Algebra univers. **55** (2006) 293–296
0002-5240/06/020293 – 04
DOI 10.1007/s00012-006-1969-x

Algebra Universalis

On profinite completions and canonical extensions

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This paper is dedicated to Walter Taylor.

Abstract. We show that if a variety V of monotone lattice expansions is finitely generated, then profinite completions agree with canonical extensions on V. The converse holds for varieties of finite type.

It is a matter of folklore that the profinite completion of a Boolean algebra B is given by the power set of the Stone space of B, or in the terminology of Jónsson and Tarski [5], by the canonical extension of B. Similarly, the profinite completion of a distributive lattice D is given by the lattice of upsets of the Priestley space of D, or equivalently, by the canonical extension of D [4].

In [1], Bezhanishvili *et. al.*, give a description of the profinite completion of a Heyting algebra in terms of its dual space. As a consequence of this, they obtain that for a variety V of Heyting algebras, profinite completions coincide with canonical extensions for all members of V if, and only if, V is finitely generated. It is our purpose here to show that this result holds in a wider setting.

Theorem. *For any variety V of monotone lattice expansions, the first condition below implies the second, and for varieties of finite type the conditions are equivalent.*

(1) *V is finitely generated.*
(2) *Profinite completions coincide with canonical extensions on V.*

Before proving the theorem, we recall a few basics. As defined by Gehrke and Harding [2], the canonical completion of a bounded lattice L is a pair (e, C) where (i) $e\colon L \to C$ is a bounded lattice embedding, (ii) C is a complete lattice, (iii) each element of C is a join of meets and a meet of joins of elements of the image $e[L]$ of L, and (iv) if F, I are a filter and ideal of L, then $\bigwedge e[F] \leq \bigvee e[I]$ implies $F \cap I \neq \emptyset$.

Suppose L is a bounded lattice and f is an n-ary operation on L that in each coordinate either preserves or reverses order. We call f a monotone operation on L, and call a bounded lattice L with a family of monotone operations a monotone

Presented by I. Hodkinson.
Received May 14, 2005; accepted in final form September 8, 2005.
2000 *Mathematics Subject Classification*: 06B23; 08B25, 06E25.
Key words and phrases: profinite completion, canonical extension, lattice expansion, finitely generated.

lattice expansion. As shown in [2], such a monotone operation f can be naturally extended to an operation f^σ on the canonical extension of L, so we may speak of the canonical extension of a monotone lattice expansion.

Proof of the Theorem. Assume V is a finitely generated variety of bounded lattice expansions and $A \in V$. Let Φ be the set of all congruences θ on A with A/θ finite and let $\hat{A}$ be the set of all elements $\alpha \in \prod_\Phi A/\theta$ such that if $\alpha(\theta) = a/\theta$ and $\theta \subseteq \theta'$, then $\alpha(\theta') = a/\theta'$. Then $\hat{A}$ is the profinite completion of A. Let $e\colon A \to \hat{A}$ be defined by $e(a)(\theta) = a/\theta$ for each $\theta \in \Phi$. We will show that $(e, \hat{A})$ is the canonical extension of A.

Claim 1. $e\colon A \to \hat{A}$ is a bounded lattice embedding.

Clearly e is a homomorphism. As V is finitely generated and congruence distributive (each algebra in V has a lattice reduct), all subdirectly irreducibles in V are finite, and it follows that e is an embedding.

Claim 2. $\hat{A}$ is a complete sublattice of $\prod_\Phi A/\theta$.

We must show that joins and meets in $\hat{A}$ are the pointwise join and meet of the product. Suppose $S \subseteq \hat{A}$ with α being the join of S in the product. To show α is the join of S in $\hat{A}$ it suffices to show α is an element of $\hat{A}$. Suppose $\alpha(\theta) = a/\theta$ and $\theta \subseteq \theta'$. Then for $S/\theta = \{b/\theta \mid b \in S\}$ we have a/θ is the join of S/θ. As A/θ is finite, there is a finite subset $S' \subseteq S$ with $\bigvee S/\theta = \bigvee S'/\theta$. Then as $\theta \subseteq \theta'$, we have $\bigvee S/\theta' = \bigvee S'/\theta'$, giving $\alpha(\theta') = a/\theta'$. So $\alpha \in \hat{A}$. Meets are handled similarly.

