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MACNEILLE COMPLETIONS OF MODAL ALGEBRAS

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Abstract. For a modal algebra (B, f), there are two natural ways to extend f to an operation on the MacNeille completion of B. The resulting structures are called the lower and upper MacNeille completions of (B, f). In this paper we consider lower and upper MacNeille completions for various varieties of modal algebras. In particular, we characterize the varieties of closure algebras and diagonalizable algebras that are closed under lower and upper MacNeille completions. We also introduce the variety of Sierpinski algebras, and show that although this variety is not closed under lower or upper MacNeille completions, it follows from the axiom of choice that each Sierpinski algebra has a MacNeille completion that is also a Sierpinski algebra, and that this result implies the Boolean ultrafilter theorem.

1. Introduction

For a modal algebra (B, f), there are two natural ways to extend f to an operation on the MacNeille completion $\overline{B}$ of B. Define $\underline{f}, \overline{f} : \overline{B} \to \overline{B}$ by setting

$$\underline{f}x = \bigvee\{fa | a \in B \text{ and } a \leq x\}$$

and

$$\overline{f}x = \bigwedge\{fa | a \in B \text{ and } x \leq a\}.$$

We call $(\overline{B}, \underline{f})$ and $(\overline{B}, \overline{f})$ the lower and upper MacNeille completions of (B, f), respectively. Lower MacNeille completions of modal algebras were first studied by Monk [33]. Givant and Venema [18] investigated the identities preserved under lower MacNeille completions when the modal operator is conjugated. Lower and

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upper MacNeille completions of lattices with additional operations were discussed by Gehrke, Harding, and Venema [17].

In this note we investigate varieties of modal algebras closed under lower and upper MacNeille completions. We show that none of the varieties of all modal algebras, all weakly derivative algebras, all derivative algebras, all closure algebras, all Grzegorczyk algebras, or all diagonalizable algebras are closed under lower MacNeille completions. We also characterize which varieties of closure algebras and diagonalizable algebras are closed under lower MacNeille completions.

We show that the variety of all modal algebras, the variety of all weakly derivative algebras, the variety of all derivative algebras, the variety of all closure algebras, and the variety of all monadic algebras are closed under upper MacNeille completions. We also characterize which varieties of closure algebras and diagonalizable algebras are closed under upper MacNeille completions.

We introduce Sierpinski algebras and Boolean triples and show that the category of Sierpinski algebras is equivalent to the category of Boolean triples. From our results on lower and upper MacNeille completions it follows that the variety of Sierpinski algebras is closed under neither lower nor upper MacNeille completions. Nevertheless, we imply from the axiom of choice, using the equivalence of the categories of Sierpinski algebras and Boolean triples, that each Sierpinski algebra has a MacNeille completion that is a Sierpinski algebra as well. Finally, we show that this result implies the Boolean ultrafilter theorem.

The paper is organized as follows. Section 2 consists of preliminaries, and in it we recall the basic definitions used throughout the paper. In Section 3 we discuss the lower and upper MacNeille completions of modal algebras. In particular, we establish that several basic varieties of modal algebras are closed under upper MacNeille completions, but not under lower MacNeille completions. We also characterize which varieties of closure algebras and diagonalizable algebras are closed under lower MacNeille completions. In Section 4 we characterize which varieties of closure algebras are closed under upper MacNeille completions, and in Section 5 we characterize which varieties of diagonalizable algebras are closed under upper MacNeille completions. In Section 6 we first introduce Sierpinski algebras and axiomatize the variety of Sierpinski algebras; it follows from our earlier results on lower and upper MacNeille completions that the variety of Sierpinski algebras is closed under neither lower nor upper MacNeille completions; then we introduce Boolean triples and show that the category of Sierpinski algebras is equivalent to the category of Boolean triples; and finally, using this equivalence, we show that the axiom of choice implies that every Sierpinski algebra has a MacNeille completion that is a Sierpinski algebra as well. We finish the paper by showing that this fact implies the Boolean ultrafilter theorem.

2. Preliminaries

Definition 1. A modal algebra is a pair $\mathfrak{A} = (B, f)$ where B is a Boolean algebra and f is a unary operation on B that satisfies

(1) $f0 = 0$,
(2) $f(a \vee b) = fa \vee fb$.

A modal algebra is called a weak derivative algebra if it satisfies

3. $ffa \leq a \vee fa$.

A modal algebra is called a derivative algebra if it satisfies

4. $ffa \leq fa$.

A derivative algebra is called a closure algebra if it satisfies

5. $a \leq fa$.

A closure algebra is called a monadic algebra if it satisfies

6. $fa \leq -f - fa$.

A closure algebra is called a Grzegorczyk algebra if it satisfies

7. $a \leq f(a - f(fa - a))$.

A derivative algebra is called a diagonalizable algebra if it satisfies

8. $fa \leq f(a - fa)$.

We use **MA**, **wDA**, **DA**, **CA**, **Mon**, **Grz**, and **Diag** for the varieties of modal algebras, weak derivative algebras, derivative algebras, closure algebras, monadic algebras, Grzegorczyk algebras, and diagonalizable algebras, respectively.

Remark. These definitions of derivative algebras and closure algebras were given by McKinsey and Tarski [31] to conduct an algebraic study of topological spaces. Given a topological space (X, τ), if one considers the power set $\mathcal{P}(X)$ with the topological closure operator $\mathbf{C}$, then the pair $(\mathcal{P}(X), \mathbf{C})$ forms a closure algebra. Similarly, if the space (X, τ) is T_1, then $\mathcal{P}(X)$ with the derived set operator δ forms a derivative algebra $(\mathcal{P}(X), \delta)$. Esakia has shown [16] that for any topological space (X, τ), that $(\mathcal{P}(X), \delta)$ is a weak derivative algebra, and that $(\mathcal{P}(X), \delta)$ is a derivative algebra if, and only if, the space (X, τ) satisfies the T_D separation axiom (i.e. every point is the intersection of an open set and a closed set).

Monadic algebras were introduced by Halmos [20] for an algebraic study of the one variable fragment of predicate logic. Grzegorczyk algebras were introduced by Esakia [14] for an algebraic study of Grzegorczyk's modal system, and diagonalizable algebras were introduced by Magari [27] for an algebraic study of the Gödel-Löb provability logic.

Due to the topological connections mentioned above, we will often use **C** for the operation of a closure algebra and δ for that of a derivative algebra.

Remark. There is a well-known duality theory for modal algebras [37]. To each modal algebra (B, f) one associates a pair (X, R) consisting of a Stone space X and a binary relation R on X that satisfies (i) for each $x \in X$, the set $R[x] = \{y \in X | xRy\}$ is closed, and (ii) for each clopen $A \subseteq X$, the set $R^{-1}[A] = \{y \in X | yRa$ for some $a \in A\}$ is clopen. The dual spaces of weak derivative algebras are exactly those (X, R) where R is weakly transitive (xRy and yRz and $x \neq z$ imply xRz) [16], and the dual spaces of derivative algebras are those (X, R) where R is transitive [24]. The dual spaces of closure algebras are those (X, R) with R reflexive and transitive [24], and the dual spaces of monadic algebras are those (X, R) where R is an equivalence relation [20]. Unfortunately there is no first-order characterization of the dual spaces of diagonalizable algebras or Grzegorczyk algebras [4, pages 130–132].

A diagram of portions of the lattice of varieties of modal algebras is given below.

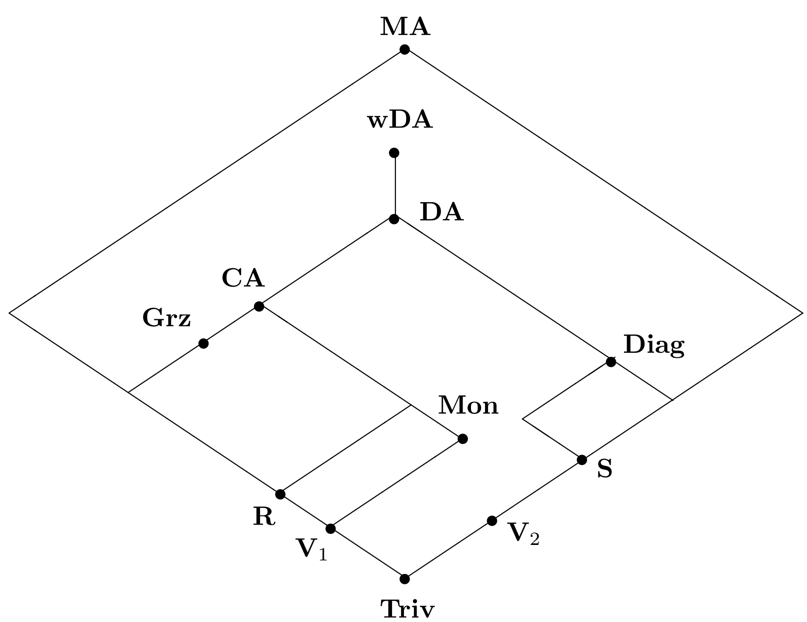

Figure 1

This diagram is intended to show only containments between varieties. It is not the case that **DA** is the join of the varieties **CA** and **Diag**. However, it does happen that **CA** and **Diag** intersect in the trivial variety, and that **Grz** and **Mon** intersect in $\mathbf{V}_1$.

The varieties $\mathbf{V}_1$, $\mathbf{V}_2$, **R**, and **S** are the varieties generated by the following algebras.

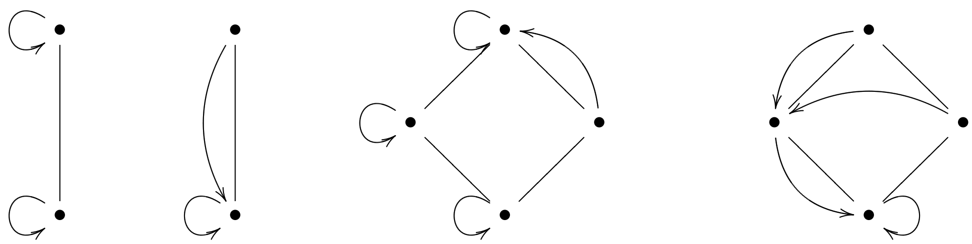

Figure 2

In each case, the action of the additional unary operation on the (two or four element) Boolean algebra is indicated by arrows.

It is known that every subvariety of **MA** other than the trivial variety contains either $\mathbf{V}_1$ or $\mathbf{V}_2$ [28]. It is also known that any variety of closure algebras that is not contained in **Mon** contains the variety **R** [29]. Consequently, every nontrivial variety of Grzegorczyk algebras different from $\mathbf{V}_1$ contains **R**. Similarly, every nontrivial variety of diagonalizable algebras different from $\mathbf{V}_2$ contains **S** [30]. We call the variety **S** the variety of *Sierpinski algebras*. This variety will play an important role in our considerations.

Definition 2. For any weak derivative algebra $\mathfrak{A} = (B, \delta)$ we define auxiliary operations **C**, **I** on B by setting

$$\mathbf{C}a = a \vee \delta a \quad \textit{and} \quad \mathbf{I}a = -\mathbf{C} - a.$$

The following basic facts are well-known [16, 32].

Proposition 2.1.

(1) *If* $\mathfrak{A} = (B, \delta)$ *is a weak derivative algebra, then* $(B, \mathbf{C})$ *is a closure algebra.*
(2) *If* $\mathfrak{A} = (B, \mathbf{C})$ *is a closure algebra, then* $H = \{\mathbf{I}a | a \in B\}$ *forms a Heyting algebra where* $a \to b = \mathbf{I}(-a \vee b)$.

These facts form the basis of various connections between the categories of weak derivative algebras **wDA**, closure algebras **CA**, and Heyting algebras **HA**. There is a well-developed theory of the connections between the categories **CA** and **HA** [5, 6, 15], and to a lesser extent that of connections between **wDA** and **CA** [16, 26]. The particular facts we will need are given below.

Theorem 2.2. *There are functors* $F : \mathbf{wDA} \to \mathbf{CA}$ *and* $G : \mathbf{CA} \to \mathbf{HA}$ *where* $F(B, \delta) = (B, \mathbf{C})$ *and* $G(B, \mathbf{C}) = \{\mathbf{I}a | a \in B\}$. *Further*

(1) *The functor* F *commutes with the class operators* **H**, **P**, *but not* **S**.
(2) *The functor* G *commutes with the class operators* **H**, **S**, **P** *and therefore provides a complete lattice homomorphism from the lattice of subvarieties of* **CA** *to the lattice of subvarieties of* **HA**.

Remark. It can further be shown [5, 13] that G provides an isomorphism between the lattice of subvarieties of **Grz** and the lattice of subvarieties of **HA**.

