proof.* Since $(\mathbb{I}, T_{\mathbf{P}})$ belongs to $\mathcal{C}$, it is only necessary to show (1) implies (2). Suppose that $a^x b^y \leq a^{p_1} b^{q_1} \vee a^{p_2} b^{q_2}$ for all $a, b \in [0,1]$. First we eliminate the possibility that both $x < p_1$ and $x < p_2$, for if this were true then for all $a, b \in (0,1]$ we would have

$$b^y \leq a^{p_1 - x} b^{q_1} \vee a^{p_2 - x} b^{q_2},$$

which fails for $b = 1$ and $a \in (0,1)$. Similarly we cannot have both $y < q_1$ and $y < q_2$. If $x \geq p_1$ and $y \geq q_1$, then

$$a^x b^y \leq a^{p_1} b^{q_1} \leq a^{p_1} b^{q_1} \vee a^{p_2} b^{q_2},$$

and similarly, if $x \geq p_2$ and $y \geq q_2$, then

$$a^x b^y \leq a^{p_2} b^{q_2} \leq a^{p_1} b^{q_1} \vee a^{p_2} b^{q_2},$$

and in either case, the inequality holds for all chains $\mathbb{C} \in \mathcal{C}$. Thus if $x \geq p_2$ we may assume that $q_2 > y$, whence also $y \geq q_1$. And from $y \geq q_1$, we may assume that $p_1 > x$. Thus we have the case $p_1 > x \geq p_2$ and $q_2 > y \geq q_1$, so

$$(\mathbb{I}, T_{\mathbf{P}}) \models a^{x - p_2} b^{y - q_1} \leq a^{p_1 - p_2} \vee b^{q_2 - q_1}$$

with $p_1 - p_2 \geq x - p_2$ and $q_2 - q_1 \geq y - q_1$, and by Lemma 12.7.3 the same inequality holds for all chains in $\mathcal{C}$. Then multiplying both sides by $a^{p_2} b^{q_1}$ gets the original inequality to hold in all members of $\mathcal{C}$. The argument for $x \geq p_1$ is similar. □

A dual argument establishes the following.

12.7.6 LEMMA
*For nonnegative integers $x, y, p_1, q_1, p_2, q_2$, these are equivalent.*

1. $(\mathbb{I}, T_{\mathbf{P}}) \models a^{p_1} b^{q_1} \wedge a^{p_2} b^{q_2} \leq a^x b^y$.
2. $\mathcal{C} \models a^{p_1} b^{q_1} \wedge a^{p_2} b^{q_2} \leq a^x b^y$.

The following lemma reduces the general case to the previous ones. Its proof relies heavily on the linear geometry of the situation.

12.7.7 LEMMA
*For nonnegative integers $x_i$, $y_i$, $p_i$, $q_i$ these are equivalent.*

1. $(\mathbb{I}, T_{\mathbf{P}}) \models a^{x_1}b^{y_1} \wedge a^{x_2}b^{y_2} \wedge \cdots \wedge a^{x_n}b^{y_n} \leq a^{p_1}b^{q_1} \vee a^{p_2}b^{q_2} \vee \cdots \vee a^{p_m}b^{q_m}$.
2. *At least one of the following is true.*
   (a) *There is* $1 \leq i \leq n$ *and* $1 \leq j \leq m$ *with*

$$(\mathbb{I}, T_{\mathbf{P}}) \models a^{x_i}b^{y_i} \leq a^{p_j}b^{q_j}.$$

   (b) *There is* $1 \leq i \leq n$ *and* $1 \leq j < k \leq m$ *with*

$$(\mathbb{I}, T_{\mathbf{P}}) \models a^{x_i}b^{y_i} \leq a^{p_j}b^{q_j} \vee a^{p_k}b^{q_k}.$$

   (c) *There is* $1 \leq i < j \leq n$ *and* $1 \leq k \leq m$ *with*

$$(\mathbb{I}, T_{\mathbf{P}}) \models a^{x_i}b^{y_i} \wedge a^{x_j}b^{y_j} \leq a^{p_k}b^{q_k}.$$

