THE SOURCE OF THE ORTHOMODULAR LAW

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1 INTRODUCTION

Beginning with Birkhoff and von Neumann [4], a central theme in quantum logic is to consider generalizations of the lattice $\mathcal{C}(\mathcal{H})$ of closed subspaces of a Hilbert space $\mathcal{H}$ as models for the propositions of a quantum mechanical system. Husimi [20] was the first to note that the ortholattices $\mathcal{C}(\mathcal{H})$ satisfied the following identity known as the orthomodular law:

(1) $A \leq B \quad \Rightarrow \quad A \vee (A^{\perp} \wedge B) = B.$

This fact was rediscovered several times in the 1950's and 1960's [23, 28, 30, 31, 35] and lead to the role of orthomodular lattices (abbreviated: OMLs) and orthomodular posets (abbreviated: OMPs) as abstract models for the propositions of a quantum mechanical system.

It is instructive to see how the validity of the orthomodular law in $\mathcal{C}(\mathcal{H})$ follows in a transparent way from basic properties of Hilbert spaces. For the non-trivial containment in (1) note that if $b \in B$, then $b = a_1 + a_2$ for some unique $a_1 \in A$ and $a_2 \in A^{\perp}$. Then if $A \subseteq B$ we have $a_1 \in B$, hence $b - a_1 = a_2$ belongs to $A^{\perp} \cap B$, and therefore $b = a_1 + a_2$ belongs to $A \vee (A^{\perp} \wedge B)$. Thus, the validity of the orthomodular law in $\mathcal{C}(\mathcal{H})$ follows as each vector in $\mathcal{H}$ can be uniquely expressed as a sum of vectors from A and $A^{\perp}$. This shall be of fundamental importance to us.

The orthomodular law has several equivalent formulations that highlight different aspects of its nature. We mention one of these that provides insight that will be helpful here. Note first that every ortholattice L is equal to the set-theoretic union of its Boolean subalgebras as each $a \in L$ lies in the Boolean subalgebra $\{0, a, a', 1\}$. Orthomodular lattices are exactly those ortholattices L where the partial ordering of L is determined by the partial orderings of its Boolean subalgebras. In this sense, OMLs are exactly the *locally Boolean* ortholattices.

While the above comments make a case that the orthomodular law is worthy of study, the over fifty year longevity of the orthomodular law must be attributed to its role in developing a large body of deep and beautiful mathematics and to its role in the theoretical foundations of quantum mechanics. We cannot describe these results in any depth, but point to the work on the dimension theory of projection lattices [33] and the subsequent development of a dimension theory for certain OMLs [28, 31]; the related development of continuous geometry [34] and its deep ties to modular ortholattices [24]; the work on Baer*-semigroups [9]; the beautiful algebraic theory of OMLs outlined in [5, 22]; generalized Hilbert spaces [35, 26] and Solèr's theorem [37]; generalized orthomodular structures [11, 12, 27] and their ties to partially ordered abelian groups [10]; and of the multitude of work on applications of orthomodularity to the foundations of quantum mechanics of which [3, 13, 30, 35] is a sample. As a meta-question, one might well ask why such an innocent looking identity as the orthomodular law should lie at the heart of so much interesting mathematics.

While the orthomodular law has its origins and many of its application centered on Hilbert spaces, we present here a very different view of orthomodularity, one that eliminates the reliance on Hilbert spaces and focuses on a more elementary mathematical property instead. We present our thesis in an informal manner below, and provide a more detailed treatment in the body of this chapter. The crucial item is the following.

Slogan I: The direct product decompositions of many familiar mathematical structures, including sets, groups, modules, and topological spaces, naturally form an orthomodular poset. Thus, orthomodularity at its root arises from considering direct product decompositions.

Of course, we will make this precise in the sequel, in particular in Theorem 13. The point of this statement is to identify decompositions as the basic mathematical process that leads to orthomodularity. What then about the about the intimate link between orthomodularity and Hilbert spaces?

Slogan II: Orthomodularity has nothing to do with Hilbert space, it is a consequence of considering direct product decompositions of a Hilbert space.

Here we mean orthomodularity is not a property of a Hilbert space $\mathcal{H}$, but arises only when we consider the lattice $\mathcal{C}(\mathcal{H})$ of closed subspaces of $\mathcal{H}$. As we noted above, the key property of Hilbert spaces is that for a closed subspace A, each vector v in $\mathcal{H}$ can be uniquely represented as the sum of a vector $v_A \in A$ and a vector $v_{A^\perp} \in A^\perp$. Viewed another way, this means that each closed subspace A induces a direct product decomposition $\mathcal{H} \simeq A \times A^\perp$ of $\mathcal{H}$ and all direct product decompositions of $\mathcal{H}$ arise in this manner. Thus the process of considering the closed subspaces of $\mathcal{H}$ is really a disguise for considering the direct product decompositions of $\mathcal{H}$.

Slogan III: The role of orthomodularity in the foundations of quantum mechanics is due, at least in part, to an underlying role of direct product decompositions in the foundations of quantum mechanics.

In support of this statement, we will give an axiomatic development of what we term an experimental system based essentially on the notion of direct product decompositions. This will include the development of a system of *yes-no* experiments, a type of logic for such experiments, a probabilistic treatment of the results of such experiments, and an approach to observables and their calculus. The standard Hilbert space approach to quantum mechanics will be shown to fit exactly into this framework.

Slogan IV: Independent of any connection to quantum mechanics, orthomodularity is worthy of study due to its role in the theory of direct product decompositions.

This chapter is organized in the following manner. In the second section we review well-known facts about the treatment of surjective images of structures such as sets and groups. This is a familiar topic that is presented from a perhaps unfamiliar viewpoint. The treatment we give provides a sort of toy model for our treatment of direct product decompositions.

In Section 3 we begin our study of direct product decompositions of sets and prove our Main Theorem — that the binary direct product decompositions of a set, with operations induced in a natural way, forms an OMP. In the fourth section we illustrate this result with several concrete examples, and in the fifth section we extend our results on decompositions of sets to decompositions of other types of mathematical structures, such as groups, modules, topological spaces, Hilbert spaces, and so forth. A number of well-known methods to construct OMPs arise as instances of this construction.

Section 6 contains a finer study of the structure of the OMPs $BDec\ X$ of decompositions of a set X. In particular, we characterize the Boolean subalgebras of such OMPs. The finite Boolean subalgebras with n atoms are shown to correspond to direct product decompositions of X with n factors, while the infinite Boolean subalgebras are shown to correspond to a certain type of continuously varying direct product decomposition known as a Boolean sheaf (or Boolean product). This characterization of the Boolean subalgebras of $BDec\ X$ leads to a complete description of compatibility in such OMPs, and the result that all such OMPs are regular. These results are exploited in numerous ways in the subsequent section, and in particular lead to a type of logic based on decompositions.