3. Lower and upper MacNeille completions

Definition 3. For $\mathfrak{A} = (B, f)$ a modal algebra, we let $\overline{B}$ be the MacNeille completion of the Boolean algebra B and define maps $\underline{f}, \overline{f} : \overline{B} \to \overline{B}$, called the lower and upper extensions of f, by setting

$$\underline{f}x = \bigvee \{fa | a \in B \text{ and } a \leq x\},$$

and

$$\overline{f}x = \bigwedge \{fa | a \in B \text{ and } x \leq a\}.$$

We call $\underline{\mathfrak{A}} = (\overline{B}, \underline{f})$ the lower MacNeille completion of $\mathfrak{A}$ and $\overline{\mathfrak{A}} = (\overline{B}, \overline{f})$ the upper MacNeille completion of $\mathfrak{A}$.

Lower MacNeille completions were introduced by Monk [33] and later studied by Givant and Venema [18]. Upper and lower MacNeille completions of lattices with additional operations were discussed by Gehrke, Harding, and Venema [17].

Remark. For $\mathfrak{A} = (B, f)$ a modal algebra with dual space (X, R), one may recover an algebra isomorphic to (B, f) by taking the Boolean algebra Clopen(X) of clopen subsets of X and the unary operation R^{-1} on this algebra. One may also realize algebras isomorphic to the lower and upper MacNeille completions of $\mathfrak{A}$ through the dual space. For the lower MacNeille completion we take the Boolean algebra Reg(X) of regular open subsets of X with additional operation $\mathbf{IC}R^{-1}$, and for the upper MacNeille completion we take Reg(X) with additional operation $\mathbf{I}R^{-1}\mathbf{C}$.

To see this we note that joins in $\mathrm{Reg}(X)$ are given by taking $\mathbf{IC}$ of the union and meets in $\mathrm{Reg}(X)$ are given by taking $\mathbf{I}$ of the intersection. Then for any regular open set U, $\bigvee\{R^{-1}K|K$ clopen and $K \subseteq U\}$ is equal to $\mathbf{IC}\bigcup\{R^{-1}K|K$ clopen and $K \subseteq U\}$, and this equals $\mathbf{IC}R^{-1}\bigcup\{K|K$ clopen and $K \subseteq U\}$, which is equal to $\mathbf{IC}R^{-1}U$. Also $\bigwedge\{R^{-1}K|K$ clopen and $U \subseteq K\}$ is equal to $\mathbf{I}\bigcap\{R^{-1}K|K$ clopen and $U \subseteq K\}$. By Esakia's Lemma (see, e.g., [8, page 350]) this is equal to $\mathbf{I}R^{-1}\bigcap\{K|K$ clopen and $U \subseteq K\}$, which is equal to $\mathbf{I}R^{-1}\mathbf{C}U$.

The following definition, closely related to the notion of residuation [7], is due to Jónsson and Tarski [24].

Definition 4. We say a unary operation f on a Boolean algebra B is conjugated if there is a unary operation g on B such that for all $a, b \in B$ we have

$$fa \wedge b = 0 \quad \textit{if, and only if,} \quad a \wedge gb = 0.$$

Givant and Venema [18] proved the following result, although in somewhat different terminology.

Theorem 3.1. *If $\mathfrak{A} = (B, f)$ is a Boolean algebra with a conjugated unary operation f, then $\underline{\mathfrak{A}} = \overline{\mathfrak{A}}$.*

Proof. Using f^d to denote the operation given by $f^d a = -f - a$, Givant and Venema proved [18, Lemmas 18 and 21] that if f is conjugated then the lower extension of f^d is equal to $(\underline{f})^d$. (Note, the notation f^+ of Givant and Venema corresponds to our $\underline{f}$.) One can then easily show, directly from the definitions, that the lower extension of f^d is equal to $(\overline{f})^d$. So $(\underline{f})^d = (\overline{f})^d$, and therefore $\underline{f} = \overline{f}$. □

Remark. Suppose (B, f) is a modal algebra whose operation f has a conjugate g. Then it is well-known that g is also an operator on B (i.e. preserves finite joins), see [24], and therefore the dual space can naturally be considered as a triple (X, R, S) where S is the relation corresponding to the operator g. Moreover, the conjugacy of f, g implies [24] that S is the converse of R. Therefore $R[K]$ is clopen for each clopen K, and it can then be shown that R^{-1} commutes with the topological closure operation $\mathbf{C}$. Therefore $\mathbf{IC}R^{-1} = \mathbf{I}R^{-1}\mathbf{C}$, providing a dual method to show that $\underline{f}$ and $\overline{f}$ coincide when f is conjugated.

Lemma 3.2. *Every subvariety of* **Mon** *is closed under lower MacNeille completions.*

Proof. Every subvariety of **Mon** can be defined by strictly positive identities [22], that is, identities that do not use the Boolean negation. As the operator of any monadic algebra is conjugated [22], it then follows from results of Givant and Venema [18] that these strictly positive identities are preserved under lower MacNeille completions. □

In conjunction with this result, the following theorem completely determines which subvarieties of **CA** and **Diag** are closed under lower MacNeille completions. There are rather few of them.

Theorem 3.3.

(1) *If a subvariety of* **MA** *contains* **R**, *then it is not closed under lower MacNeille completions.*
(2) *If a subvariety of* **MA** *contains* **S**, *then it is not closed under lower MacNeille completions.*
(3) *The varieties* $\mathbf{V}_1$ *and* $\mathbf{V}_2$ *are closed under lower MacNeille completions.*
(4) *If* $\mathfrak{A}$ *is a monadic algebra, then* $\underline{\mathfrak{A}} = \overline{\mathfrak{A}}$.

PROOF. (1) Let B be the Boolean algebra of all finite and cofinite subsets of the natural numbers ω and consider the algebra $\mathfrak{A} = (B \times 2, f)$ where $f(a,b) = (a,0)$ if a is a finite subset of ω and $b = 0$, and $f(a,b) = (a,1)$ in all other cases.

For each natural number n let φ_n be the map from $\mathfrak{A}$ to the two-element closure algebra $(2, \mathrm{id})$ defined by $\varphi_n(a,b) = 0$ if $n \notin a$ and $\varphi_n(a,b) = 1$ if $n \in a$. It is routine to check that each φ_n is a homomorphism.

Next, let $\mathfrak{R}$ be the four-element closure algebra $(\{0,p,q,1\}, \mathbf{C})$ where $\mathbf{C}0 = 0$, $\mathbf{C}p = p$, and $\mathbf{C}q = \mathbf{C}1 = 1$. As mentioned in the preliminaries, $\mathfrak{R}$ generates the variety **R**. Define $\varphi_\omega : \mathfrak{A} \to \mathfrak{R}$ by setting $\varphi_\omega(a,b) = 0$ if a is finite and b is 0, $\varphi_\omega(a,b) = p$ if a is finite and b is 1, $\varphi_\omega(a,b) = q$ if a is cofinite and b is 0, and $\varphi_\omega(a,b) = 1$ if a is cofinite and b is 1. One can check that φ_ω is a homomorphism and that the set of maps $\{\varphi_n | n \in \omega\} \cup \{\varphi_\omega\}$ separates points. Therefore $\mathfrak{A}$ is a subalgebra of the product of copies of the two-element closure algebra, which is a subalgebra of $\mathfrak{R}$, and $\mathfrak{R}$, so $\mathfrak{A}$ belongs to **R**.

One sees easily that $\overline{B \times 2}$ is the product of the power set of ω and 2. Let Odd and Even be the sets of odd and even natural numbers, respectively. Then

$$\underline{f}(\mathrm{Odd}, 0) = \bigvee \{f(a,b) | (a,b) \in B \times 2 \text{ and } (a,b) \leq (\mathrm{Odd}, 0)\},$$

$$\underline{f}(\mathrm{Even}, 0) = \bigvee \{f(a,b) | (a,b) \in B \times 2 \text{ and } (a,b) \leq (\mathrm{Even}, 0)\}.$$

Therefore $\underline{f}(\mathrm{Odd}, 0) = (\mathrm{Odd}, 0)$ and $\underline{f}(\mathrm{Even}, 0) = (\mathrm{Even}, 0)$. But $\underline{f}(\omega, 0) = f(\omega, 0) = (\omega, 1)$. So the lower MacNeille completion $\underline{\mathfrak{A}}$ is not a modal algebra. As any subvariety of **MA** that contains **R** will contain the algebra $\mathfrak{A}$, our result follows.

(2) Again let B be the Boolean algebra of all finite and cofinite subsets of ω and consider the algebra $\mathfrak{B} = (B \times 2, g)$ where $g(a,b) = (0,0)$ if a is finite and $g(a,b) = (0,1)$ if a is cofinite. We note that for $f(a,b)$ the function from Part 1, that $f(a,b) = (a,b) \vee g(a,b)$. Using the same maps φ_n and φ_ω as in Part 1, we can see that each φ_n is a homomorphism from $\mathfrak{B}$ to the two-element diagonalizable algebra (where the operation δ must satisfy $\delta 1 = \delta 0 = 0$), and that φ_ω is a homomorphism from $\mathfrak{B}$ to the algebra $\mathfrak{S} = (\{0,p,q,1\},\delta)$ where $\delta 0 = \delta p = 0$ and $\delta q = \delta 1 = p$. As mentioned in the preliminaries, the algebra $\mathfrak{S}$ generates the variety **S**. As the set of maps $\{\varphi_n | n \in \omega\} \cup \{\varphi_\omega\}$ separates points, and the two-element diagonalizable algebra is a homomorphic image of $\mathfrak{S}$, the algebra $\mathfrak{B}$ belongs to the variety **S**.

If we again compute the action of the lower MacNeille extension $\underline{g}$ on $(\mathrm{Odd},0)$ and $(\mathrm{Even},0)$ we obtain that $\underline{g}(\mathrm{Odd},0) = (0,0)$ and $\underline{g}(\mathrm{Even},0) = (0,0)$. As $\underline{g}(\omega,0) = g(\omega,0) = (0,1)$, we see that $\underline{\mathfrak{B}}$ is not even a modal algebra. Again, it follows that if a subvariety of **MA** contains **S**, then it is not closed under lower MacNeille completions.

(3) One sees easily (and it is well-known [28]) that the variety $\mathbf{V}_1$ is defined by the Boolean algebra identities and the identity $fa = a$; and the variety $\mathbf{V}_2$ is defined by the Boolean algebra identities and the identity $f1 = 0$. Clearly if $\mathfrak{A}$ satisfies $fa = a$, then $\underline{\mathfrak{A}}$ satisfies $\underline{f}x = x$ since each $x \in \overline{B}$ is equal to $\bigvee\{a \in B | a \leq x\}$. And obviously if $\mathfrak{A}$ satisfies $f1 = 0$, then $\underline{\mathfrak{A}}$ satisfies $\underline{f}1 = 0$. So $\mathbf{V}_1$ and $\mathbf{V}_2$ are closed under lower MacNeille completions.

(4) This follows from Theorem 3.1 as the unary operation of any monadic algebra is conjugated [22]. □

Corollary 3.4.

(1) *None of the varieties* **MA**, **wDA**, **DA**, **CA**, **Grz**, **R**, **Diag**, **S** *are closed under lower MacNeille completions.*

(2) *A subvariety of* **CA** *is closed under lower MacNeille completions if, and only if, it is a subvariety of* **Mon**.

(3) *A subvariety of* **Grz** *is closed under lower MacNeille completions if, and only if, it is equal to the trivial variety or the variety* $\mathbf{V}_1$.

(4) *A subvariety of* **Diag** *is closed under lower MacNeille completions if, and only if, it is equal to the trivial variety or the variety* $\mathbf{V}_2$.

Proof. (1) Each of these varieties contains either **R** or **S**, so the result follows by the first and second parts of Theorem 3.3. (2) Any subvariety of **Mon** is closed under lower MacNeille completions by Lemma 3.2. Any subvariety of **CA** not contained in **Mon** contains **R** [29] so is not closed under lower MacNeille completions. (3) Clearly the trivial variety is closed under lower MacNeille completions, and by Part 3 of Theorem 3.3 $\mathbf{V}_1$ is closed under lower MacNeille completions. Any subvariety of **Grz** other than these contains **R** so is not closed under lower MacNeille completions. (4) Clearly the trivial variety is closed under lower MacNeille completions, and by Part 3 of Theorem 3.3, so also is $\mathbf{V}_2$. Any subvariety of **Diag** other than these contains **S** [30] so by Part 2 of Theorem 3.3 is not closed under lower MacNeille completions. □

Therefore, lower MacNeille completions are of limited use when applied to varieties of closure algebras or diagonalizable algebras. In fact, the only such varieties that are closed under lower MacNeille completions are also closed under upper MacNeille completions, and the lower and upper MacNeille completions of each algebra in the variety coincide. Matters for upper MacNeille completions are somewhat better, as is shown in the following result.