*Proof.* We observe first that

$$(\mathbb{I}, T_{\mathbf{P}}) \models a^{x_1}b^{y_1} \wedge a^{x_2}b^{y_2} \wedge \cdots \wedge a^{x_n}b^{y_n} \leq a^{p_1}b^{q_1} \vee a^{p_2}b^{q_2} \vee \cdots \vee a^{p_m}b^{q_m} \tag{12.1}$$

is equivalent to requiring that for all $0 \leq \lambda < \infty$

$$\max\{x_i + \lambda y_i\}_{i=1}^{n} \geq \min\{p_j + \lambda q_j\}_{j=1}^{m}.$$

To see this, note the inequality (12.1) holding for all $a, b \in [0,1]$ is equivalent, via a continuity argument, to it holding for all $a \in (0,1)$, $b \in (0,1]$, hence it is equivalent to it holding for all $a \in (0,1)$ and all $b = a^{\lambda}$ for some $0 \leq \lambda < \infty$. Thus the inequality (12.1) being valid in $(\mathbb{I}, T_{\mathbf{P}})$ is equivalent to requiring that for all $a \in (0,1)$ and all $0 \leq \lambda < \infty$

$$a^{x_1+\lambda y_1} \wedge a^{x_2+\lambda y_2} \wedge \cdots \wedge a^{x_n+\lambda y_n} \leq a^{p_1+\lambda q_1} \vee a^{p_2+\lambda q_2} \vee \cdots \vee a^{p_m+\lambda q_m}.$$

As $a \in (0,1)$ this is equivalent to requiring that for all $0 \leq \lambda < \infty$

$$\max\{x_i + \lambda y_i\}_{i=1}^{n} \geq \min\{p_j + \lambda q_j\}_{j=1}^{m}.$$

Thus, our task is reduced to showing that if for all $0 \leq \lambda < \infty$

$$\max\{x_i + \lambda y_i\}_{i=1}^{n} \geq \min\{p_j + \lambda q_j\}_{j=1}^{m}$$

then either there are $1 \leq i < j \leq m$ and $1 \leq k \leq m$ so that for all $0 \leq \lambda < \infty$

$$\max\{x_i + \lambda y_i, x_j + \lambda y_j\} \geq p_k + \lambda q_k,$$

or there are $1 \leq i < j \leq m$ and $1 \leq k \leq m$ so that for all $0 \leq \lambda < \infty$

$$x_i + \lambda y_i \geq \min\{p_j + \lambda q_j, p_k + \lambda q_k\}.$$

Define functions $f, g, h$ on $[0, \infty)$ by setting

$$\begin{aligned} f(\lambda) &= \max\{x_i + \lambda y_i\}_{i=1}^{n}, \\ g(\lambda) &= \min\{p_j + \lambda q_j\}_{j=1}^{m}, \\ h(\lambda) &= f(\lambda) - g(\lambda). \end{aligned}$$

Then $f, g$ are continuous and made up of finitely many linear functions. Therefore $h$ also has these properties. Further, our assumption that $f \geq g$ implies that $h \geq 0$. As $h \geq 0$ and is made up of finitely many linear functions, it must have an absolute minimum on $[0,\infty)$. This minimum can be chosen to occur at a value $\lambda_0$ where either $\lambda_0 = 0$ or $h$ has a vertex at $\lambda_0$. Note that $h$ having a vertex at $\lambda_0$ implies that either $f$ or $g$, or both, have a vertex at $\lambda_0$. Consider several cases. Suppose $\lambda_0 = 0$. Then if $x_i + \lambda y_i$ is the first linear segment of $f$ and $p_j + \lambda q_j$ is the first linear segment of $g$, we have $x_i \geq p_j$ and $y_i \geq q_j$ since $h \geq 0$ and $\lambda_0 = 0$ is a minimum of $h$. Therefore, for all $0 \leq \lambda < \infty$, $x_i + \lambda y_i \geq p_j + \lambda q_j$. Suppose $g$ has a vertex at $\lambda_0$ and $f$ does not have a vertex at $\lambda_0$. Then there is a neighborhood of $\lambda_0$ on which $f = f'$ and $g = g'$ where

$$\begin{aligned} f'(\lambda) &= x_i + \lambda y_i, \\ g'(\lambda) &= \min\{p_j + \lambda q_j, p_k + \lambda q_k\} \end{aligned}$$