In section 7 we give an axiomatic presentation of what we term an experimental system focused very tightly on the notion of direct product decompositions. Roughly, the key ingredient is the requirement that each n-ary experiment on a system induces an n-ary direct product decomposition on the state space of the system. From this we obtain that the binary experiments (the so-called questions of the system) form an OMP, and that this OMP of questions has a sane interpretation of its logic. With a few more axioms we have the notion of an experimental system with probabilities. From this we develop an interpretation of the probability of an experiment yielding a certain result when the system is in a given state, and then a theory of observables, their expected values, and their calculus. The standard Hilbert space model for quantum mechanics is seen as a specific instance of such an experimental system.

The material in these first seven sections provides enough of the core work on decompositions for the reader to gain a good feel for the subject. A number of additional results not necessary for a first view are given briefly in Section 8 along with a number of open problems for the reader who wishes to pursue the subject. A brief conclusion is then presented as Section 9.

This chapter is presented as an invitation to those who may wish to further explore the subject. It is written in a somewhat informal way, and includes only a very few proofs whose purpose is to better illustrate the nature of the mathematics involved in the underlying theory. This subject has unfortunately almost entirely been my own project, and the results here are (nearly) all contained in the papers [14, 15, 16, 17, 18] where complete proofs can be found.

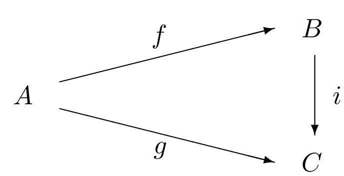
2 SURJECTIVE MAPS AND QUOTIENTS

In this section we review some basics about surjective (onto) mappings. While this material is in some way familiar to us all, it may familiar be at a more subconscious level, and a more organized treatment of these ideas may not be immediately at hand. As our treatment of decompositions will in many ways mirror our treatment of surjective mappings, this section serves as both a review and a preview for later developments.

DEFINITION 1. Given a set A, a surjection with domain A consists of a set B and a surjective (onto) mapping $f : A \to B$.

When considering surjections with domain A, or any type of mapping for that matter, one is often not interested in the particular nature of the elements involved, but only in the way elements are transformed into others. We make this precise in the following.

DEFINITION 2. Define an equivalence relation $\approx$ on the collection of all surjections with domain A by setting $f : A \to B$ to be $\approx$-related to $g : A \to C$ if there is an isomorphism (bijection) $i : B \to C$ with $i \circ f = g$.



We use $[f : A \to B]$ to denote the $\approx$-equivalence class of the surjection $f : A \to B$, and we let $Surj\ A$ be the set of all equivalence classes of surjections with domain A.

Putting structure on the set $Surj\ A$ is key.

DEFINITION 3. Define a relation $\leq$ on $Surj\ A$ by setting $[f : A \to B] \leq [g : A \to C]$ if there is a map $h : B \to C$ with $h \circ f = g$.

Of course, one must verify that the above definition of $\leq$ is independent of the representatives of the equivalence classes involved, but this is routine. Our aim is to prove the following.

THEOREM 4. *$(Surj\ A, \leq)$ is a complete lattice.*

Here we could proceed directly, using obvious arguments to show $\leq$ is reflexive and transitive, and the fact that surjections are epic for anti-symmetry, to obtain that $\leq$ is a partial ordering. The existence of arbitrary joins can be obtained from properties of products, and then the existence of meets follows from general principals. However, it is more common to treat surjections by showing $(Surj\ A, \leq)$ is dually isomorphic to the lattice of equivalence relations on A. In any event, one must establish this dual isomorphism as it is necessary to make computations tractable.

DEFINITION 5. For $f : A \to B$ set $\ker f$, the kernel of f, to be $\{(x, y)|f(x) = f(y)\}$.

THEOREM 6. *The structure $(Surj\ A, \leq)$ is dually isomorphic to $(Eq\ A, \subseteq)$, the set of equivalence relations on A partially ordered by set inclusion, via the map that takes $[f : A \to B]$ to $\ker f$.*

One uses this dual isomorphism and the following description of joins and meets in the lattice of equivalence relations to effect computations with surjections.

PROPOSITION 7. *For A a set, $(Eq\ A, \subseteq)$ is a complete lattice where*

1. *Meets are given by intersections.*

2. *Joins are given by the transitive closure of the union.*

One additional detail regarding computations in $Eq\ A$ will be important. We recall that for relations θ, ϕ on A that the relational product $\theta \circ \phi$ is the relation $\{(x, z)|x\theta y \text{ and } y\phi z \text{ for some } y\}$. The following is well known [6].

PROPOSITION 8. *For θ, ϕ equivalence relations on A these are equivalent.*

1. *$\theta \circ \phi = \phi \circ \theta$, i.e. θ and ϕ permute.*

2. *$\theta \circ \phi$ is the join $\theta \vee \phi$ of θ, ϕ in $Eq\ A$.*

To summarize, we believe that the lattice $(Surj\ A, \leq)$ is the primitive notion. The more familiar $(Eq\ A, \subseteq)$ arises as a means to effectively work with this primitive notion. Essentially, from each equivalence class of surjections $[f : A \to B]$, we choose a canonical representative $\kappa_\theta : A \to A/\theta$ where $\theta = \ker f$ and κ_θ is the natural quotient map. Then rather than study the surjections $\kappa_\theta : A \to A/\theta$, we simply study the equivalence relations θ that determine them.

3 DECOMPOSITIONS

This section contains the core material around which the chapter is built — the notion of direct product decompositions of a set. Our treatment will mirror the treatment of surjections given in the previous section.

DEFINITION 9. A direct product decomposition, or simply a decomposition, of a set A consists of a finite sequence of sets $A_1, \ldots, A_n$ and an isomorphism $f : A \to A_1 \times \cdots \times A_n$.

When the map f is clear from the context, and cumbersome to write, we use $A \simeq A_1 \times \cdots \times A_n$. We need some notation for working with maps and products.

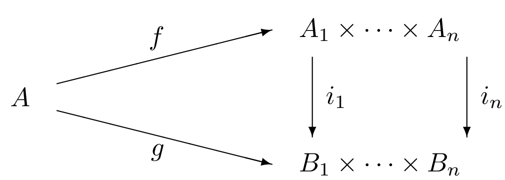
DEFINITION 10. Let $A, A_1, \ldots, A_n$ be sets, $f : A \to A_1 \times \cdots \times A_n$ and $g_i : A \to A_i$.

1. Define $\pi_i : A_1 \times \cdots \times A_n \to A_i$ to be the i^{th} projection map.
2. Define $f_i : A \to A_i$ to be the composite $\pi_i \circ f$.
3. Define $g_1 \times \cdots \times g_n : A \to A_1 \times \cdots \times A_n$ by $(g_1 \times \cdots \times g_n)(a) = (g_1(a), \ldots, g_n(a))$.

Thus for $f : A \to A_1 \times \cdots \times A_n$ we have $f = f_1 \times \cdots \times f_n$.