Theorem 3.5. *Each of the varieties* **MA***,* **wDA***,* **DA***, and* **CA** *is closed under upper MacNeille completions.*

Proof. Let $\mathfrak{A} = (B, f)$ be a modal algebra. Then as $f0 = 0$ we have $\overline{f}0 = 0$. For any $x, y \in \overline{B}$ we have $\overline{f}x \vee \overline{f}y$ is equal to $\bigwedge\{fa|a \in B \text{ and } x \leq a\} \vee \bigwedge\{fb|b \in B \text{ and } y \leq b\}$. Using the infinite distributive law for the complete Boolean algebra $\overline{B}$ we have $\overline{f}x \vee \overline{f}y = \bigwedge\{fa \vee fb|a, b \in B \text{ and } x \leq a, y \leq b\}$. Then as $\mathfrak{A}$ satisfies $fa \vee fb = f(a \vee b)$ we have $\overline{f}x \vee \overline{f}y = \bigwedge\{f(a \vee b)|a, b \in B \text{ and } x \leq a, y \leq b\}$. As $\{a \vee b|a, b \in B \text{ and } x \leq a, y \leq b\}$ is equal to $\{c|c \in B \text{ and } x \vee y \leq c\}$, we have $\overline{f}x \vee \overline{f}y = \bigwedge\{fc|c \in B \text{ and } x \vee y \leq c\}$, hence $\overline{f}x \vee \overline{f}y = \overline{f}(x \vee y)$. So the upper MacNeille completion of $\mathfrak{A}$ is a modal algebra.

Suppose next that the modal algebra $\mathfrak{A}$ satisfies the identity $ffa \leq a \vee fa$. Then for any $x \in \overline{B}$, as x is equal to the meet of the elements of B above it, we have $x \vee \overline{f}x = \bigwedge\{a|a \in B \text{ and } x \leq a\} \vee \bigwedge\{fb|b \in B \text{ and } x \leq b\}$. Using complete distributivity, $x \vee \overline{f}x = \bigwedge\{a \vee fb|a, b \in B \text{ and } x \leq a, b\}$. As $\{c|c \in B \text{ and } x \leq c\}$ is down-directed, we then have that $x \vee \overline{f}x = \bigwedge\{c \vee fc|c \in B \text{ and } x \leq c\}$. Then as $\mathfrak{A}$ satisfies $ffa \leq a \vee fa$ we have $\bigwedge\{ffc|c \in B \text{ and } x \leq c\} \leq x \vee \overline{f}x$. But for any $c \in B$ with $x \leq c$ we have $\overline{f}\,\overline{f}x \leq \overline{f}\,\overline{f}c = ffc$, so $\overline{f}\,\overline{f}x \leq x \vee \overline{f}x$. So the variety **wDA** is closed under upper MacNeille completions.

Suppose that the modal algebra $\mathfrak{A}$ satisfies the identity $ffa \leq fa$. Then for any $x \in \overline{B}$ we have $\overline{f}\,\overline{f}x \leq \bigwedge\{ffa|a \in B \text{ and } x \leq a\}$ since $a \in B$ and $x \leq a$ imply $\overline{f}\,\overline{f}x \leq \overline{f}\,\overline{f}a = ffa$. Thus as $\mathfrak{A}$ satisfies $ffa \leq fa$, we have

$\overline{f}\,\overline{f}x \leq \bigwedge\{fa | a \in B \text{ and } x \leq a\}$, so $\overline{f}\,\overline{f}x \leq \overline{f}x$. So the variety **DA** is closed under upper MacNeille completions.

Suppose that the modal algebra $\mathfrak{A}$ satisfies the identity $a \leq fa$. Then for any $x \in \overline{B}$ we have $x = \bigwedge\{a | a \in B \text{ and } x \leq a\}$, so $x \leq \bigwedge\{fa | a \in B \text{ and } x \leq a\}$, giving $x \leq \overline{f}x$. As we have already seen the variety **DA** is closed under upper MacNeille completions, it follows that **CA** is closed under upper MacNeille completions. □

Remark. Based on Jónsson's treatment of the preservation of Sahlqvist identities under canonical completions [23], Givant and Venema [18] have conducted a study of which identities are preserved under lower MacNeille completions when the operators involved are conjugated. Recently Theunissen and Venema [38] have considered the non-conjugated setting, and among other results have produced a Sahlqvist-type theorem that encompasses the results in Theorem 3.5 above.

In the next section we will classify those subvarieties of **CA** that are closed under upper MacNeille completions. In the fifth section we will show that the variety **Diag** is not closed under upper MacNeille completions and classify those subvarieties of **Diag** that are closed under upper MacNeille completions.

4. Upper MacNeille completions of closure algebras

As mentioned in Section 2, there is a close connection between closure algebras and Heyting algebras. Here we exploit this connection, and existing results about MacNeille completions of Heyting algebras, to determine which subvarieties of **CA** are closed under upper MacNeille completions. We begin with the following well-known result [3, Page 238, Exercise 11].

Theorem 4.1. *The MacNeille completion of the lattice reduct of a Heyting algebra H naturally forms a Heyting algebra we denote $\overline{H}$.*

We note that the implication $\rightarrow$ of a Heyting algebra is completely determined by the underlying lattice structure. So, there is one and only one way to extend the implication $\rightarrow$ of a Heyting algebra H to an implication on the MacNeille completion of the lattice reduct of H.

We next consider the connection between the functor $G : \mathbf{CA} \rightarrow \mathbf{HA}$ discussed in the preliminaries and MacNeille completions. The key result is the following.

Proposition 4.2. *For $\mathfrak{A} = (B, \mathbf{C})$ a closure algebra, $G\overline{\mathfrak{A}} = \overline{G\mathfrak{A}}$.*

Proof. Let $H = G\mathfrak{A}$ and $\widehat{H} = G\overline{\mathfrak{A}}$ be the Heyting algebras of open elements of $\mathfrak{A}$ and $\overline{\mathfrak{A}}$ respectively. As $\mathfrak{A}$ is a subalgebra of $\overline{\mathfrak{A}}$, we have that H is a subalgebra of $\widehat{H}$. It is well known, and not difficult to show, that in a complete closure algebra, the join of open elements is open. Thus $\widehat{H}$ is complete. Using the well-known abstract characterization of the MacNeille completion as the unique (to isomorphism) join and meet dense completion [3], to show that $\widehat{H} = \overline{H}$ it is sufficient to show that H is join and meet dense in $\widehat{H}$.

Suppose $x \in \widehat{H}$, so x is open in $\overline{\mathfrak{A}} = (\overline{B}, \overline{\mathbf{C}})$. For convenience, we use $\widehat{\mathbf{I}}$ for the interior operator of $\overline{\mathfrak{A}}$, so $\widehat{\mathbf{I}} = -\overline{\mathbf{C}}-$. As B is meet dense in $\overline{B}$ we have $x = \bigwedge\{a | a \in B \text{ and } x \leq a\}$. But $a \in B$ and $x \leq a$ imply $\widehat{\mathbf{I}}x \leq \mathbf{I}a$. So $x = \widehat{\mathbf{I}}x \leq \bigwedge\{\mathbf{I}a | a \in B \text{ and } x \leq a\} \leq \bigwedge\{a | a \in B \text{ and } x \leq a\} = x$. Therefore x is the meet (in $\overline{B}$) of elements of H, and as $x \in \widehat{H}$, it is the meet (in $\widehat{H}$) of elements of H. Set $y = -x$. Then $x = \widehat{\mathbf{I}}x$ yields $y = \overline{\mathbf{C}}y$. By the definition of $\overline{\mathbf{C}}$ we have $\overline{\mathbf{C}}y = \bigwedge\{\mathbf{C}a | a \in B \text{ and } y \leq a\}$, so $x = -y = \bigvee\{-\mathbf{C}a | a \in B \text{ and } y \leq a\}$. As each $-\mathbf{C}a$ is open, we have x is the join (in $\overline{B}$) of elements of H, and as $x \in \widehat{H}$, it follows that x is the join (in $\widehat{H}$) of elements of H. □

Definition 5. We let $\Lambda(\mathbf{CA})$ be the lattice of subvarieties of $\mathbf{CA}$, and $\Lambda(\mathbf{HA})$ be the lattice of subvarieties of $\mathbf{HA}$.

The following is well known [6, 15].

Theorem 4.3. *For any subvariety* $\mathbf{V}$ *of* $\mathbf{CA}$ *the class* $\{G\mathfrak{A} | \mathfrak{A} \in \mathbf{V}\}$ *is a subvariety of* $\mathbf{HA}$. *Moreover, the map* $\Phi : \Lambda(\mathbf{CA}) \to \Lambda(\mathbf{HA})$ *defined by*

$$\Phi(\mathbf{V}) = \{G\mathfrak{A} | \mathfrak{A} \in \mathbf{V}\}$$

is a complete lattice homomorphism.

As Φ is a complete lattice homomorphism, the preimage of any point in $\Lambda(\mathbf{HA})$ is an interval in the lattice $\Lambda(\mathbf{CA})$. It is known (see, e.g., [9]) that the preimage of the variety $\mathbf{BA}$ of Boolean algebras is the interval $[\mathbf{V}_1, \mathbf{Mon}]$ of $\Lambda(\mathbf{CA})$ and that the preimage of the variety $\mathbf{HA}$ is the interval $[\mathbf{Grz}, \mathbf{CA}]$ of $\Lambda(\mathbf{CA})$. These facts are illustrated in the following figure.

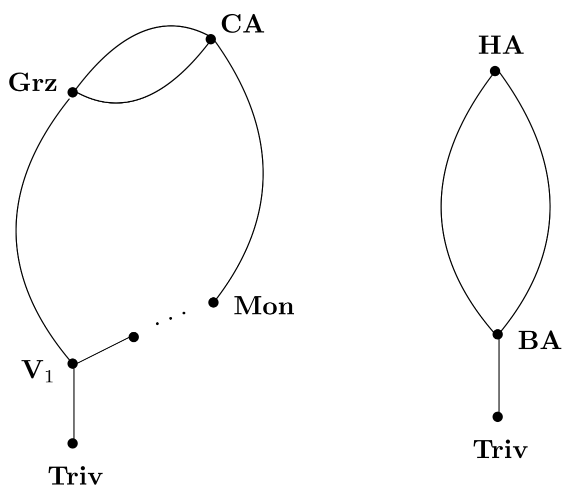

Figure 3

Lemma 4.4. *If* $\mathbf{V}$ *is a subvariety of* $\mathbf{CA}$*, then* $\mathbf{V}$ *being closed under upper MacNeille completions implies* $\Phi(\mathbf{V})$ *is closed under MacNeille completions.*

PROOF. Suppose $H \in \Phi(\mathbf{V})$. Then $H = G\mathfrak{A}$ for some $\mathfrak{A} \in \mathbf{V}$. By Proposition 4.2 we have $\overline{H} = G\overline{\mathfrak{A}}$. Then as $\mathbf{V}$ is closed under upper MacNeille completions, $\overline{\mathfrak{A}} \in \mathbf{V}$, hence $\overline{H} \in \Phi(\mathbf{V})$. □

Lemma 4.5. *Every subvariety of* $\mathbf{Mon}$ *is closed under upper MacNeille completions.*

PROOF. Theorem 3.3 shows that the lower and upper MacNeille completions of monadic algebras coincide. Now apply Lemma 3.2. □

Lemma 4.6. *The only subvariety of* $\mathbf{CA}$ *that contains* $\mathbf{Grz}$ *and is closed under upper MacNeille completions is the variety* $\mathbf{CA}$.

PROOF. Let T denote the countably infinite binary tree with root r where each node has exactly two children. We consider T to be a poset with least element r (and no greatest element). Let $\mathfrak{U}$ be the collection of upsets of T and $B(\mathfrak{U})$ be the Boolean subalgebra of the powerset $\mathcal{P}(T)$ of T generated by $\mathfrak{U}$. One can easily see that for each $t \in T$ the singleton $\{t\}$ belongs to $B(\mathfrak{U})$, so the MacNeille completion $\overline{B(\mathfrak{U})}$ of $B(\mathfrak{U})$ is $\mathcal{P}(T)$.

For any $A \subseteq T$ let $\downarrow A = \{t \in T | t \leq a \text{ for some } a \in A\}$. It is well known that $\mathfrak{A} = (B(\mathfrak{U}), \downarrow)$ belongs to **Grz** [14]. Consider the operation $\overline{\downarrow}$ on $\overline{B(\mathfrak{U})} = \mathcal{P}(T)$. For any $X \subseteq T$ we have $\overline{\downarrow} X = \bigcap\{\downarrow A | A \in B(\mathfrak{U}) \text{ and } X \subseteq A\}$, and this is easily seen to equal $\downarrow X$. Thus $(\overline{B(\mathfrak{U})}, \overline{\downarrow}) = (\mathcal{P}(T), \downarrow)$. However, it is known (see, e.g., [19]) that $(\mathcal{P}(T), \downarrow)$ generates the variety **CA**, and our result follows. □

Theorem 4.7. *The only subvarieties of* **CA** *that are closed under upper MacNeille completions are the subvarieties of* **Mon** *and the variety* **CA**.