Claim 3. Each $\alpha \in \hat{A}$ is a join of meets and a meet of joins of elements of $e[A]$.

For each $\alpha \in \hat{A}$ and $\theta \in \Phi$, define k_α^θ in $\hat{A}$ by setting $k_\alpha^\theta = \bigwedge\{e(a) \mid \alpha(\theta) \leq e(a)(\theta)\}$. As meets in $\hat{A}$ are componentwise, $k_\alpha^\theta(\theta) = \alpha(\theta)$. Suppose $\phi \in \Phi$ and set $\psi = \theta \wedge \phi$. As $A/\psi \leq A/\theta \times A/\phi$, we have A/ψ is finite, so $\psi \in \Phi$. Let $a \in A$ be such that $\alpha(\psi) = a/\psi$. Then as $\psi \subseteq \theta, \phi$, we have $\alpha(\theta) = a/\theta$ and $\alpha(\phi) = a/\phi$. Then as $\alpha(\theta) = e(a)(\theta)$, we have $k_\alpha^\theta \leq e(a)$, hence $k_\alpha^\theta(\phi) \leq a/\phi = \alpha(\phi)$. Therefore $k_\alpha^\theta \leq \alpha$ and $k_\alpha^\theta(\theta) = \alpha(\theta)$. As joins in $\hat{A}$ are componentwise, $\alpha = \bigvee\{k_\alpha^\theta \mid \theta \in \Phi\}$, hence α is a join of meets of elements of $e[A]$. Similarly, α is a meet of joins of elements of $e[A]$.

Claim 4. If F, I are a filter and ideal of A then $\bigwedge e[F] \leq \bigvee e[I]$ implies $F \cap I \neq \emptyset$.

Suppose $F \cap I = \emptyset$. We must show $\bigwedge e[F] \not\leq \bigvee e[I]$, or equivalently, that there is some $\theta \in \Phi$ with $a/\theta \not\leq b/\theta$ for all $a \in F$ and $b \in I$. Let $A \leq \prod_X A_x$ be a representation of A as a subdirect product of subdirectly irreducibles. As V is congruence distributive and finitely generated, there is a finite upper bound N on the size of the subdirectly irreducibles in V, hence on the size of the algebras A_x. Set

$$\mathcal{I} = \{S \subseteq X \mid \text{ there exists } a \in F,\, b \in I \text{ with } a(x) \leq b(x) \text{ for all } x \in S\}.$$

If $S \in \mathcal{I}$ and $S' \subseteq S$, then surely $S' \in \mathcal{I}$. Also, as F is a filter and I is an ideal, it follows that $\mathcal{I}$ is closed under finite unions, and as $F \cap I = \emptyset$, it follows that $X \notin I$. Thus there is an ultrafilter $\mathcal{U}$ on X that is disjoint from $\mathcal{I}$. Set $A_{\mathcal{U}}$ to be the ultraproduct of the A_x by $\mathcal{U}$ and note that as each A_x has at most N elements, so does $A_{\mathcal{U}}$. Let θ be the kernel of the natural map from A to $A_{\mathcal{U}}$, so $a\theta b \Leftrightarrow \{x \mid a(x) = b(x)\} \in \mathcal{U}$. Then A/θ is finite, and by construction $a/\theta \not\leq b/\theta$ for all $a \in F$ and $b \in I$.