Just as with surjections, when considering a decomposition $f : A \to A_1 \times \cdots \times A_n$ one is often not interested in the particular elements of the sets $A_1, \ldots, A_n$, but only in how the bijection f maps elements of A into the elements of the product (for instance, which elements of A are mapped to ones with identical first components). This is made precise by the following.

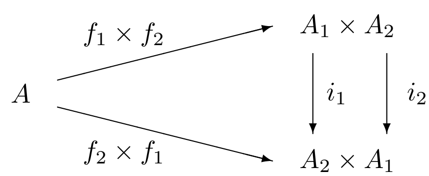
DEFINITION 11. Define an equivalence relation $\approx$ on the decompositions of A by setting $f : A \to A_1 \times \cdots \times A_m$ to be $\approx$-related to $g : A \to B_1 \times \cdots \times B_n$ if $m = n$ and there are there isomorphisms $i_1, \ldots, i_n$ with $i_k : A_k \to B_k$ and $(i_1 \times \cdots \times i_n) \circ f = g$.



Then $[f : A \to A_1 \times \cdots \times A_n]$ denotes the $\approx$-equivalence class of $f : A \to A_1 \times \cdots \times A_n$.

As with $Surj\ A$, it is key to place structure on the equivalence classes of decompositions. We begin by putting structure on the equivalence classes $BDec\ A$ of binary decompositions of a set A, though later we will also place structure on the collection of all equivalence classes of decompositions. At heart, we use the obvious connection between $A_1 \times A_2$ and $A_2 \times A_1$ to define a unary operation on *BDec A*, and we use the relationship between $A_1 \times (A_2 \times A_3)$ and $(A_1 \times A_2) \times A_3$ to define a relation $\leq$ on *BDec A*. Before giving the precise definitions, we review a few facts about decompositions and the corresponding notation.

The decomposition $f : A \to A_1 \times A_2$ is literally equal to $f_1 \times f_2 : A \to A_1 \times A_2$ (as $f = f_1 \times f_2$ by Definition 10). One sees that $f_2 \times f_1 : A \to A_2 \times A_1$ is also a decomposition. However, this decomposition is not even $\approx$-related to the original! This fact seems unusual as there is an obvious isomorphism $i : A_1 \times A_2 \to A_2 \times A_1$. But to have these decompositions $\approx$-related, Definition 11 requires a pair of isomorphisms i_1, i_2 making the following diagram commute, and clearly this is not the case.



Note aslo that from a ternary decomposition $f : A \to A_1 \times A_2 \times A_3$ we can build several binary decompositions such as $f_1 \times (f_2 \times f_3) : A \to A_1 \times (A_2 \times A_3)$ and the quite different $(f_1 \times f_2) \times f_3 : A \to (A_1 \times A_2) \times A_3$. Again, there is an isomorphism between $A_1 \times (A_2 \times A_3)$ and $(A_1 \times A_2) \times A_3$, but this is certainly not sufficient to make these decompositions $\approx$-related.

DEFINITION 12. For a set A, let *BDec A* be the collection of all equivalence classes of binary decompositions of A. Define a unary operation $*$ on *BDec A* by setting

$$[f : A \to A_1 \times A_2]^* = [f_2 \times f_1 : A \to A_2 \times A_1].$$

And define $\leq$ on *BDec A* by setting $[f : A \to A_1 \times A_2] \leq [g : A \to B_1 \times B_2]$ if

$$\begin{aligned} [f : A \to A_1 \times A_2] &= [h_1 \times (h_2 \times h_3) : A \to C_1 \times (C_2 \times C_3)] \\ [g : A \to B_1 \times B_2] &= [(h_1 \times h_2) \times h_3 : A \to (C_1 \times C_2) \times C_3] \end{aligned}$$

for some ternary decomposition $h : A \to C_1 \times C_2 \times C_3$.

The crucial definition of the relation $\leq$ can be expressed in a somewhat different way. Whenever A is isomorphic to a ternary direct product $C_1 \times C_2 \times C_3$, then the equivalence class of the binary decomposition $A \simeq C_1 \times (C_2 \times C_3)$ is $\leq$ that of the decomposition $A \simeq (C_1 \times C_2) \times C_3$. In a sense, it is this careful treatment of commutativity and associativity, or rather the lack of these properties, that gives our structure. We now present our primary result.

THEOREM 13. *For A a set, $(BDec\ A, \leq, *)$ is an* OMP.

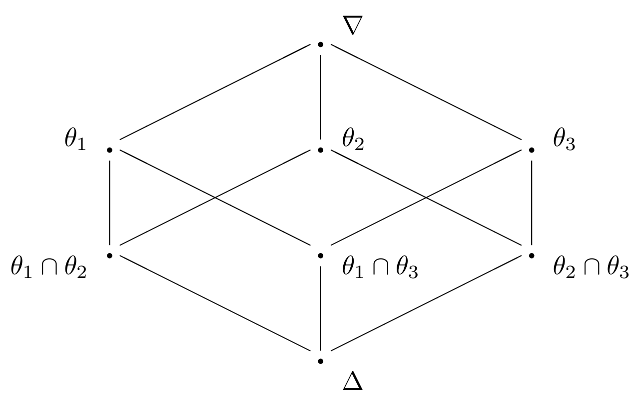
To prove *Surj* A is a lattice, and to find tractable methods to compute in this lattice, we showed each equivalence class of surjections has a canonical representative $A \to A/\theta$ uniquely determined by an equivalence relation on A. Computations in *Surj* A are then reduced to computations with equivalence relations. We now follow a similar path to show *BDec* A is an OMP, and to give tractable methods to compute in this OMP.

DEFINITION 14. Two equivalence relations θ, ϕ on a set A permute if $\theta\circ\phi = \phi\circ\theta$. A set of equivalence relations on A is pairwise permuting if any two members of the set permute. A set of equivalence relations on A is called a Boolean subsystem of *Eq* A if it is pairwise permuting and forms a Boolean sublattice of the lattice *Eq* A.

As we will see, Boolean subsystems are closely linked to direct product decompositions. We require one further definition.

DEFINITION 15. A factor n-tuple of a set A is a sequence $(\theta_1, \ldots, \theta_n)$ of equivalence relations on A whose members that differ from the universal (largest) relation ∇ on A are distinct and comprise exactly the coatoms of a Boolean subsystem of *Eq* A.

So a factor pair is an ordered pair (θ_1, θ_2) of permuting equivalence relations that are complements in the lattice *Eq* A. A factor triple $(\theta_1, \theta_2, \theta_3)$ is formed either by inserting ∇ into a factor pair, or by taking the coatoms of an eight-element Boolean subsystem as shown below.



The following result is the key link between factor tuples and decompositions. In the binary case it is well known and easily found in the literature [6]. The general case follows easily, and is found in [29, pg. 161].

PROPOSITION 16. *Let A be a set.*

1. *If $f : A \to A_1 \times \cdots \times A_n$ is an n-ary decomposition, then for $\theta_i = \ker f_i$ we have that $(\theta_1, \ldots, \theta_n)$ is a factor n-tuple of A.*

2. *If* $(\theta_1, \ldots, \theta_n)$ *is a factor* n*-tuple, then the natural map* $\kappa_{\theta_1} \times \cdots \times \kappa_{\theta_n}$ *provides an* n*-ary decomposition* $A \simeq A/\theta_1 \times \cdots \times A/\theta_n$.