PROOF. We have seen in Theorem 3.5 that **CA** is closed under upper MacNeille completions, and in Lemma 3.2 that each subvariety of **Mon** is closed under upper MacNeille completions. We wish to show these are the only such varieties. Suppose **V** is a subvariety of **CA** that is closed under upper MacNeille completions. Then by Lemma 4.2 $\Phi(\mathbf{V})$ is a variety of Heyting algebras that is closed under MacNeille completions. By [21] $\Phi(\mathbf{V})$ is either the trivial variety, the variety **BA**, or the variety **HA**. In the first two cases **V** is contained in **Mon**. In the final case, $\mathbf{Grz} \subseteq \mathbf{V}$, so by Lemma 4.6 $\mathbf{V} = \mathbf{CA}$. □

Remark. In this section we made use of the homomorphism $\Phi : \Lambda(\mathbf{CA}) \to \Lambda(\mathbf{HA})$ provided by the functor $G : \mathbf{CA} \to \mathbf{HA}$ to characterize varieties of closure algebras closed under upper MacNeille completions. Key were the facts that G commuted with **HSP**, so $\Phi(\mathbf{V}) = \{G\mathfrak{A} | \mathfrak{A} \in \mathbf{V}\}$, and that $G\overline{\mathfrak{A}} = \overline{G\mathfrak{A}}$.

For any weak derivative algebra $\mathfrak{A} = (B, \delta)$ we similarly have $F\overline{\mathfrak{A}} = \overline{F\mathfrak{A}}$. Indeed, as $F\mathfrak{A} = (B, \mathbf{C})$, where $\mathbf{C}a = a \vee \delta a$, then for any $x \in \overline{B}$ we have $x \vee \overline{\delta} x = \bigwedge\{a | x \leq a\} \vee \bigwedge\{\delta b | x \leq b\} = \bigwedge\{a \vee \delta b | x \leq a, b\} = \bigwedge\{e \vee \delta e | x \leq e\} = \bigwedge\{\mathbf{C}e | x \leq e\} = \overline{\mathbf{C}} x$.

However, the homomorphism $\Psi : \Lambda(\mathbf{wDA}) \to \Lambda(\mathbf{CA})$ provided by the functor F takes the more complicated form $\Psi(\mathbf{V}) = \mathbf{S}(\{F\mathfrak{A} | \mathfrak{A} \in \mathbf{V}\})$ as F commutes with **H** and **P**, but not **S**. This prevents us from duplicating the proof of Lemma 4.4 for varieties of weak derivative algebras and closure algebras. Indeed, as we will see in the next section, there are varieties **V** of weak derivative algebras with **V** closed under upper MacNeille completions, but $\Psi(\mathbf{V})$ not closed under upper MacNeille completions.

5. Upper MacNeille completions of diagonalizable algebras

We begin by reviewing known results [1, 30] about the lattice of subvarieties of **Diag**. For this, we need the following definition.

Definition 6. For each natural number n, let $\mathfrak{O}_n = (\mathcal{P}(\omega^n + 1), \delta)$ where δ is the derived set operator of the interval topology on the ordinal $\omega^n + 1$, and let $\mathbf{O}_n$ be the variety generated by $\mathfrak{O}_n$.

Definition 7. For each natural number n, let $\mathfrak{S}_n$ be the subalgebra of $\mathfrak{O}_n$ generated by the bounds $0 = \emptyset$ and $1 = \omega^n + 1$ and let $\mathbf{S}_n$ be the variety generated by $\mathfrak{S}_n$. Let $\mathbf{S}_\omega$ be the variety generated by $\{\mathfrak{S}_n | n \in \omega\}$.

Note that $\mathfrak{O}_0$ and $\mathfrak{S}_0$ are the two-element diagonalizable algebra depicted second in Figure 2, so $\mathbf{O}_0 = \mathbf{S}_0$ is the variety we called $\mathbf{V}_2$ in Section 2. Also, $\mathfrak{S}_1$ is the algebra depicted fourth in Figure 2, so $\mathbf{S}_1$ is the variety we called $\mathbf{S}$ in Section 2. It is not difficult to see, and well known, that $\mathbf{S}_n$ is properly contained in $\mathbf{S}_{n+1}$ for each natural number n. Therefore, the following definition is valid.

Definition 8. For $\mathbf{V}$ a subvariety of **Diag**, we say $\mathbf{V}$ is of order n if $\mathbf{S}_n \subseteq \mathbf{V}$ and $\mathbf{S}_{n+1} \not\subseteq \mathbf{V}$. If there is no such natural number n, then $\mathbf{S}_n \subseteq \mathbf{V}$ for each $n \in \omega$, and we say $\mathbf{V}$ is of order ω.

Theorem 5.1. *Suppose* $\mathbf{V}$ *is a subvariety of* **Diag**. *Then*

(1) $\mathbf{V}$ *is of order* n *if, and only if,* $\mathbf{S}_n \subseteq \mathbf{V} \subseteq \mathbf{O}_n$.
(2) $\mathbf{V}$ *is of order* ω *if, and only if,* $\mathbf{S}_\omega \subseteq \mathbf{V} \subseteq \mathbf{Diag}$.

A figure depicting the lattice of subvarieties of **Diag** is given below. This diagram is intended only to show containments, and it is not the case that $\mathbf{S}_n \vee \mathbf{O}_{n-1} = \mathbf{O}_n$.

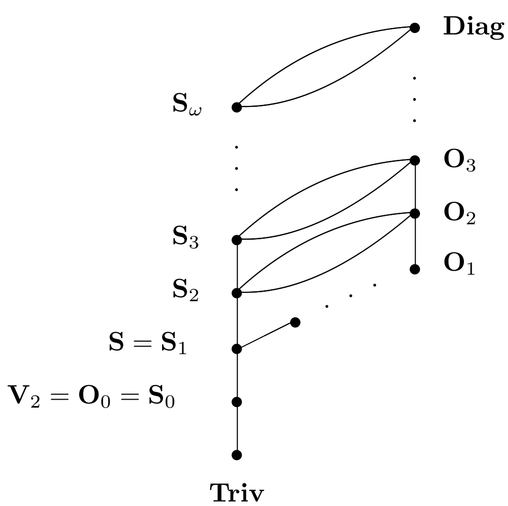

Figure 4

It is worthwhile to note that the variety **Diag** is generated by $\{\mathfrak{O}_n | n \in \omega\}$ hence is the join of the varieties $\mathbf{O}_n$, $n \in \omega$. The following result on equational definitions of these varieties is known (see, e.g., [30]).

Proposition 5.2. *In the presence of the axioms defining* **Diag***,*

(1) $\mathbf{O}_n$ *is defined by the identity* $\delta^{n+1}1 = 0$.
(2) $\mathbf{S}_\omega$ *is defined by the identity* $\delta(a - \mathbf{C}b) \wedge \delta(b - \mathbf{C}a) = 0$.
(3) $\mathbf{S}_n$ *is defined by the identities* $\delta^{n+1}1 = 0$ *and* $\delta(a - \mathbf{C}b) \wedge \delta(b - \mathbf{C}a) = 0$.

We will need the somewhat technical result that each variety $\mathbf{O}_n$ is defined by only the **wDA** axioms and the identity $\delta^{n+1}1 = 0$. This likely belongs to folklore, but we cannot find it in the literature. The proof presented below is likely somewhat novel, as it is based on algebraic techniques, rather than techniques based on duality.

Definition 9. For $\mathfrak{A} = (B, \delta)$ a weak derivative algebra we say an ideal $I \subseteq B$ is a δ-ideal if $a \in I$ implies $\delta a \in I$.

The definition of δ-ideals, and the following well-known result [37] about them, are valid in the more general setting of modal algebras.

Proposition 5.3. *For $\mathfrak{A} = (B, \delta)$ a weak derivative algebra, there is a bijective correspondence between congruences of $\mathfrak{A}$ and δ-ideals of $\mathfrak{A}$. In particular, a subdirectly irreducible weak derivative algebra has a least nontrivial δ-ideal.*

Lemma 5.4. *For any $n \geq 1$, equations 1–4 imply equation 5.*

(1) $\delta 0 = 0$,
(2) $\delta(a \vee b) = \delta a \vee \delta b$,
(3) $\delta\delta a \leq a \vee \delta a$,
(4) $\delta^n 1 = 0$,
(5) $\delta a = \delta(a - \delta a)$.

Proof. For $n = 1$ this is trivial as $\delta a = 0$ and $\delta(a - \delta a) = 0$. So assume $n \geq 2$. It is enough to show that if equations 1–4 hold in a subdirectly irreducible algebra $\mathfrak{A}$, then equation 5 holds in $\mathfrak{A}$ as well.

Assume $\mathfrak{A} = (B, \delta)$ is subdirectly irreducible and that equations 1–4 hold in $\mathfrak{A}$. Let I be the least non-trivial δ-ideal of $\mathfrak{A}$. Note that equations 1–2 imply that if $d \in B$ and $\delta d \leq d$, then $\downarrow d$ is a δ-ideal. Then as $\delta^n 1 = 0$ we have $\downarrow(\delta^{n-1}1)$ is a δ-ideal, and assuming $\delta^{n-1}1 \neq 0$, we have $I \subseteq\downarrow(\delta^{n-1}1)$. Thus $\delta a = 0$ for each $a \in I$, and it follows that $I \subseteq\downarrow a$ for each non-zero $a \in I$, so I is equal to $\downarrow c$ for some atom c of $\mathfrak{A}$ with $\delta c = 0$. We claim $c =\downarrow(\delta^{n-1}1)$. If not, then there is some

d with $c \wedge d = 0$ and $c \vee d = \delta^{n-1}1$. But then as $d \leq \delta^{n-1}1$, we have $\delta d = 0$, hence $\downarrow d$ is a δ-ideal, so $c \leq d$, a contradiction.

So $I = \downarrow(\delta^{n-1}1)$. As $\mathfrak{A}/I$ satisfies $\delta^{n-1}1 = 0$, the inductive hypothesis gives that for any $a \in B$, $\delta(a/I) = \delta((a/I) - \delta(a/I))$. Therefore $\delta a \leq \delta(a - \delta a) \vee \delta^{n-1}1$. But for any $b \in B$ we have $\downarrow(b \vee \delta b)$ is a δ-ideal as equation 3 gives $\delta(b \vee \delta b) = \delta b \vee \delta\delta b \leq b \vee \delta b$. So $\downarrow((a - \delta a) \vee \delta(a - \delta a))$ is a δ-ideal, giving $\delta^{n-1}1 \leq (a - \delta a) \vee \delta(a - \delta a)$. Therefore as $\delta a \leq \delta(a - \delta a) \vee \delta^{n-1}1$ we have $\delta a \leq (a - \delta a) \vee \delta(a - \delta a)$, and taking the meet of both sides of this equation with δa gives $\delta a = \delta(a - \delta a)$. □

Proposition 5.5. *For each natural number n, the variety $\mathbf{O}_n$ is closed under upper MacNeille completions.*

Proof. The previous lemma shows that $\mathbf{O}_n$ is defined by the **wDA** identities and the identity $\delta^{n+1}1 = 0$. We have seen that the **wDA** identities are preserved under upper MacNeille completions. For any weak derivative algebra $\mathfrak{A} = (B, \delta)$, as $\mathfrak{A}$ is a subalgebra of $\overline{\mathfrak{A}}$, we have $(\overline{\delta})^{n+1}1 = \delta^{n+1}1$. Therefore the identity $\delta^{n+1}1 = 0$ is also preserved under upper MacNeille completions. □

Proposition 5.6. *If $\mathbf{V}$ is a subvariety of* **Diag** *that is closed under upper MacNeille completions and contains $\mathbf{S}_n$, then $\mathbf{V}$ contains $\mathbf{O}_n$.*

Proof. Assume $\mathbf{V}$ is a subvariety of **Diag** that is closed under upper MacNeille completions, and that for some natural number n, that $\mathbf{V}$ contains the variety $\mathbf{S}_n$. Let $\mathfrak{T}$ be the subalgebra of $(\mathcal{P}(\omega^n + 1), \delta)$ generated by $\{\downarrow\alpha | \alpha \leq \omega^n\}$ where $\downarrow\alpha = \{\beta | \beta \leq \alpha\}$.