Claims 1–4 establish that as a lattice, $(e, \hat{A})$ is the canonical extension of A. Suppose f is an additional monotone operation of A. We wish to show that the canonical extension f^σ agrees with the coordinatewise operation $\hat{f}$ on $\hat{A}$ associated with f. We do so under the assumption that f is unary and order preserving, the general case is no different, only more cumbersome. We briefly recall the definition of f^σ as given in [2]. We say $k \in \hat{A}$ is closed if there is a (necessarily unique) filter F in A such that $k = \bigwedge e[F]$. For k closed, $f^\sigma(k)$ is defined to be $\bigwedge\{ef(a) \mid a \in F\}$, and for an arbitrary $\alpha \in \hat{A}$, one defines $f^\sigma(\alpha)$ to be the join of all $f^\sigma(k)$ where k is closed and $k \leq \alpha$.

Claim 5. For f a basic operation of A, $f^\sigma = \hat{f}$.

We first show that f^σ and $\hat{f}$ agree on closed elements of $\hat{A}$. Suppose $k = \bigwedge e[F]$, that $\theta \in \Phi$, and that $b \in A$ is such that b/θ is least in $\{a/\theta \mid a \in F\}$. Then $k(\theta) = b/\theta$, and so $\hat{f}(k)(\theta) = f(b)/\theta$. But as meets in $\hat{A}$ are componentwise, we have that $f^\sigma(k)(\theta) = \bigwedge\{f(a)/\theta \mid a \in F\}$. As f is order preserving, we have $f(b)/\theta$ is least in $\{f(a)/\theta \mid a \in F\}$, and it follows that $f^\sigma(k)(\theta) = \hat{f}(k)(\theta)$. So f^σ and $\hat{f}$ agree on closed elements. Suppose α is an arbitrary element of $\hat{A}$. As f^σ and $\hat{f}$ agree on closed elements, the definition of f^σ gives $f^\sigma(\alpha) = \bigvee\{\hat{f}(k) \mid k \text{ is closed and } k \leq \alpha\}$. As f is order preserving, $k \leq \alpha$ implies $\hat{f}(k) \leq \hat{f}(\alpha)$, so $f^\sigma(\alpha) \leq \hat{f}(\alpha)$. For any $\theta \in \Phi$ we have k^θ_α is closed and $k^\theta_\alpha \leq \alpha$. So $\hat{f}(k^\theta_\alpha) \leq f^\sigma(\alpha)$, and in particular $\hat{f}(k^\theta_\alpha)(\theta) \leq f^\sigma(\alpha)(\theta)$. As $k^\theta_\alpha(\theta) = \alpha(\theta)$, we have $\hat{f}(\alpha)(\theta) \leq f^\sigma(\alpha)(\theta)$. So $\hat{f}(\alpha) \leq f^\sigma(\alpha)$, hence equality.

We have shown that if V is finitely generated, then profinite completions and canonical extensions coincide for all members of V. Suppose that V is of finite type (only finitely many basic operations) and that profinite completions and canonical extensions coincide for all members of V. Then in particular, for each $A \in V$ the map $e\colon A \to \hat{A}$ is an embedding. So V cannot have any infinite subdirectly irreducibles, or in other words, V is residually finite. It then follows from a result of Kearnes and Willard [6] that V is finitely generated. This concludes the proof of the theorem. □

Remark 1. The notion of a β-canonical extension, introduced in [3], allows some operations of a monotone lattice expansion to be extended via a join of meets as above, and others to be extended via a meet of joins. The completion type β says which operations are to be extended in which way. Our result applies to β-canonical extensions with only the obvious modifications needed to the proof.

Remark 2. The Theorem provides an alternate proof of the result found in [2] that every finitely generated variety of monotone lattice expansions is closed under canonical extensions.

Remark 3. Matters are not fully settled for varieties that are not of finite type. The above proof shows that any variety in which profinite completions coincide with canonical extensions must be residually finite. For a variety that is residually $< N$ (all subdirectly irreducibles are bounded in size by a finite number N), the above proof shows that profinite completions agree with canonical extensions. However, we do not know whether profinite completions must coincide with canonical extensions for varieties that are residually finite but not residually $< N$.

Acknowledgements. I thank Keith Kearnes for several helpful communications during the preparation of this note.

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