This result implies that each equivalence class $[f : A \to A_1 \times \cdots \times A_n]$ has a canonical representative $A \simeq A/\theta_1 \times \cdots \times A/\theta_n$ and this leads to a bijective correspondence between equivalence classes of n-ary decompositions and factor n-tuples where $[f : A \to A_1 \times \cdots \times A_n]$ corresponds to $(\ker f_1, \ldots, \ker f_n)$. So instead of working with the set $BDec\ A$ of equivalence classes of binary decompositions of A, we can work instead with the set of all factor pairs (θ_1, θ_2) of A. For this to be advantageous, the structure on $BDec\ A$, particularly the relation $\leq$, must have a tractable description in terms of factor pairs. Fortunately, this is the case.

PROPOSITION 17. *Let* (θ_1, θ_2) *and* (ϕ_1, ϕ_2) *be factor pairs of* A*. These are equivalent.*

1. *In* $BDec\ A$ *we have* $[A \simeq A/\theta_1 \times A/\theta_2] \leq [A \simeq A/\phi_1 \times A/\phi_2]$.
2. $\theta_1, \theta_2, \phi_1, \phi_2$ *belong to a Boolean subsystem of* $Eq\ A$ *and* $\phi_1 \subseteq \theta_1$.
3. $\phi_1 \subseteq \theta_1$, $\theta_2 \subseteq \phi_2$ *and* $\phi_1 \circ \theta_2 = \theta_2 \circ \phi_1$.

A proof of this result is found in [14, Lemma 3.3]. It provides a very simple method to work with the relation $\leq$, something that seems quite intractable when dealing directly with decompositions.

DEFINITION 18. For a set A, let $Fact\ A$ be the set of all factor pairs (θ_1, θ_2) of A. Define a unary operation $*$ on $Fact\ A$ by setting

$$(\theta_1, \theta_2)^* = (\theta_2, \theta_1)$$

and define a relation $\leq$ on $Fact\ A$ by setting

$$(\theta_1, \theta_2) \leq (\phi_1, \phi_2) \quad \text{if} \quad \phi_1 \subseteq \theta_1, \theta_2 \subseteq \phi_1 \text{ and } \phi_1 \circ \theta_2 = \theta_2 \circ \phi_1.$$

When convenient, we refer to the structure $(Fact\ A, \leq, *)$ simply as $Fact\ A$.

The manner in which the above structure is defined on $Fact\ A$, in conjunction with Proposition 17, then immediately provides the following result, which is of course, the reason we are working with the structure $Fact\ A$ in the first place.

PROPOSITION 19. *For* A *a set,* $(BDec\ A, \leq, *)$ *is isomorphic to* $(Fact\ A, \leq, *)$.

Due to this isomorphism, the following result obviously provides our primary objective of showing that $(BDec\ A, \leq, *)$ in an OMP.

THEOREM 20. *For a set* A*, the structure* $(Fact\ A, \leq, *)$ *is an* OMP.

Proof. We refer to [14, Theorem 3.5] for a complete proof in a more general setting, but we will show that $\leq$ is a partial ordering as this illustrates the nature of the arguments involved and may encourage the reader to fill the missing details themselves.

Note that the definition of a factor pair shows $\leq$ is reflexive, and anti-symmetry follows trivially from the definition of $\leq$. For transitivity, suppose $(\theta_1, \theta_2) \leq (\phi_1, \phi_2)$ and $(\phi_1, \phi_2) \leq (\psi_1, \psi_2)$. Then $\psi_1 \subseteq \phi_1 \subseteq \theta_1$, $\theta_2 \subseteq \phi_2 \subseteq \psi_2$, and both ϕ_1, θ_2 and ψ_1, ϕ_2 permute. To show $(\theta_1, \theta_2) \leq (\psi_1, \psi_2)$ we must show $\psi_1 \subseteq \theta_1$ and $\theta_2 \subseteq \psi_2$, which are obvious, and that ψ_1, θ_2 permute.

Suppose $(a, b) \in \psi_1 \circ \theta_2$. As $\psi_1 \subseteq \phi_1$ we have $(a, b) \in \phi_1 \circ \theta_2$, and as $\theta_2 \subseteq \phi_2$ we have $(a, b) \in \psi_1 \circ \phi_2$. As ϕ_1, θ_2 and ψ_1, ϕ_2 both permute, we have $(a, b) \in \theta_2 \circ \phi_1$ and $(a, b) \in \phi_2 \circ \psi_1$. So there are c, d with $a\theta_2 c\phi_1 b$ and $a\phi_2 d\psi_1 b$. Then as $\theta_2 \subseteq \phi_2$ we have $c\phi_2 a\phi_2 d$ so $c\phi_2 d$, and as $\psi_1 \subseteq \phi_1$ we have $c\phi_1 b\phi_1 d$, so $c\phi_1 d$. Then (c, d) belongs to $\phi_1 \cap \phi_2$, which is the diagonal relation Δ, so $c = d$. Thus $a\theta_2 c = d\psi_1 b$, showing $(a, b) \in \theta_2 \circ \psi_1$. So $\psi_1 \circ \theta_2 \subseteq \theta_2 \circ \psi_1$ and a similar argument shows the other containment. ∎

Above we have seen how to tractably work with $*$ and $\leq$ in *Fact A*. We next discuss computation of orthogonal joins $\oplus$ in *Fact A* and *BDec A*. Later, in Section 6, we discuss compatibility in these structures.

PROPOSITION 21. *For a set A, two elements (θ_1, θ_2) and (ϕ_1, ϕ_2) of Fact A are orthogonal if, and only if, $\phi_2 \subseteq \theta_1$, $\theta_2 \subseteq \phi_1$ and θ_2, ϕ_2 permute. In this case,*

$$(\theta_1, \theta_2) \oplus (\phi_1, \phi_2) = (\theta_1 \cap \phi_1, \theta_2 \circ \phi_2).$$

Two elements of BDec A are orthogonal if, and only if, they are of the form

$$[A \simeq A_1 \times (A_2 \times A_3)] \quad \textit{and} \quad [A \simeq A_3 \times (A_1 \times A_2)]$$

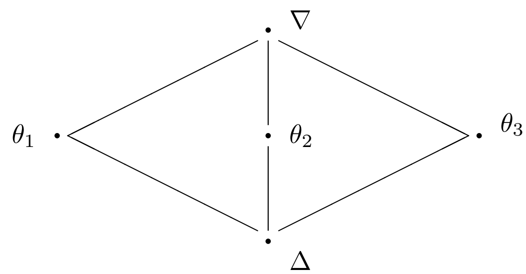
for some ternary decomposition $A \simeq A_1 \times A_2 \times A_3$. In this case, their orthogonal join is given by $[A \simeq (A_1 \times A_3) \times A_2]$.