Claim 1. *For each $\alpha \leq \omega^n$ the singleton $\{\alpha\}$ belongs to $\mathfrak{T}$.*

Proof. For $\alpha \leq \omega^n$ we say the degree of α is the largest k for which $\alpha \in \delta^k 1$. Note that if α is of order k, then as $\alpha \in \delta^k 1$ and α is not a limit point of $\delta^k 1$, the set $\{\beta | \beta \in \delta^k 1 \text{ and } \beta < \alpha\}$ is either empty or has a largest element γ. If this set is empty, then $\{\alpha\} = \downarrow\alpha \cap \delta^k 1$, and if this set has a largest element γ, then $\{\alpha\} = (\downarrow\alpha \cap \delta^k 1) - \downarrow\gamma$. □

Claim 2. *$(\mathcal{P}(\omega^n + 1), \delta)$ is the upper MacNeille completion of $\mathfrak{T}$.*

Proof. It follows from the previous claim that the Boolean algebra $\mathcal{P}(\omega^n + 1)$ is the MacNeille completion of the Boolean algebra underlying $\mathfrak{T}$. Using δ for the derivative operation on $\mathcal{P}(\omega^n + 1)$ and $\overline{\delta}$ for the upper MacNeille extension of the derivative operation $\delta|_{\mathfrak{T}}$ on $\mathfrak{T}$, we must show $\delta = \overline{\delta}$.

For any $X \subseteq \omega^n + 1$, the definition of $\overline{\delta}$ gives

$$\overline{\delta}X = \bigcap\{\delta A | A \in \mathfrak{T} \text{ and } X \subseteq A\}.$$

It then follows directly that $\delta X \subseteq \overline{\delta}X$. Suppose $\alpha \notin \delta X$. We must show that there is $A \in \mathfrak{T}$ with $X \subseteq A$ and $\alpha \notin \delta A$. As $\alpha \notin \delta X$, the set $(X \cap \downarrow\alpha) - \{\alpha\}$ is either empty or is contained in $\downarrow\beta$ for some $\beta < \alpha$. If this set is empty, then for $A = (1 - \downarrow\alpha) \cup \{\alpha\}$ we have $A \in \mathfrak{T}$, $X \subseteq A$ and $\alpha \notin \delta A$. If this set is contained in $\downarrow\beta$ for some $\beta < \alpha$, then for $A = (1 - \downarrow\alpha) \cup \{\alpha\} \cup \downarrow\beta$ we have $A \in \mathfrak{T}$, $X \subseteq A$ and $\alpha \notin \delta A$. □

Claim 3. *For each $\alpha \leq \omega^n$, $I_\alpha = \{A \in \mathfrak{T} | \alpha \notin \mathbf{C}A\}$ is a δ-ideal of $\mathfrak{T}$.*

Proof. Surely I_α is a downset, and as $\mathbf{C}(A \cup B) = \mathbf{C}A \cup \mathbf{C}B$ we have I_α is closed under finite joins. As $\omega^n + 1$ is Hausdorff, hence a T_D-space, we have $\mathbf{C}\delta A = \delta A \cup \delta\delta A \subseteq \delta A \subseteq \mathbf{C}A$, so I_α is closed under δ. □

Claim 4. *$\mathfrak{T}$ is in the variety $\mathbf{S}_n$.*

Proof. The variety $\mathbf{S}_n$ is generated by the algebra $\mathfrak{S}_n$, which is the subalgebra of $(\mathcal{P}(\omega^n + 1), \delta)$ generated by $\{0, 1\}$. Note that $\bigcap\{I_\alpha | \alpha \leq \omega^n\} = \{0\}$ since A belonging to this intersection implies $\alpha \notin \mathbf{C}A$ for all $\alpha \leq \omega^n$. So $\mathfrak{T}$ is isomorphic to a subalgebra of the product $\prod\{\mathfrak{T}/I_\alpha | \alpha \leq \omega^n\}$.

Consider I_α in terms of the generators $\downarrow\beta$ of $\mathfrak{T}$. For $\alpha, \beta \leq \omega^n$ we have $\beta < \alpha$ implies $\alpha \notin \mathbf{C}(\downarrow\beta)$, so $\downarrow\beta \in I_\alpha$, and therefore $\downarrow\beta/I_\alpha = 0/I_\alpha$. If $\alpha \leq \beta$ then $\alpha \notin \mathbf{C}(1 - \downarrow\beta)$, so $1 - \downarrow\beta \in I_\alpha$, and this implies $\downarrow\beta/I_\alpha = 1/I_\alpha$. Thus, as $\mathfrak{T}$ is generated by $\{\downarrow\beta | \beta \leq \omega^n\}$, we have $\mathfrak{T}/I_\alpha$ is generated by $\{0/I_\alpha, 1/I_\alpha\}$, so $\mathfrak{T}/I_\alpha$ is isomorphic to a quotient of $\mathfrak{S}_n$. □

To conclude the proof of our proposition, as $\mathbf{V}$ contains $\mathbf{S}_n$, we have $\mathfrak{T} \in \mathbf{V}$. Then as $\mathbf{V}$ is closed under upper MacNeille completions, $\mathfrak{O}_n = (\mathcal{P}(\omega^n + 1), \delta) \in \mathbf{V}$. Then as $\mathbf{O}_n$ is generated by $\mathfrak{O}_n$, we have $\mathbf{O}_n \subseteq \mathbf{V}$. □

Proposition 5.7. *There are no subvarieties of* **Diag** *that contain $\mathbf{S}_\omega$ and are closed under upper MacNeille completions.*

Proof. Define an ordering $\sqsubseteq$ on the set ω of natural numbers by setting $m \sqsubseteq n$ if one of the following holds: (i) m is odd and n is even, (ii) m, n are both odd and $m \leq n$, or (iii) m, n are both even and $n \leq m$. Let $\mathfrak{B}$ be the Boolean algebra of finite and cofinite subsets of ω and define δ on $\mathfrak{B}$ by setting $\delta(A) = \{n | n \sqsubset a$ for some $a \in A\}$. Here $n \sqsubset a$ means $n \sqsubseteq a$ and $n \neq a$. Using Odd and Even for the sets of odd and even numbers respectively, if $A \cap \text{Even} \neq \emptyset$ then δA is cofinite, and if $A \cap \text{Even} = \emptyset$, then as $A \in \mathfrak{B}$ we have that A is a finite subset of Odd, so δA is finite. Thus δ is a well-defined operation on $\mathfrak{B}$.

Clearly $\delta 0 = 0$, and from the form of its definition, one sees easily that $\delta(C \cup D) = \delta C \cup \delta D$. For any $C \in \mathfrak{B}$ we have $\delta\delta C \subseteq \delta C$, so $(\mathfrak{B}, \delta)$ is a derivative algebra. Note also, for any $C \in \mathfrak{B}$, that $\mathbf{C}C = \{n|n \sqsubseteq a \text{ for some } a \in C\}$. But $(\omega, \sqsubseteq)$ is a chain, so for any $C, D \in \mathfrak{B}$ we have that either $\mathbf{C}C \subseteq \mathbf{C}D$ or $\mathbf{C}D \subseteq \mathbf{C}C$. If $\mathbf{C}C \subseteq \mathbf{C}D$, then $C - \mathbf{C}D = 0$, so $\delta(C - \mathbf{C}D) = 0$, and if $\mathbf{C}D \subseteq \mathbf{C}C$, then $\delta(D - \mathbf{C}C) = 0$. In any event, $\delta(C - \mathbf{C}D) \wedge \delta(D - \mathbf{C}C) = 0$, showing that $(\mathfrak{B}, \delta)$ belongs to the variety $\mathbf{S}_\omega$.

Clearly the powerset $\mathcal{P}(\omega)$ is the MacNeille completion of the Boolean algebra $\mathfrak{B}$. Consider the action of $\overline{\delta}$ on the element Odd of $\mathcal{P}(\omega)$. Note that if $A \in \mathfrak{B}$ and $\text{Odd} \subseteq A$, then A is cofinite, so A contains some even number, giving $\text{Odd} \subseteq \delta A$. But for each even n, if we set $A_n = \{k|k \sqsubseteq n\}$ we have $A_n \in \mathfrak{B}$, $\text{Odd} \subseteq A_n$, and $\delta A_n = A_{n+2}$. As $\overline{\delta}\text{Odd} = \bigcap\{A|A \in \mathfrak{B} \text{ and } \text{Odd} \subseteq A\}$, we then have $\overline{\delta}\text{Odd} = \text{Odd}$. But then $\overline{\delta}(\text{Odd} - \overline{\delta}(\text{Odd})) = \overline{\delta}0 = 0$, so $\overline{\delta}\text{Odd} \not\leq \overline{\delta}(\text{Odd} - \overline{\delta}(\text{Odd}))$, showing that the upper MacNeille completion of $(\mathfrak{B}, \delta)$ is not a diagonalizable algebra.

So if $\mathbf{V}$ is a subvariety of **Diag** that contains $\mathbf{S}_\omega$, then $\mathfrak{A} = (\mathfrak{B}, \delta)$ belongs to $\mathbf{V}$, but $\overline{\mathfrak{A}}$ is not diagonalizable, so does not belong to $\mathbf{V}$. □

Theorem 5.8. *The trivial variety and the varieties $\mathbf{O}_n$ for $n \in \omega$ are exactly the subvarieties of* **Diag** *that are closed under upper MacNeille completions.*

Proof. This follows directly from the previous three propositions and the description of the lattice of subvarieties of **Diag** given in Theorem 5.1. □

6. MacNeille completions of Sierpinski algebras

We call the variety $\mathbf{S}$ from Section 2 the variety of Sierpinski algebras, and we call an algebra $\mathfrak{A} \in \mathbf{S}$ a Sierpinski algebra. The name, suggested to us by Leo Esakia, is due to the fact that the algebra $\mathfrak{S}$ that generates the variety $\mathbf{S}$ is isomorphic to the power set of the Sierpinski space $X = \{x, y\}$ with open sets $\emptyset, \{y\}, X$ where the additional operation δ is the derived set operator of X.

This section is split into three subsections. In the first subsection we give axioms defining the variety $\mathbf{S}$. These results likely belong to folklore, but are not easily found in the literature, and our algebraic proofs are likely novel. In the second subsection we develop a triple representation for Sierpinski algebras along the lines of the well-known triple representation of Stone algebras [10, 11]. Key to this triple representation is the fact that each Sierpinski algebra has a least dense element. In the final subsection we employ the triple representation to consider MacNeille completions of Sierpinski algebras.

6.1. **Axiomatics.**

Theorem 6.1. *An algebra $\mathfrak{A} = (B, \delta)$ consisting of a Boolean algebra B with unary operation δ is a Sierpinski algebra if, and only if, it satisfies*

(1) $\delta 0 = 0$,
(2) $\delta(a \vee b) = \delta a \vee \delta b$,
(3) $\delta\delta 1 = 0$,
(4) $\delta a \wedge \delta(-a) = 0$.

PROOF. Let $\mathbf{V}$ be the variety defined by the above equations. One easily checks that these equations are valid in $\mathfrak{S}$, so $\mathfrak{S} \in \mathbf{V}$, and therefore $\mathbf{S} \subseteq \mathbf{V}$. To show the other containment, it is enough to show that every subdirectly irreducible algebra in $\mathbf{V}$ belongs to $\mathbf{S}$.

Suppose $\mathfrak{A} = (B, \delta)$ is a subdirectly irreducible algebra in $\mathbf{V}$. Then as $\mathfrak{A}$ satisfies equations 1 and 2 above, congruences on $\mathfrak{A}$ correspond to δ-ideals of $\mathfrak{A}$. So, as $\mathfrak{A}$ is subdirectly irreducible, it has a least non-trivial δ-ideal, say I.

If $\delta 1 = 0$, then it follows that each ideal of $\mathfrak{A}$ is a δ-ideal, so $\mathfrak{A}$ has a least non-zero element. Hence, $\mathfrak{A}$ is a two-element Boolean algebra with $\delta 1 = 0$. It follows that $\mathfrak{A}$ is a quotient of $\mathfrak{S}$, so $\mathfrak{A} \in \mathbf{S}$. We therefore assume that $\delta 1 \neq 0$ in $\mathfrak{A}$.

Note that for any $b \in B$ we have $\delta(b \vee \delta b) = \delta b \vee \delta\delta b$ by equation 2, and as equation 3 gives $\delta\delta b = 0$, that $\delta(b \vee \delta b) \leq b \vee \delta b$. So for any $b \in B$, the principal ideal $\downarrow(b \vee \delta b)$ is a δ-ideal. The minimality of I yields $I \subseteq\downarrow(b \vee \delta b)$ for each $b \neq 0$. Specializing, we obtain

$$\text{if } b \neq 0 \text{ and } \delta b = 0 \text{ then } I \subseteq\downarrow b, \qquad (*)$$
$$\text{if } \delta b \neq 0 \text{ then } I \subseteq\downarrow(\delta b). \qquad (**)$$

Here $(**)$ is obtained from $(*)$ as equation 3 gives $\delta\delta b = 0$.

As $\delta 1 \neq 0$, condition $(*)$ gives $I \subseteq\downarrow(\delta 1)$, hence $\delta b = 0$ for all $b \in I$. Then $(*)$ gives $I \subseteq\downarrow b$ for all non-zero $b \in I$. This implies $I =\downarrow a$ for some atom a of $\mathfrak{A}$ with $\delta a = 0$.