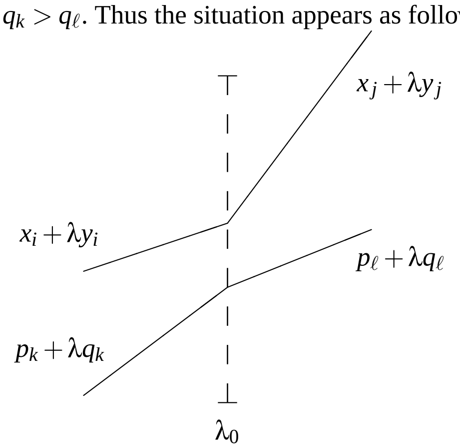
for some $1 \leq i \leq m$ and $1 \leq j < k \leq n$. Then $f' - g'$ agrees with $h$ on a neighborhood of $\lambda_0$, and it follows that $f' - g'$ has a local minimum at $\lambda_0$. But $f' - g'$ is comprised of two linear functions. Hence this local minimum is an absolute minimum. It follows that $f' - g' \geq 0$. So for all $0 \leq \lambda < \infty$

$$x_i + \lambda y_i \geq \min\{p_j + \lambda q_j, p_k + \lambda q_k\}.$$

Similarly, if $f$ has a vertex at $\lambda_0$ and $g$ does not, we find there are $1 \leq i < j \leq m$ and $1 \leq k \leq n$ so that for all $0 \leq \lambda < \infty$

$$\max\{x_i + \lambda y_i, x_j + \lambda y_j\} \geq p_k + \lambda q_k.$$

Finally, assume both $f$ and $g$ have vertices at $\lambda_0$. Then there is a neighborhood of $\lambda_0$ on which $f = f'$ and $g = g'$ where $f'(\lambda) = \max\{x_i + \lambda y_i, x_j + \lambda y_j\}$ and $g'(\lambda) = \min\{p_j + \lambda q_j, p_k + \lambda q_k\}$ for some $1 \leq i < j \leq m$ and some $1 \leq k < \ell \leq n$. Without loss of generality we assume $y_i < y_j$ and $q_k > q_\ell$. Thus the situation appears as follows.




As $h$ has a minimum at $\lambda_0$, it follows that $y_i - q_k \leq 0$ and $y_j - q_\ell \geq 0$, hence $y_i \leq q_k$ and $y_j \geq q_\ell$. It follows that either $y_i \leq q_\ell \leq y_j$ or $q_\ell \leq y_i \leq q_k$. In the first case we have for all $0 \leq \lambda < \infty$

$$x_i + \lambda y_i \geq \min\{p_j + \lambda q_j, p_k + \lambda q_k\}.$$

This completes the proof, as the second case is similar. □

Note that $(\mathbb{I}, T_{\mathbf{P}}) \models a^x b^y \leq a^p b^q$ if and only if $x \geq p$ and $y \geq q$ if and only if $\mathcal{C} \models a^x b^y \leq a^p b^q$. Therefore the previous lemma reduces the general case to known cases, hence proving Theorem 12.7.2.

The difficulty in extending this proof to three or more variables seems to lie, in part, in reducing the general case to a simple situation as in Lemma 12.7.3.

## 12.8 Varieties generated by De Morgan systems

12.8.1 DEFINITION
A *De Morgan system* is an algebra $(\mathbb{I}, T, \eta)$, where $T$ is a t-norm, and $\eta$ is a negation. We call a De Morgan system *strict* if $T$ is strict, and *nilpotent* if $T$ is nilpotent.

There are two families of De Morgan systems that play an important role.