In sum, we have treated decompositions much the way we treated surjections. We gave a definition of decompositions, and defined an equivalence relation on the collection of all decompositions. We then put structure on the collection *BDec A* of equivalence classes of binary decompositions of A. To prove this structure gave an OMP and to provide tractable methods to work with this OMP we passed to an auxiliary set *Fact A* built from equivalence relations on A. The key idea is that each equivalence class of decompositions has a canonical representative that can be described in terms of equivalence relations.

4 SURJECTIONS AND DECOMPOSITIONS FOR FINITE SETS

Here we provide some concrete examples decompositions to give the reader a more definite feel for the subject. We begin with one small example of *Surj A*, primarily to emphasize a certain point that sometimes causes difficulty.

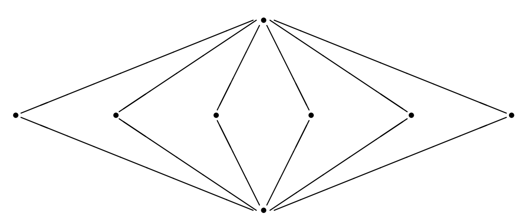
EXAMPLE 22. *Consider the 3-element set $A = \{a, b, c\}$. There are 5 equivalence relations on A; the diagonal relation Δ, the universal relation ∇, and the relations $\theta_1 = \Delta \cup \{(a, b), (b, a)\}$, $\theta_2 = \Delta \cup \{(a, c), (c, a)\}$, and $\theta_3 = \Delta \cup \{(b, c), (c, a)\}$. So the lattice Eq A, which is dually isomorphic to Surj A, is as depicted below.*



A point to emphasize is that an equivalence class of surjections $[f : A \to B]$ is not simply determined by the cardinality of the image set B, but depends also on how elements of A are collapsed together into elements of B. In this case there are 3 distinct equivalence classes $[f : A \to B]$ where B is a 2*-element set as there are 3 ways to choose a pair of elements of A to collapse together.*

Before giving examples of *Fact A*, we note that if A is a finite set and θ belongs to a factor pair of A, then θ is a regular equivalence relation, which means that each equivalence class (also called block) of θ has the same number of elements. Thus if (θ_1, θ_2) is a factor pair of A, then A is isomorphic to $A/\theta_1 \times A/\theta_2$, and the number of blocks of θ_1 is the cardinality of A/θ_1, while the number of elements in each block of θ_1 is the cardinality of A/θ_2 as both are given by the cardinality of A divided by the cardinality of A/θ_1.

EXAMPLE 23. *Suppose that A is the* 4*-element set $\{a, b, c, d\}$. Then there are 5 regular equivalence relations on this set; Δ, ∇, the equivalence relation θ_1 whose blocks are $\{a, b\}, \{c, d\}$, the equivalence relation θ_2 whose blocks are $\{a, c\}, \{b, d\}$, and the equivalence relation θ_3 whose blocks are $\{a, d\}, \{b, c\}$. Each of the 6 pairs (θ_i, θ_j) where $i \neq j$ yields a factor pair in this case, and it follows that Fact A is the height two* OMP *MO_6 shown below.*



To reiterate, there are 8 equivalence classes of decompositions of a 4*-element set. There is one as the product of a* 4*-element set and a* 1*-element set, one as the product of a* 1*-element set and a* 4*-element set, and 6 different decompositions as the a product of two* 2*-element sets.*

EXAMPLE 24. *Suppose A is the 6-element set $\{a,b,c,d,e,f\}$. Aside from Δ, ∇, each regular equivalence relation θ will have either two or three blocks, depending on whether A/θ is a two or three-element set. Factor pairs (θ_1, θ_2) not involving Δ, ∇ will consist of one regular equivalence relation with two blocks and one with three. But not all such pairs of regular equivalence relations will provide a factor pair. For the relation θ_1 with blocks $\{a,b,c\}, \{d,e,f\}$, θ_2 with blocks $\{a,b\}, \{c,d\}, \{e,f\}$, and θ_3 with blocks $\{a,d\}, \{b,e\}, \{c,f\}$, we can see that (θ_1, θ_2) is not a factor pair as $\theta_1 \cap \theta_2 \neq \Delta$, while (θ_1, θ_3) is a factor pair. Again, the* OMP *Fact A will be an MO_n for some finite n (the combinatorial exercise of finding n is left to the reader).*

At this point it may seem that the OMPs *Fact A* are of limited interest as each of OMPs in the previous two examples is of height two, consisting of blocks (maximal Boolean subalgebras) having exactly two atoms each. This however, is because each of the primary decompositions $4 = 2 \times 2$ and $6 = 2 \times 3$ involves exactly two prime factors. The following description of *Fact A* for a finite set A follows from results in Section 6.

PROPOSITION 25. *Suppose A is an n-element set where $n = p_1 \cdot p_2 \cdots p_k$ is the primary decomposition of n. Then the* OMP *Fact A is of height k (the number of factors in the primary decomposition of n) and consists of blocks all having k atoms each.*

It is useful to consider things from the perspective of *BDec A* rather than *Fact A*.

EXAMPLE 26. *Let $A = \{0,1\}^3$ and let $f, g : A \to \{0,1\} \times \{0,1\} \times \{0,1\}$ be defined by $f(x,y,z) = (x,y,z)$ and $g(x,y,z) = (x+y, y, z)$, where addition is modulo 2. These two ternary decompositions give rise to the following binary decompositions*

$$
\begin{aligned}
D &= [f_1 \times (f_2 \times f_3) : A \to \{0,1\} \times (\{0,1\} \times \{0,1\})] \\
E &= [g_1 \times (g_2 \times g_3) : A \to \{0,1\} \times (\{0,1\} \times \{0,1\})] \\
F &= [(f_1 \times f_2) \times f_3 : A \to (\{0,1\} \times \{0,1\}) \times \{0,1\}] \\
G &= [(g_1 \times g_2) \times g_3 : A \to (\{0,1\} \times \{0,1\}) \times \{0,1\}].
\end{aligned}
$$

By definition, $D \leq F$ and $E \leq G$. Surely $D \neq E$ as there can be no isomorphism $i_1 : \{0,1\} \to \{0,1\}$ with $i_1 \circ f_1 = g_1$ as this would require $i_1(x) = x + y$ for all $x, y, z \in \{0,1\}$. But $F = G$ as the isomorphism $j_1 : \{0,1\} \times \{0,1\} \to \{0,1\}$ defined by $j_1(x,y) = (x+y, y)$ satisfies $j_1 \circ (f_1 \times f_2) = g_2 \times g_2$, and the isomorphism $j_2 : \{0,1\} \to \{0,1\}$ defined by $j_2 = id$ satisfies $j_2 \circ f_3 = g_3$.