Note next that for any $b \in B$ we have $\delta b \vee \delta(-b) = \delta 1 \neq 0$, so at most one of δb, $\delta(-b)$ is zero. We can not have both δb, $\delta(-b)$ non-zero as $(**)$ would give $I \subseteq\downarrow(\delta b)$ and $I \subseteq\downarrow(\delta(-b))$, contrary to equation 4 which gives $\delta b \wedge \delta(-b) = 0$. So for any $b \in B$ exactly one of δb, $\delta(-b)$ is zero. It follows easily that $J = \{b | \delta b = 0\}$ is a maximal ideal of $\mathfrak{A}$. But condition $(*)$ yields that the atom a is the least non-zero element of the maximal ideal J, and this implies that $J =\downarrow a$. It follows that $\mathfrak{A}$ has exactly four elements, $0, a, -a, 1$.

We have seen that $\delta a = 0$ and clearly $\delta 0 = 0$. Also, we have seen that $a \leq \delta 1$, and as $\delta\delta 1 = 0$ we have $\delta 1 \neq 1$. Thus $\delta 1 = a$. Finally, as exactly one of $\delta a, \delta(-a)$ is zero, we have $\delta(-a) \neq 0$, and as $\delta(-a) \leq \delta 1$, we have $\delta(-a) = a$. Thus $\mathfrak{A}$ is isomorphic to $\mathfrak{S}$. □

6.2. **Triples and Sierpinski algebras.**

Definition 10. For $\mathfrak{A} = (B, \delta)$ a Sierpinski algebra and $a \in B$ we say

(1) a is dense if $a \vee \delta a = 1$.
(2) a is closed if $a \vee \delta a = a$.
(3) a is clopen if both a and $-a$ are closed.

We use $\mathrm{Dense}\,\mathfrak{A}$ and $\mathrm{Clopen}\,\mathfrak{A}$ for the set of dense elements of $\mathfrak{A}$ and the set of clopen elements of $\mathfrak{A}$, respectively.

Note, using the auxiliary operations $\mathbf{C}$ and $\mathbf{I}$ of closure and interior, a being dense means $\mathbf{C}a = 1$, a being closed means $\mathbf{C}a = a$, and a being clopen means $\mathbf{C}a = a$ and $\mathbf{I}a = a$.

Lemma 6.2. *For $\mathfrak{A} = (B, \delta)$ a Sierpinski algebra and $a \in B$, these are equivalent.*

(1) *a is dense.*
(2) $a \geq -\delta 1$.

Proof. $1 \Rightarrow 2$. If a is dense, then $a \vee \delta a = 1$, so $a \vee \delta 1 = 1$, giving that $a \geq -\delta 1$. $2 \Rightarrow 1$. One checks that $-\delta 1 \vee \delta(-\delta 1) = 1$ in $\mathfrak{S}$, hence $-\delta 1 \vee \delta(-\delta 1) = 1$ in all Sierpinski algebras. Now $a \geq -\delta 1$ implies $a \vee \delta a \geq -\delta 1 \vee \delta(-\delta 1) = 1$. □

Lemma 6.3. *If $\mathfrak{A} = (B, \delta)$ is a Sierpinski algebra, then*

(1) $\mathrm{Dense}\,\mathfrak{A}$ *is a Boolean sublattice of B.*
(2) $\mathrm{Clopen}\,\mathfrak{A}$ *is a Boolean subalgebra of B.*

Proof. 1. By Lemma 6.2 we have $\mathrm{Dense}\,\mathfrak{A}$ is the interval $[-\delta 1, 1]$ of B.

2. From its definition, $\mathrm{Clopen}\,\mathfrak{A}$ is closed under complementation. Also one easily sees that both $0, 1$ are clopen. Suppose a, b are closed. Then $a \vee b \vee \delta(a \vee b) = (a \vee \delta a) \vee (b \vee \delta b) = a \vee b$, so the join of closed elements is closed. Also as a, b are closed, $\delta a \leq a$ and $\delta b \leq b$, so $\delta(a \wedge b) \leq \delta a \wedge \delta b \leq a \wedge b$, showing $a \wedge b$ is closed. It then follows easily that $\mathrm{Clopen}\,\mathfrak{A}$ is closed under finite joins and meets. □

Definition 11. For $\mathfrak{A} = (B, \delta)$ a Sierpinski algebra, define a mapping $\varphi_{\mathfrak{A}} : \mathrm{Clopen}\,\mathfrak{A} \to \mathrm{Dense}\,\mathfrak{A}$ by

$$\varphi_{\mathfrak{A}}(a) = a \vee -\delta 1.$$

We recall that by Lemma 6.2, $-\delta 1$ is the least dense element of $\mathfrak{A}$.

Lemma 6.4. *For $\mathfrak{A} = (B, \delta)$ a Sierpinski algebra, $\varphi_{\mathfrak{A}}$ is a Boolean algebra homomorphism.*

Proof. Clearly $\varphi_{\mathfrak{A}}(a \vee b) = \varphi_{\mathfrak{A}} a \vee \varphi_{\mathfrak{A}} b$, and $\varphi_{\mathfrak{A}}(-a) = -a \vee -\delta 1$ which is the complement of $a \vee -\delta 1$ in the interval $[-\delta 1, 1]$. □

We have seen that to each Sierpinski algebra $\mathfrak{A}$ one may associate a triple $(\operatorname{Clopen}\mathfrak{A}, \operatorname{Dense}\mathfrak{A}, \varphi_{\mathfrak{A}})$ where $\operatorname{Clopen}\mathfrak{A}$ and $\operatorname{Dense}\mathfrak{A}$ are Boolean algebras and $\varphi_{\mathfrak{A}} : \operatorname{Clopen}\mathfrak{A} \to \operatorname{Dense}\mathfrak{A}$ is a Boolean algebra homomorphism. We next consider abstractly such triples.

Definition 12. A Boolean triple is an ordered triple $\mathcal{T} = (C, D, \varphi)$ where

(1) C, D are Boolean algebras.
(2) $\varphi : C \to D$ is a Boolean algebra homomorphism.

A morphism between triples $\mathcal{T} = (C, D, \varphi)$ and $\mathcal{T}' = (C', D', \varphi')$ is an ordered pair $h = (h_1, h_2)$ where $h_1 : C \to C'$ and $h_2 : D \to D'$ are Boolean algebra homomorphisms with $\varphi' \circ h_1 = h_2 \circ \varphi$.

$$\begin{array}{ccc} C & \xrightarrow{\varphi} & D \\ {\scriptstyle h_1}\downarrow & & \downarrow{\scriptstyle h_2} \\ C' & \xrightarrow{\varphi'} & D' \end{array}$$

Finally, if $\mathcal{T}, \mathcal{T}'$ and $\mathcal{T}''$ are triples and $h : \mathcal{T} \to \mathcal{T}'$ and $g : \mathcal{T}' \to \mathcal{T}''$ are triple morphisms, we define $g \circ h : \mathcal{T} \to \mathcal{T}''$ to be $(g_1 \circ h_1, g_2 \circ h_2)$.

Definition 13. We let **S** be the category of Sierpinski algebras and their homomorphisms and **T** be the category of Boolean triples and their morphisms.

The reader will observe that for **BA** the category of Boolean algebras, **T** is none other than the arrow category $\mathbf{BA}^{\to}$.

Definition 14. For $\mathfrak{A}, \mathfrak{A}'$ Sierpinski algebras and $f : \mathfrak{A} \to \mathfrak{A}'$ a homomorphism, define

$F\mathfrak{A} = (\operatorname{Clopen}\mathfrak{A}, \operatorname{Dense}\mathfrak{A}, \varphi_{\mathfrak{A}})$ where $\varphi_{\mathfrak{A}}(c) = c \vee -\delta 1$.
$Ff = (f_1, f_2)$ where $f_1 = f|\operatorname{Clopen}\mathfrak{A}$ and $f_2 = f|\operatorname{Dense}\mathfrak{A}$.

Definition 15. For $\mathcal{T} = (C, D, \varphi)$ and $\mathcal{T}' = (C', D', \varphi')$ Boolean triples and $h : \mathcal{T} \to \mathcal{T}'$ a triple morphism define

$G\mathcal{T} = (C \times D, \delta)$ where δ is given by $\delta(c, d) = (0, \varphi c)$.
Gh is the map $h_1 \times h_2 : G\mathcal{T} \to G\mathcal{T}'$ given by $Gh(c, d) = (h_1 c, h_2 d)$.

Theorem 6.5. $F : \mathbf{S} \to \mathbf{T}$ *and* $G : \mathbf{T} \to \mathbf{S}$ *are functors.*

Proof. Suppose $\mathfrak{A} = (B, \delta)$ and $\mathfrak{A}' = (B', \delta')$ are Sierpinski algebras and $f : \mathfrak{A} \to \mathfrak{A}'$ is a homomorphism. Lemmas 6.3 and 6.4 show $F\mathfrak{A}$ and $F\mathfrak{A}'$ are triples. As f is a homomorphism, $f\delta a = \delta' f a$, and it follows that f maps $\operatorname{Clopen}\mathfrak{A}$ to $\operatorname{Clopen}\mathfrak{A}'$ and $\operatorname{Dense}\mathfrak{A}$ to $\operatorname{Dense}\mathfrak{A}'$. Thus $f_1 = f|\operatorname{Clopen}\mathfrak{A}$ and $f_2 = f|\operatorname{Clopen}\mathfrak{A}$

are maps $f_1 : \text{Clopen}\,\mathfrak{A} \to \text{Clopen}\,\mathfrak{A}'$ and $f_2 : \text{Dense}\,\mathfrak{A} \to \text{Dense}\,\mathfrak{A}'$. By Lemma 6.3 $\text{Clopen}\,\mathfrak{A}$ and $\text{Clopen}\,\mathfrak{A}'$ are subalgebras of B and B' respectively, and as f_1 is the restriction of a homomorphism, it follows that f_1 is a homomorphism. Lemma 6.3 shows that $\text{Dense}\,\mathfrak{A}$ is the interval $[-\delta 1, 1]$ of $\mathfrak{A}$ and $\text{Dense}\,\mathfrak{A}'$ is the interval $[-\delta' 1, 1]$ of $\mathfrak{A}'$. Then as f_2 is the restriction of the homomorphism f and f satisfies $f(-\delta 1) = -\delta' f1$, we have that f_2 is a Boolean algebra homomorphism. For $\varphi_{\mathfrak{A}} : \text{Clopen}\,\mathfrak{A} \to \text{Dense}\,\mathfrak{A}$ and $\varphi_{\mathfrak{A}'} : \text{Clopen}\,\mathfrak{A}' \to \text{Dense}\,\mathfrak{A}'$ defined by $\varphi_{\mathfrak{A}} a = a \vee -\delta 1$ and $\varphi_{\mathfrak{A}'} a = a \vee -\delta' 1$ we have $\varphi_{\mathfrak{A}'} f_1 a = \varphi_{\mathfrak{A}'} fa = fa \vee -\delta' 1 = fa \vee f(-\delta 1) = f(a \vee -\delta 1) = f\varphi_{\mathfrak{A}} a = f_2 \varphi_{\mathfrak{A}} a$. So $\varphi_{\mathfrak{A}'} \circ f_1 = f_2 \circ \varphi_{\mathfrak{A}}$, and this shows that $Ff = (f_1, f_2)$ is a triple morphism from $F\mathfrak{A}$ to $F\mathfrak{A}'$. Finally, it is a simple matter to show that F preserves composition of morphisms and the identity maps. Thus $F : \mathbf{S} \to \mathbf{T}$ is a functor.

Suppose $\mathcal{T} = (C, D, \varphi)$ is a triple. Then $C \times D$ is a Boolean algebra and the map $\delta : C \times D \to C \times D$ defined by $\delta(c, d) = (0, \varphi c)$ is a unary operation on $C \times D$. To show $(C \times D, \delta)$ is a Sierpinski algebra it is enough to show that equations 1 through 4 of Theorem 6.1 hold. As φ is a homomorphism $\delta(0, 0) = (0, \varphi 0) = (0, 0)$, so equation 1 holds. Also $\delta(c \vee c', d \vee d') = (0, \varphi(c \vee c')) = (0, \varphi c) \vee (0, \varphi c') = \delta(c, d) \vee \delta(c', d')$, so equation 2 holds. As $\delta\delta(1, 1) = \delta(0, \varphi 1) = (0, \varphi 0) = (0, 0)$, equation 3 holds, and as $\delta(c, d) \wedge \delta(-c, -d) = (0, \varphi c) \wedge (0, \varphi(-c)) = (0, \varphi c \wedge -\varphi c) = (0, 0)$, equation 4 holds. Thus $G\mathcal{T}$ is a Sierpinski algebra.