12.8.2 DEFINITION
For $\eta$ a negation on $\mathbb{I}$, set

1. $\mathbb{I}_\eta = (\mathbb{I}, T_{\mathbf{P}}, \eta)$ where $T_{\mathbf{P}}$ is the product t-norm.
2. $\mathbb{J}_\eta = (\mathbb{I}, T_{\mathbf{L}}, \eta)$ where $T_{\mathbf{L}}$ is the Łukasiewicz t-norm.

Note that each $\mathbb{I}_\eta$ is strict and each $\mathbb{J}_\eta$ is nilpotent.

We use the symbol $\cong$ to denote isomorphism of algebras.

12.8.3 THEOREM ( [8])
*Let $\mathbb{A}$ be a De Morgan system.*

1. *If $\mathbb{A}$ is strict, then $\mathbb{A} \cong \mathbb{I}_\eta$ for some negation $\eta$.*
2. *If $\mathbb{A}$ is nilpotent, then $\mathbb{A} \cong \mathbb{J}_\eta$ for some negation $\eta$.*

Thus, to determine the equations valid in all strict (nilpotent) De Morgan systems, one may restrict attention to De Morgan systems having the usual product (Łukasiewicz) t-norm. Still, the situation is quite complicated. The following result shows that for any two non-isomorphic strict De Morgan systems, there are equations valid in one, and not the other. In particular, $(\mathbb{I}, T_P, ')$ does not play the fundamental role for De Morgan systems we conjecture $(\mathbb{I}, T_P)$ plays for t-norms.

12.8.4 THEOREM ( [12])
*For any negations $\gamma$ and $\beta$, these are equivalent.*

1. $\mathbb{I}_\gamma \cong \mathbb{I}_\beta$.
2. $\mathbb{I}_\gamma$ *and* $\mathbb{I}_\beta$ *generate the same variety.*
3. $\mathbb{I}_\gamma$ *and* $\mathbb{I}_\beta$ *generate comparable varieties.*
4. $\mathbb{I}_\gamma$ *and* $\mathbb{I}_\beta$ *satisfy the same inequalities in the family*

$$(\eta((\eta((x \wedge \eta(x))^m))^n))^l \leq (y \vee \eta(y))^k \tag{12.2}$$

*where $m, n, l, k$ range over the nonnegative integers.*

For any negation $\eta$ the variety $\mathcal{V}(\mathbb{I}_\eta)$ is not generated by its finite members. In fact, $\mathcal{V}(\mathbb{I}_\eta)$ has a largest proper subvariety and this subvariety contains all finite members of $\mathcal{V}(\mathbb{I}_\eta)$ [12]. We suspect that no $\mathcal{V}(\mathbb{I}_\eta)$ can be defined by a finite set of equations, and that each $\mathcal{V}(\mathbb{I}_\eta)$ is defined by a family of equations in (12.2) together with the equations defining a *dl*-monoid and a Kleene algebra.

12.8.5 DEFINITION
A nilpotent De Morgan system satisfying $T(x,\eta(x)) = 0$ is called a *Boolean system*.

12.8.6 THEOREM ( [11])
*For $T$ a nilpotent t-norm, the residual*

$$\eta_T = \bigvee\{y \mid T(x,y) = 0\}$$

*is a negation and $(\mathbb{I}, T, \eta_T)$ is a Boolean system. Further, each Boolean system arises in this manner.*

As a Boolean system is a nilpotent De Morgan system, by Theorem 12.8.3 each is isomorphic to some $\mathbb{J}_\eta$. The following is not unexpected [12].

12.8.7 THEOREM
*Each Boolean system is isomorphic to $\mathbb{J}_\alpha$ where $\alpha(x) = 1 - x$ is the usual negation. Therefore, each Boolean system generates the variety of all MV-algebras.*

Thus, there is a finite set of equations defining the variety generated by any Boolean system—the well-known equations defining MV-algebras [4]. We wonder if there is a finite set of equations defining the variety generated by all De Morgan systems. Perhaps the equations defining *dl*-monoids together with those defining Kleene algebras comprise such a finite set, we don't know. This problem is similar to one raised by P. Hájek, R. Cignoli, F. Esteva, and others [14,5,6] asking whether the algebras consisting of a continuous t-norm and its residuum generate the variety of all BL-algebras.

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