One then obtains that D, F generate one of the 8-element blocks of BDec A, that E, G generate another distinct block, and that these two blocks have in common the coatom $F = G$ and the atom $F^* = G^*$. In particular, for this 8-element set A, the OMP *BDec A is of height three and consists of 8-element Boolean algebras linked in a rather intricate way.*

The OMPs *Fact A* for A a finite set are combinatorially very interesting structures with many interesting symmetries. Of course, the structure *Fact A* for A

an infinite set is surely of primary interest, as are sub-OMPs of *Fact A* induced by placing various types of structure on A and restricting to decompositions that respect this structure. This shall be the focus of the following section.

5 DECOMPOSITIONS OF SETS WITH STRUCTURE

The term "set with structure" is a broad one, meant to include such familiar objects as groups, rings, vector spaces, and other algebras with finitary or infinatary operations; relational structures such as posets and graphs; topological structures and uniform spaces; and so forth. Our aim is to consider structure preserving decompositions of such objects. As before, it is instructive to begin with surjections.

DEFINITION 27. For a group $\mathcal{G}$, a surjection of $\mathcal{G}$ consists of a group $\mathcal{H}$ and an onto homomorphism $f : \mathcal{G} \to \mathcal{H}$. An equivalence relation $\approx$ is defined on the surjections of $\mathcal{G}$ as in Definition 2 and a relation $\leq$ and is defined on the equivalence classes of surjections of $\mathcal{G}$ as in Definition 3.

Of course, a similar definition would apply for any other type of algebra $\mathcal{A}$, such as a ring, vector space, monoid, and so forth. Key to studying $(Surj\ \mathcal{A}, \leq)$ for an algebra $\mathcal{A}$ is the correspondence between surjections of $\mathcal{A}$ and congruences of $\mathcal{A}$ (certain equivalence relations on the underlying set of the algebra that are compatible with the basic operations of the algebra [6]).

PROPOSITION 28. *For an algebra $\mathcal{A}$ with underlying set A, the structure $(Surj\ \mathcal{A}, \leq)$ is dually isomorphic to the sublattice $(Con\ \mathcal{A}, \leq)$ of the lattice $(Eq\ A, \leq)$ consisting of all congruence relations on $\mathcal{A}$.*

In specific settings, there are alternate, but equivalent, approaches to working with the lattice of surjections. For example, for groups one often works with the lattice of normal subgroups, for rings the lattice of ideals, and for vector spaces the lattice of subspaces. We turn now to decompositions.

DEFINITION 29. A decomposition of a group $\mathcal{G}$ consists of a finite sequence $\mathcal{G}_1, \ldots, \mathcal{G}_n$ of groups and a group isomorphism $f : \mathcal{G} \to \mathcal{G}_1 \times \cdots \times \mathcal{G}_n$. We define an equivalence relation $\approx$ on the decompositions of $\mathcal{G}$ as in Definition 11 and we define a unary operation $*$ and binary relation $\leq$ on the collection $BDec\ \mathcal{G}$ of all equivalence classes of binary decompositions of $\mathcal{G}$ as in Definition 12.

Of course, similar definitions apply for rings, vector spaces, or indeed any type of algebra. A proof of the following is found in [14, Remark 4.2].

PROPOSITION 30. *For an algebra $\mathcal{A}$ with A as its underlying set, $(BDec\ \mathcal{A}, \leq, *)$ is isomorphic to the sub-OMP $(Fact\ \mathcal{A}, \leq, *)$ of $(Fact\ A, \leq, *)$ consisting of all factor pairs (θ_1, θ_2) where both θ_1, θ_2 are congruences.*

For relational structures and topological spaces, the situation is similar (see [14, Theorem 4.4 and 4.6]). A decomposition consists of a finite sequence of relational structures or topological spaces, and an isomorphism or homeomorphism, as the case may be, betweem the original structure and the product of this family of structures. Exactly as above, one obtains a structure $(BDec\ \mathcal{A}, \leq, *)$ for any relational or topological structure $\mathcal{A}$.

PROPOSITION 31. *If $\mathcal{A}$ is a relational structure (topological space) with underlying set A, then $(BDec\ \mathcal{A}, \leq, *)$ is isomorphic to the sub-*OMP *$(Fact\ \mathcal{A}, \leq, *)$ of $(Fact\ A, \leq, *)$ consisting of all factor pairs (θ_1, θ_2) for which there are relations (topologies) on A/θ_1 and A/θ_2 making $\mathcal{A}$ naturally isomorphic (homeomorphic) to their product.*

To be precise, the above result requires the underlying set of the structure to be non-empty, and for relational structures, we require also that the relation on the set be non-empty. Next, some examples.

EXAMPLE 32. *Consider a vector space $\mathcal{V}$. As all congruences of $\mathcal{V}$ permute and congruences of $\mathcal{V}$ correspond to subspaces of $\mathcal{V}$, it follows that $Fact\ \mathcal{V}$ is isomorphic to the collection of all ordered pairs (S_1, S_2) of subspaces of $\mathcal{V}$ satisfying $S_1 \cap S_2 = 0$ and $S_1 + S_2 = V$. The unary operation on such pairs is given by $(S_1, S_2)^* = (S_2, S_1)$, the partial ordering is given by $(S_1, S_2) \leq (T_1, T_2)$ if $S_1 \subseteq T_1$ and $T_2 \subseteq S_2$, and orthogonal joins are given by $(S_1, S_2) \oplus (T_1, T_2) = (S_1 + T_1, S_2 \cap T_2)$.*

EXAMPLE 33. *For a ring $\mathcal{R}$, if we consider $\mathcal{R}$ to be a left $\mathcal{R}$-module $\mathcal{R}_{\mathcal{R}}$ over itself, then $Fact\ \mathcal{R}_{\mathcal{R}}$ is isomorphic to the* OMP *of idempotents $E(\mathcal{R})$ of the ring $\mathcal{R}$. This* OMP *$E(\mathcal{R})$ has been well studied and is known as the logic of idempotents of $\mathcal{R}$ [8, 25]. In it, the orthocomplement e^* is given by $1 - e$, we have $e \leq f$ if e, f commute and $ef = e$, and orthogonal joins are given by $e \oplus f = e + f$.*

There are many examples where the decompositions of a structure that has a mix of algebraic, relational and topological features form an OMP. One case of obvious interest is that of a Hilbert space.

EXAMPLE 34. *A Hilbert space $\mathcal{H}$ consists of a vector space with an inner product associating a real or complex number to each pair of vectors in $\mathcal{H}$. There is a standard definition of the product $\mathcal{H}_1 \times \mathcal{H}_2$ of two Hilbert spaces where the inner product on $\mathcal{H}_1 \times \mathcal{H}_2$ is the sum of the componentwise inner products. So we may define the structure $BDec\ \mathcal{H}$ as above. Equivalence classes of binary decompositions of $\mathcal{H}$ correspond to closed subspaces of $\mathcal{H}$. To see this we observe that each closed subspace A of $\mathcal{H}$ gives a decomposition $\mathcal{H} \simeq A \times A^{\perp}$ as described in the introduction, and for each decomposition $\mathcal{H} \simeq \mathcal{H}_1 \times \mathcal{H}_2$ we have that $\{(a, 0) | A \in \mathcal{H}_1\}$ gives a closed subspace of $\mathcal{H}$. Upon examining the definitions of $\leq$ and $*$, it follows that the structure $BDec\ \mathcal{H}$ is isomorphic to the familiar* OML *of closed subspaces of $\mathcal{H}$.*

A related example is worthwhile examining.