Suppose $\mathcal{T} = (C, D, \varphi)$ and $\mathcal{T}' = (C', D', \varphi')$ are triples with $G\mathcal{T} = (C \times D, \delta)$ and $G\mathcal{T}' = (C' \times D', \delta')$. Suppose $f : \mathcal{T} \to \mathcal{T}'$ is a triple morphism with $f = (f_1, f_2)$. Then it is well known that $f_1 \times f_2 : C \times D \to C' \times D'$ is a Boolean algebra homomorphism. Then using the definitions of δ and δ', as well as the facts that f_1, f_2 are Boolean algebra homomorphisms with $\varphi' \circ f_1 = f_2 \circ \varphi$, we have $(f_1 \times f_2)\delta(c, d) = (f_1 \times f_2)(0, \varphi c) = (f_1 0, f_2 \varphi c) = (0, \varphi' f_1 c) = \delta'(f_1 c, f_2 d) = \delta'(f_1 \times f_2)(c, d)$. Thus $Gf = f_1 \times f_2$ is a Sierpinski algebra homomorphism from $G\mathcal{T}$ to $G\mathcal{T}'$. Finally, it is a simple matter to show that G preserves compositions of morphisms and identity maps. So $G : \mathbf{T} \to \mathbf{S}$ is a functor. □

Lemma 6.6. *For $\mathfrak{A} = (B, \delta)$ a Sierpinski algebra and $\mathcal{T} = (C, D, \varphi)$ a triple:*

(1) $GF\mathfrak{A} = (\text{Clopen}\,\mathfrak{A} \times \text{Dense}\,\mathfrak{A}, \delta')$ *where* $\delta'(c, d) = (0, c \vee -\delta 1)$.

(2) $FG\mathcal{T} = (\{(c, \varphi c) | c \in C\}, \{1\} \times D, \varphi')$ *where* $\varphi'(c, \varphi c) = (1, \varphi c)$.

Proof. 1. For a Sierpinski algebra $\mathfrak{A} = (B, \delta)$ we have $F\mathfrak{A} = (\text{Clopen}\,\mathfrak{A}, \text{Dense}\,\mathfrak{A}, \varphi)$ where $\varphi a = a \vee -\delta 1$. Then $GF\mathfrak{A} = (\text{Clopen}\,\mathfrak{A} \times \text{Dense}\,\mathfrak{A}, \delta')$ where $\delta'(c, d) = (0, \varphi c)$, and upon substituting, $\delta'(c, d) = (0, c \vee -\delta 1)$.

2. For a triple $\mathcal{T} = (C, D, \varphi)$ we have $G\mathcal{T} = (C \times D, \delta)$ where $\delta(c, d) = (0, \varphi c)$. Note that (c, d) is closed if, and only if, $\delta(c, d) \leq (c, d)$, which is equivalent to

requiring that $\varphi c \le d$. Thus (c,d) and $-(c,d)$ are closed if, and only if, $\varphi c \le d$ and $\varphi(-c) \le -d$, or equivalently, if $\varphi c = d$. So $\text{Clopen}(G\mathcal{T}) = \{(c,\varphi c) | c \in C\}$. Note also that (c,d) is dense if, and only if, $(c,d) \vee \delta(c,d) = (1,1)$, which is equivalent to $(c,d) \vee (0,\varphi c) = (1,1)$, and hence to $c = 1$. So $\text{Dense}(G\mathcal{T})$ is given by $\{1\} \times D$. From the above descriptions of $\text{Clopen}(G\mathcal{T})$ and $\text{Dense}(G\mathcal{T})$, as well as the definition of $FG\mathcal{T}$, we have $FG\mathcal{T} = (\{(c,\varphi c)|c \in C\}, \{1\} \times D, \varphi')$ where $\varphi' : \{(c,\varphi c)|c \in C\} \to \{1\} \times D$ is the map $\varphi_{G\mathcal{T}}$ described in Definition 14. We then have $\varphi'(c,\varphi c) = (c,\varphi c) \vee -\delta(1,1)$. As the derivative operation δ of $G\mathcal{T}$ is given by $\delta(c,d) = (0,\varphi c)$, then $-\delta(1,1) = -(0,\varphi 1) = -(0,1) = (1,0)$, and so $\varphi'(c,\varphi c) = (1,\varphi c)$. ☐

Lemma 6.7. *For each Sierpinski algebra* $\mathfrak{A} = (B,\delta)$, *the mapping* $\eta_{\mathfrak{A}} : \mathfrak{A} \to GF\mathfrak{A}$ *defined by*

$$\eta_{\mathfrak{A}} a = (\mathbf{CI}a, a \vee -\delta 1)$$

is an isomorphism. Here $\mathbf{C}$, $\mathbf{I}$ *are the closure and interior operators of* $\mathfrak{A}$.

Proof. Recall $\mathfrak{S}$ is the algebra constructed from the Sierpinski space $X = \{x,y\}$ with open sets $\emptyset, \{y\}, X$. From the earlier description of the derived set operator δ of $\mathfrak{S}$ we have that the closure and interior operators $\mathbf{C}$, $\mathbf{I}$ of $\mathfrak{S}$ are given by $\mathbf{C}\emptyset = \emptyset$, $\mathbf{C}\{x\} = \{x\}$, $\mathbf{C}\{y\} = X$, $\mathbf{C}X = X$ and $\mathbf{I}\emptyset = \emptyset$, $\mathbf{I}\{x\} = \emptyset$, $\mathbf{I}\{y\} = \{y\}$, $\mathbf{I}X = X$. One can then see that in $\mathfrak{S}$ we have $\mathbf{CI}a = \emptyset$ if, and only if, $a \subseteq \{x\}$, and $\mathbf{CI}a = X$ if, and only if, $\{y\} \subseteq a$. Similarly, $\mathbf{IC}a = \emptyset$ if, and only if, $a \subseteq \{x\}$, and $\mathbf{IC}a = X$ if, and only if, $\{y\} \subseteq a$.

One may then see that the following identities hold in $\mathfrak{S}$, and therefore hold in all Sierpinski algebras: (i) $\delta\mathbf{CI}a \le \mathbf{CI}a$, (ii) $\mathbf{CI}(-a) = -\mathbf{CI}a$, (iii) $\mathbf{CI}(a \vee b) = \mathbf{CI}a \vee \mathbf{CI}b$, (iv) $(\mathbf{CI}a \vee \delta 1) \wedge (a \vee -\delta 1) = a$, (v) $\mathbf{CI}a = \mathbf{IC}a$, (vi) $\mathbf{CI}\delta 1 = 0$, and (vii) $\delta a \vee -\delta 1 = \mathbf{CI}a \vee -\delta 1$.

As these identities hold in the Sierpinski algebra $\mathfrak{A} = (B,\delta)$, we have by (i) that $\delta\mathbf{CI}a \le \mathbf{CI}a$, showing that $\mathbf{CI}a$ is closed for any $a \in B$. In particular, $\mathbf{CI}(-a)$ is closed, and by (ii) $\mathbf{CI}(-a) = -\mathbf{CI}a$, giving that $-\mathbf{CI}a$ is closed, hence $\mathbf{CI}a$ is open. Thus $\mathbf{CI}a$ is clopen for each $a \in B$. So $a \longmapsto \mathbf{CI}a$ provides a mapping from $\mathfrak{A}$ to $\text{Clopen}\,\mathfrak{A}$, and identities (ii) and (iii) show that this map preserves finite joins and complements. Similarly, Lemma 6.2 shows that $a \longmapsto a \vee -\delta 1$ is a mapping from $\mathfrak{A}$ to $\text{Dense}\,\mathfrak{A}$. This map clearly preserves finite joins, and as $\text{Dense}\,\mathfrak{A}$ is the interval $[-\delta 1, 1]$ of $\mathfrak{A}$, it preserves complementation as well. Thus $\eta_{\mathfrak{A}}$ is a mapping from $\mathfrak{A}$ to $GF\mathfrak{A}$ that preserves finite joins and complementations.

Note that if $a \in B$ then $\eta_{\mathfrak{A}}(\delta a) = (\mathbf{CI}\delta a, \delta a \vee -\delta 1)$. Then as $\delta a \le \delta 1$, equations (vi) and (vii) give us that $\eta_{\mathfrak{A}}(\delta a) = (0, \mathbf{CI}a \vee -\delta 1)$. Then by Lemma

6.6, $\eta_{\mathfrak{A}}(\delta a) = \delta'(\eta_{\mathfrak{A}} a)$ where δ' is the derivative operation of $GF\mathfrak{A}$. So $\eta_{\mathfrak{A}}$ is a Sierpinski algebra homomorphism.

Next note that equation (iv) implies that each element $a \in B$ is uniquely determined by $\eta_{\mathfrak{A}} a$, so $\eta_{\mathfrak{A}}$ is one-one. To see that $\eta_{\mathfrak{A}}$ is onto, suppose that $c \in \operatorname{Clopen}\mathfrak{A}$ and $d \in \operatorname{Dense}\mathfrak{A}$. We claim that $\eta_{\mathfrak{A}}((c \wedge -\delta 1) \vee (d \wedge \delta 1)) = (c, d)$. Using the fact that $a \longmapsto \mathbf{CI}a$ preserves finite joins, meets, and complements, as well as identity (v) which says $\mathbf{CI}a = \mathbf{IC}a$, and identity (vi) which says $\mathbf{CI}\delta 1 = 0$, we have $\mathbf{CI}((c \wedge -\delta 1) \vee (d \wedge \delta 1)) = (\mathbf{CI}c \wedge (-\mathbf{CI}\delta 1)) \vee (\mathbf{CI}d \wedge \mathbf{CI}\delta 1) = \mathbf{CI}c$. Then as c is clopen, this expression reduces simply to c. Also, $(c \wedge -\delta 1) \vee (d \wedge \delta 1) \vee -\delta 1 = d \vee -\delta 1$, and by Lemma 6.2 this reduces to d as d is dense. Thus $\eta_{\mathfrak{A}}((c \wedge -\delta 1) \vee (d \wedge \delta 1)) = (c, d)$ showing that $\eta_{\mathfrak{A}}$ is onto, hence an isomorphism. □

Lemma 6.8. *For each triple $\mathcal{T} = (C, D, \varphi)$ the maps $\varepsilon_{\mathcal{T}_1} : C \to \operatorname{Clopen} G\mathcal{T}$ and $\varepsilon_{\mathcal{T}_2} : D \to \operatorname{Dense} G\mathcal{T}$ defined by*

$$\varepsilon_{\mathcal{T}_1} c = (c, \varphi c),$$
$$\varepsilon_{\mathcal{T}_2} d = (1, d)$$

provide a triple isomorphism $\varepsilon_{\mathcal{T}} = (\varepsilon_{\mathcal{T}_1}, \varepsilon_{\mathcal{T}_2})$ from $\mathcal{T}$ to $FG\mathcal{T}$.

Proof. Note that by Lemma 6.6, both $\varepsilon_{\mathcal{T}_1} : C \to \operatorname{Clopen} G\mathcal{T}$ and $\varepsilon_{\mathcal{T}_2} : D \to \operatorname{Dense} G\mathcal{T}$ are well defined, and clearly both $\varepsilon_{\mathcal{T}_1}$ and $\varepsilon_{\mathcal{T}_2}$ are Boolean algebra isomorphisms. Lemma 6.6 also provides $FG\mathcal{T} = (\operatorname{Clopen} G\mathcal{T}, \operatorname{Dense} G\mathcal{T}, \varphi')$ where $\varphi'(c, \varphi c) = (1, \varphi c)$. Then as $\varphi' \circ \varepsilon_{\mathcal{T}_1}(c) = (1, \varphi c) = \varepsilon_{\mathcal{T}_2} \circ \varphi(c)$ for each $c \in C$, we have $\varphi' \circ \varepsilon_{\mathcal{T}_1} = \varepsilon_{\mathcal{T}_2} \circ \varphi$. Thus $\varepsilon_{\mathcal{T}} = (\varepsilon_{\mathcal{T}_1}, \varepsilon_{\mathcal{T}_2})$ is a triple morphism from $\mathcal{T}$ to $FG\mathcal{T}$. Then as $\varphi' \circ \varepsilon_{\mathcal{T}_1} = \varepsilon_{\mathcal{T}_2} \circ \varphi$ we have $(\varepsilon_{\mathcal{T}_2})^{-1} \circ \varphi' = \varphi \circ (\varepsilon_{\mathcal{T}_1})^{-1}$, hence $((\varepsilon_{\mathcal{T}_1})^{-1}, (\varepsilon_{\mathcal{T}_2})^{-1})$ is also a triple morphism, and this is easily seen to be the inverse of $\varepsilon_{\mathcal{T}}$. Thus $\varepsilon_{\mathcal{T}}$ is a triple isomorphism. □

Theorem 6.9. *The functors F, G provide an equivalence between the categories $\mathbf{S}$ and $\mathbf{T}$ with $\eta : 1_{\mathbf{S}} \to GF$ and $\varepsilon : 1_{\mathbf{T}} \to FG$ as natural isomorphisms.*

Proof. In view of Lemmas 6.7 and 6.8, all that remains is to show the naturality of η and ε. This means that for each Sierpinski algebra homomorphism $f : \mathfrak{A} \to \mathfrak{B}$ we must show $(GFf) \circ \eta_{\mathfrak{A}} = \eta_{\mathfrak{B}} \circ f$, and for each triple morphism $g : \mathcal{T} \to \mathcal{U}$ that $(FGg) \circ \varepsilon_{\mathcal{T}} = \varepsilon_{\mathcal{U}} \circ g$.