EXAMPLE 35. *An inner product space $\mathcal{E}$ is a vector space with an inner product associating to each pair of vectors a real or complex number. Hilbert spaces are inner product spaces where the metric induced by this inner product is complete. If one considers incomplete inner product spaces, also known as pre-Hilbert or Euclidean spaces, it is well known that the closed subspaces no longer satisfy the* *orthomodular law. In fact, having the closed subspaces be orthomodular characterizes Hilbert spaces among inner product spaces as is shown by Amemiya and Araki [1]. This was long used as an argument that orthomodularity captured the essence of Hilbert space. However, one can obtain an* OMP *from even an incomplete inner product space by considering not the closed subspaces, but the splitting subspaces (see [7]). This is a standard method to construct* OMP*s. The crucial observation to make is that these splitting spaces correspond exactly to the direct product decompositions of the inner product space. Thus, for an inner product space $\mathcal{E}$, the structure BDec $\mathcal{E}$ is isomorphic to the well-known* OMP *of splitting subspaces of $\mathcal{E}$.*

There are many other examples of structures whose decompositions form OMPs, such as uniform spaces and topological groups. One gets the feeling that most familiar mathematical structures have this property. The matter of developing the theory of these OMPs of decompositions for particular classes of structures lies essentially wide open, and seems a worthwhile task.

6 COMPATIBILITY OF DECOMPOSITIONS

Here we begin a study of the fine structure of OMPs *BDec A* for a set A. Our results will apply equally to *BDec* $\mathcal{A}$ for many classes of structures $\mathcal{A}$. We begin with several definitions.

DEFINITION 36. For an OMP P, we say a subset S of P is a subalgebra of P if S is closed under orthocomplementation and finite orthogonal joins.

Clearly a subalgebra of an OMP P is itself an OMP. So we will often call a subalgebra S a sub-OMP of P.

DEFINITION 37. A subalgebra S of an OMP P is called a Boolean subalgebra of P if, when considered as an OMP, S forms a a Boolean algebra.

By definition, any two elements in a Boolean subalgebra B of an OMP P have a join and meet in B. One might ask whether these elements also have a join and meet in P. Fortunately, the situation works out as nicely as one would hope.

PROPOSITION 38. *If B is a Boolean subalgebra of an* OMP *P, then any two elements of B have a join and meet in P, and these agree with the join and meet taken in B.*

Characterizing the Boolean subalgebras of *BDec A*, and its alter ego *Fact A*, will be a main task for us. We begin by determining when two elements of *Fact A* lie in a Boolean subalgebra. For an account of the following important notion, see [36].

DEFINITION 39. We say elements x, y of an OMP P are compatible if they lie in a Boolean subalgebra of P.

PROPOSITION 40. *For a set A, two elements (θ_1, θ_2) and (ϕ_1, ϕ_2) of Fact A are compatible if, and only if, $\theta_1, \theta_2, \phi_1, \phi_2$ lie in a Boolean subsystem of Eq A. In* *this case, their join and meet are given by*

$$\begin{aligned}(\theta_1,\theta_2)\vee(\phi_1,\phi_2) &= (\theta_1\cap\phi_1,\theta_2\circ\phi_2),\\ (\theta_1,\theta_2)\wedge(\phi_1,\phi_2) &= (\theta_1\circ\phi_1,\theta_2\cap\phi_2).\end{aligned}$$

Further, if (θ_1,θ_2) *and* (ϕ_1,ϕ_2) *are compatible, the members of* $\theta_1\circ\phi_1$, $\theta_2\circ\phi_1$, $\theta_2\circ\phi_1$ *and* $\theta_2\circ\phi_2$ *that are distinct from* ∇ *are the coatoms of a Boolean subsystem of Eq A that contains* $\theta_1,\theta_2,\phi_1,\phi_2$.

This result, found in [15, Lemma 3.1], has the following consequence.

PROPOSITION 41. *For A a set and B a subset of Fact A, these are equivalent.*

1. *B is a Boolean subalgebra of Fact A.*
2. $\{\theta|$ *there is some* θ' *with* $(\theta,\theta')\in B\}$ *is a Boolean subsystem of Eq A.*

While this yields a workable method to deal with Boolean subalgebras of *Fact A*, it is not the kind of insightful characterization we seek. This comes when we pass back to our object of primary interest, the OMP *BDec A* of decompositions of *A*. Key to translating the above results into results about *BDec A* is the connection between finite Boolean subsystems of *Eq A* and direct product decompositions of *A* that was described at the end of Section 3.

PROPOSITION 42. *For a set A, two elements of BDec A are compatible if, and only if, there is a decomposition* $f : A\to A_1\times\cdots\times A_4$ *with the given elements equal to*

$$[A\simeq(A_1\times A_2)\times(A_3\times A_4)] \quad \textit{and} \quad [A\simeq(A_1\times A_3)\times(A_2\times A_4)].$$

In this case, the join and meet of these elements are given respectively by

$$[A\simeq(A_1\times A_2\times A_3)\times A_4] \quad \textit{and} \quad [A\simeq A_1\times(A_2\times A_3\times A_4)].$$

Thus, compatible decompositions are ones that are built from a common decomposition. We then obtain the following characterization of the finite Boolean subalgebras of *BDec A*.

PROPOSITION 43. *For a decomposition* $f : A\to A_1\times\cdots\times A_n$ *and* $S\subseteq\{1,\ldots,n\}$,

$$A\simeq\prod_{i\in S}A_i\times\prod_{j\notin S}A_j$$

is a binary decomposition of A. The collection of all equivalence classes of such binary decompositions, where S ranges over all subsets of $\{1,\ldots,n\}$*, forms a Boolean subalgebra of BDec A. Further, all finite Boolean subalgebras of BDec A arise in this manner.*

In this result, the product of the empty family is by definition a one-element set. If we further require that in the decomposition $f : A\to A_1\times\cdots\times A_n$ none of the factors A_i is a one-element set (whose removal would yield a decomposition with one fewer factor), then the corresponding Boolean subalgebra $BDec\ A$ is isomorphic to the power set of $\{1,\ldots,n\}$.

We now turn our attention to infinite Boolean subalgebras of $BDec\ A$. The obvious path, that infinite Boolean subalgebras correspond to infinite direct product decompositions, is clearly not sufficient as countable sets can have no infinite direct product decompositions. Further, for instances where they do exist, infinite direct product decompositions will yield only subalgebras of $BDec\ A$ that are complete and atomic. We need a more general notion, a type of continuously varying direct product decomposition known as a Boolean sheaf decomposition. Those familiar with the closely related notion of Boolean products [6] will have no trouble placing our treatment in this other context.