From Definitions 14 and 15 we have $Ff = (f|\operatorname{Clopen}\mathfrak{A}, f|\operatorname{Dense}\mathfrak{A})$ and so $GFf = (f|\operatorname{Clopen}\mathfrak{A}) \times (f|\operatorname{Dense}\mathfrak{A})$. Thus for any $a \in \mathfrak{A}$, $(GFf) \circ \eta_{\mathfrak{A}}(a) = (GFf)(\mathbf{CI}a, a \vee -\delta 1) = (f\mathbf{CI}a, f(a \vee -\delta 1))$. Using the fact that f is a homomorphism, $(GFf) \circ \eta_{\mathfrak{A}}(a) = (\mathbf{CI}fa, fa \vee -\delta 1) = \eta_{\mathfrak{B}} f(a)$. Thus $(GFf) \circ \eta_{\mathfrak{A}} = \eta_{\mathfrak{B}} \circ f$.

Again using Definitions 14 and 15, we have $Gg = g_1 \times g_2$ where $g = (g_1, g_2)$, and so $FGg = ((g_1 \times g_2)|\operatorname{Clopen} G\mathcal{T}, (g_1 \times g_2)|\operatorname{Dense} G\mathcal{T})$. Suppose $\mathcal{T} = (C, D, \varphi)$

and $\mathcal{U} = (P, Q, \psi)$. So by Lemma 6.8 $\varepsilon_{\mathcal{T}} = (\varepsilon_{\mathcal{T}_1}, \varepsilon_{\mathcal{T}_2})$ where $\varepsilon_{\mathcal{T}_1} c = (c, \varphi c)$ and $\varepsilon_{\mathcal{T}_2} d = (1, d)$; and $\varepsilon_{\mathcal{U}} = (\varepsilon_{\mathcal{U}_1}, \varepsilon_{\mathcal{U}_2})$ where $\varepsilon_{\mathcal{U}_1} p = (p, \psi p)$ and $\varepsilon_{\mathcal{U}_2} q = (1, q)$. From Definition 12 we have that $(FGg) \circ \varepsilon_{\mathcal{T}}$ is given by $(((g_1 \times g_2)|\mathrm{Clopen}\, G\mathcal{T}) \circ \varepsilon_{\mathcal{T}_1}, ((g_1 \times g_2)|\mathrm{Dense}\, G\mathcal{T}) \circ \varepsilon_{\mathcal{T}_2})$ and $\varepsilon_{\mathcal{U}} \circ g$ is given by $(\varepsilon_{\mathcal{U}_1} \circ g, \varepsilon_{\mathcal{U}_2} \circ g)$. For any $c \in C$ we have that $((g_1 \times g_2)|\mathrm{Clopen}\, G\mathcal{T}) \circ \varepsilon_{\mathcal{T}_1}(c) = (g_1 \times g_2)(c, \varphi c) = (g_1 c, g_2 \varphi c)$, and g being a triple morphism gives $g_2 \circ \varphi = \psi \circ g$. So this expression is equal to $(g_1 c, \psi g_1 c) = \varepsilon_{\mathcal{U}_1} \circ g_1(c)$. For any $d \in D$ we have $((g_1 \times g_2)|\mathrm{Dense}\, G\mathcal{T}) \circ \varepsilon_{\mathcal{T}_2}(d) = (g_1 \times g_2)(1, d) = (1, g_2 d) = \varepsilon_{\mathcal{U}_2} \circ g_2(d)$. Thus $(FGg) \circ \varepsilon_{\mathcal{T}} = \varepsilon_{\mathcal{U}} \circ g$. □

Remark. We suspect this triple construction and resulting categorical equivalence for Sierpinski algebras can be extended to the variety of diagonalizable algebras whose associated Heyting algebras satisfy Stone's identity. It would also be worthwhile to determine the relationship between this triple construction other triple constructions in the literature [10, 11, 35, 25, 36, 12].

6.3. **MacNeille completions of Sierpinski algebras.** Since the variety of Sierpinski algebras is a subvariety of **Diag**, it follows from Corollary 3.4 and Theorem 5.8 that it is not closed under lower or upper MacNeille completions. Nevertheless, it is, in a sense, closed under MacNeille completions. We formulate this statement precisely in the following theorem.

Theorem 6.10. *For each Sierpinski algebra $\mathfrak{A} = (B, \delta)$, there is an operation δ^* on the MacNeille completion $\overline{B}$ of B such that the algebra $\mathfrak{A}^* = (\overline{B}, \delta^*)$ is a Sierpinski algebra that contains $\mathfrak{A}$ as a subalgebra.*

PROOF. As every Sierpinski algebra is isomorphic to one of the form $G\mathcal{T}$ for some triple $\mathcal{T}$, it is enough to prove this result under the assumption that $\mathfrak{A} = G\mathcal{T}$ for the triple $\mathcal{T} = (C, D, \varphi)$. Thus $\mathfrak{A} = (C \times D, \delta)$ where $\delta(c, d) = (0, \varphi c)$.

As $\varphi : C \to D$ is a Boolean algebra homomorphism and D is a subalgebra of its MacNeille completion $\overline{D}$, we may consider φ to be a homomorphism from C to $\overline{D}$. Then as $\overline{D}$ is a complete Boolean algebra, hence an injective Boolean algebra [3, Page 113, Theorem 2], and C is a subalgebra of its MacNeille completion $\overline{C}$, there is a Boolean algebra homomorphism $\overline{\varphi} : \overline{C} \to \overline{D}$ with $\overline{\varphi}|C = \varphi$.

Consider the triple $\overline{\mathcal{T}} = (\overline{C}, \overline{D}, \overline{\varphi})$ and let $\mathfrak{A}^*$ be the Sierpinski algebra $G\overline{\mathcal{T}}$. So $\mathfrak{A}^* = (\overline{C} \times \overline{D}, \delta^*)$ where $\delta^*(x, y) = (0, \overline{\varphi} x)$. Then $\overline{C} \times \overline{D}$ is the MacNeille completion of $C \times D$, and for $(c, d) \in C \times D$ we have $\delta^*(c, d) = (0, \overline{\varphi} c) = (0, \varphi c) = \delta(c, d)$. Thus $\mathfrak{A}$ is a subalgebra of $\mathfrak{A}^*$, proving our result. □

We have used the axiom of choice, or somewhat weaker statements, in our proof of Theorems 6.1 and 6.10. In Theorem 6.1, we prove that HSP$\mathfrak{S}$ is equal to the equational class $\mathrm{Eq}\Sigma$ where Σ is the set consisting of equations 1–4 of Theorem 6.1 as well as the equations defining Boolean algebras. It may be possible to prove $\mathbf{HSP}\mathfrak{S} = \mathrm{Eq}\Sigma$ solely from the ZF axioms [2], but this is not our main interest and we shall sidestep this issue by working directly with $\mathrm{Eq}\Sigma$. We then note that if we interpret the term "Sierpinski algebra" to mean a member of $\mathrm{Eq}\Sigma$, then all of the above results relating Sierpinski algebras and triples remain valid without any form of the axiom of choice. One must, however, give direct syntactic proofs of several identities used in Lemmas 6.2 and 6.7 from Σ, and this is a somewhat cumbersome job.

The use of some form of the axiom of choice in Theorem 6.10 is more fundamental as the following result shows.

Theorem 6.11. *Let Σ be the set consisting of equations 1–4 of Theorem 6.1 and the identities defining Boolean algebras, and consider the following statements.*

(1) *The axiom of choice.*
(2) *Sikorski's theorem that complete Boolean algebras are injective.*
(3) *For any $\mathfrak{A} = (B, \delta)$ in $\mathrm{Eq}\Sigma$ there is $\mathfrak{A}' = (B', \delta')$ in $\mathrm{Eq}\Sigma$ with $\mathfrak{A}$ a subalgebra of $\mathfrak{A}'$ and B' the MacNeille completion of B.*
(4) *The Boolean ultrafilter theorem.*

Then in ZF we have $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$.

Proof. The standard proof of Sikorski's theorem follows from ZF using choice, so $1 \Rightarrow 2$. Our proof of Theorem 6.10 uses only our results on triples, which we have noted remain valid without choice when we consider instead algebras in $\mathrm{Eq}\Sigma$, and Sikorski's theorem. So $2 \Rightarrow 3$. It remains only to show $3 \Rightarrow 4$. We first establish the following.

Claim 5. *Suppose $\mathfrak{A} = (B, \delta)$ and $\mathfrak{A}' = (B', \delta')$ are members of $\mathrm{Eq}\Sigma$ with $\mathfrak{A}$ a subalgebra of $\mathfrak{A}'$ and B' the MacNeille completion of B. Then for $F\mathfrak{A} = (C, D, \varphi)$ and $F\mathfrak{A}' = (C', D', \varphi')$ we have C' is the MacNeille completion of C, D' is the MacNeille completion of D, and $\varphi'|C = \varphi$.*

Proof. To see this, note that $\eta_{\mathfrak{A}} : B \to C \times D$ and $\eta_{\mathfrak{A}'} : B' \to C' \times D'$ are isomorphisms. So, as B' is complete, $C' \times D'$ is complete, and it follows that C' and D' are complete. Also, as $\mathfrak{A} \leq \mathfrak{A}'$ we have $\mathrm{Clopen}\mathfrak{A} \leq \mathrm{Clopen}\mathfrak{A}'$ and $\mathrm{Dense}\mathfrak{A} \leq \mathrm{Dense}\mathfrak{A}'$, so $C \leq C'$ and $D \leq D'$. Let $i : B \to B'$ and $j : C \times D \to C' \times D'$ be the identical embeddings. Note that for $a \in B$ we have $\eta_{\mathfrak{A}} a = (\mathbf{CI}a, a \vee -\delta 1)$ and $\eta_{\mathfrak{A}'} a = (\mathbf{C}'\mathbf{I}'a, a \vee -\delta' 1)$, and as $\mathfrak{A} \leq \mathfrak{A}'$ we have $\eta_{\mathfrak{A}} a = \eta_{\mathfrak{A}'} a$. It follows that $j \circ \eta_{\mathfrak{A}} = \eta_{\mathfrak{A}'} \circ i$, and hence that $j = \eta_{\mathfrak{A}'} \circ i \circ \eta_{\mathfrak{A}}^{-1}$. Then as $\eta_{\mathfrak{A}}^{-1}, \eta_{\mathfrak{A}'}$ are isomorphisms and i is join and meet dense, $j : C \times D \to C' \times D'$

is join and meet dense as well. So for $x \in C'$ and $y \in D'$ we have $(x, 0)$ and $(0, y)$ are joins of elements of $C \times D$, hence x, y are joins of elements of C, D respectively, and similarly, x, y are meets of elements of C, D respectively. So C' is the MacNeille completion of C, and D' is the MacNeille completion of D. Finally, for $a \in C$ we have $\varphi a = a \vee -\delta 1$ and $\varphi' a = a \vee -\delta' 1$. Then as $\mathfrak{A} \leq \mathfrak{A}'$ we have $\varphi' a = \varphi a$. Thus $\varphi' | C = \varphi$. □

To show $3 \Rightarrow 4$, let X be an infinite set, C be the Boolean algebra of finite and cofinite subsets of X, and $\varphi : C \to 2$ be the homomorphism mapping all finite subsets of X to 0 and all cofinite subsets of X to 1. Then $(C, 2, \varphi)$ is a triple, and from our discussion of triples, one can construct an algebra $\mathfrak{A} = (B, \delta)$ in $\mathrm{Eq}\Sigma$ that has $F\mathfrak{A} = (C, 2, \varphi)$ as its triple. By assumption 3 there is $\mathfrak{A}' = (B', \delta')$ in $\mathrm{Eq}\Sigma$ with $\mathfrak{A} \leq \mathfrak{A}'$ and B' the MacNeille completion of B. So, by the claim, if $F\mathfrak{A}' = (C', D', \varphi')$, then C' is the MacNeille completion of C, D' is the MacNeille completion of 2, and $\varphi' | C = \varphi$. In particular, C' is the power set $\mathcal{P}X$ of X, $D' = 2$, and $\varphi' : \mathcal{P}X \to 2$ is a Boolean algebra homomorphism. Thus, for any infinite set X we have produced a finitely additive 2-valued measure on $\mathcal{P}X$ taking value 0 on each atom, and by [34, page 328], this is equivalent to the Boolean ultrafilter theorem. □

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