DEFINITION 44. A Boolean sheaf $\tilde{S} = (S, X, \pi)$ consists of two topological spaces, S and X, and a local homeomorphism $\pi : S \to X$. We often call S the sheaf space and X the base space. Given any subset $U \subseteq X$ we then set

$$\Gamma U = \{f : U \to S | f \text{ is continuous and } \pi \circ f = id_U\}.$$

We call members of ΓU sections over U. Finally, we call a sheaf a Boolean sheaf if the base space X is a Boolean space, i.e. a space that is homeomorphic to the Stone space of a Boolean algebra [2, 6].

For an element $x \in X$, we call $S_x = \pi^{-1}\{x\}$ the stalk over x. The requirement $\pi \circ f = id_U$ is then equivalent to saying that $f(x) \in S_x$ for each $x \in U$. Thus ΓU is a subset of the product $\prod_{x\in U} S_x$. Indeed, it is the subset of this product consisting of all those choice functions that are continuous with respect to the topologies involved.

PROPOSITION 45. *Suppose A is a set, (S, X, π) is a Boolean sheaf, and $f : A \to \Gamma X$ is an isomorphism. Then for each clopen set $U \subseteq X$ we have*

$$A \simeq \Gamma U \times \Gamma(X - U)$$

is a binary direct product decomposition of A. The collection of all equivalence classes of such binary decompositions, where U ranges over all clopen subsets of X, forms a Boolean subalgebra of $BDec\ A$. Further, all Boolean subalgebras of $BDec\ A$ arise in this manner.

This result will be of essential use in the following section where we incorporate analytic features into the discussion of $BDec\ A$. We should note that the various results in this section lift to the setting of $BDec\ \mathcal{A}$ for various types of structures $\mathcal{A}$, but in the infinite case it can be a bit delicate to phrase things precisely. We refer the reader to [15, Section 6] for a complete account. We conclude this section with a final note of interest.

DEFINITION 46. A subset S of an OMP P is called a compatible set if S is contained in a Boolean subalgebra of P. The set S is called pairwise compatible if every 2-element subset of S is compatible. The OMP P is called regular if every pairwise compatible subset of P is compatible.

For background on the important notion of regularity see [36].

THEOREM 47. *For a set A, the* OMP *$BDec\ A$ is regular.*

Again, this result extends to OMPs $BDec\ \mathcal{A}$ for many types of structures $\mathcal{A}$. It was this way that the regularity of OMPs of splitting subspaces of an inner product space, and of OMPs formed from idempotents of a ring , was first established. See [15] for a complete account.

7 DECOMPOSITIONS AND QUANTUM LOGIC

Beside the fact that the structures $BDec\ \mathcal{A}$ provide a rich source of the OMPs used in quantum logic, we can make a more direct argument that decompositions are of interest in foundational studies of quantum mechanics. We present an axiomatic development of what we term an experimental system. Our aim is not to give a precise definition of the elusive idea of an experiment, but to axiomatize basic behavior of the collection of experiments of a physical system. One final introductory comment. The experiments we consider here are intended to be finitary ones, meaning that each has only finitely many possible outcomes.

DEFINITION 48. A leveled set consists of a non-empty set E together with a map from E to the natural numbers. The natural number associated to an element $e \in E$ is called the arity of e.

The collection of all experiments E of a physical system naturally forms a leveled set where the arity of an experiment $e \in E$ is the number of possible outcomes of e. For example, consider the experiment where a particle is sent through a magnetic device, then follows one of two paths with a detector placed in an appropriate spot on each path. This experiment has two possible outcomes, either detector 1 goes off, or detector 2 goes off, and is therefore a binary experiment.

DEFINITION 49. An ordered partition of a natural number $n \geq 1$ is a finite sequence σ of pairwise disjoint (possibly empty) subsets of $\{1, \ldots, n\}$ that cover $\{1, \ldots, n\}$. The number of terms in the sequence is denoted $\|\sigma\|$ and $\sigma(i)$ denotes the i^{th} term in the sequence.

This notion of ordered partitions is key to our development of an experimental system. As an example of their intended use, suppose we have an experiment where a particle is sent through a magnetic device, and depending on whether the particle is spin $+1$, spin 0, or spin -1, the particle follows one of three paths. Detectors are placed along the three paths, and we label the spin $+1$, 0, -1 outcomes as outcomes one, two and three, respectively. This describes a ternary experiment e.

If we consider the ordered partition $(\{1\}, \{3\}, \{2\})$, we intend $(\{1\}, \{3\}, \{2\})e$ to be the ternary experiment obtained from e by labeling the spin $+1$, 0, -1 outcomes as outcomes one, three, and two respectively. Of more interest is applying the ordered partition $(\{1,2\},\{3\})$ to e. Here we form a new experiment by placing a single detector over the spin +1 and spin 0 paths, calling this outcome one, and leaving in place the detector over the spin -1 path, and calling this outcome two. The resulting binary experiment is denoted $(\{1,2\},\{3\})e$.

DEFINITION 50. Let $\mathcal{O}$ be the set of all ordered partitions of natural numbers, and define a partial binary operation on $\mathcal{O}$ as follows. If σ is an ordered partition of n and ϕ is an ordered partition of $\|\sigma\|$, let $\phi\sigma$ be the ordered partition of n whose i^{th} member is $\bigcup\{\sigma(j)|j \in \phi(i)\}$. Finally, define i_n to be the ordered partition $(\{1\},\ldots,\{n\})$.

Thus $(\{1,3\},\{2\})(\{2,4\},\{1\},\{5\}) = (\{2,4,5\},\{1\})$, for example. One easily checks that when defined, this partial binary operation is associative.

DEFINITION 51. An action of $\mathcal{O}$ on a leveled set E associates to each n-ary element $e \in E$ and each ordered partition σ of n, an $\|\sigma\|$-ary element σe of E such that $(\phi\sigma)e = \phi(\sigma e)$ and $i_n e = e$.

Before the definition of an experimental system we require two further definitions.

DEFINITION 52. Let E be a leveled set acted on by $\mathcal{O}$ and let e, f be elements of E. Then f is said to be built from e if there is some σ with $\sigma e = f$. A subset $K \subseteq E$ is called compatible if for each finite $K' \subseteq K$ there is some member of E from which all members of K' can be built.

Next comes the anticipated link to decompositions.

DEFINITION 53. For a set A, let $Dec\ A$ be the collection of all equivalence classes of decompositions of A. Note that $Dec\ A$ naturally forms a leveled set. Define an action of $\mathcal{O}$ on $Dec\ A$ by setting $\sigma[f : A \to A_1 \times \cdots \times A_n]$ to be the equivalence class of the obvious $\|\sigma\|$-ary decomposition of A whose i^{th} factor is $\prod\{A_j|j \in \sigma(i)\}$.

As a simple example, we have that $(\{1,3\},\{2\})[f : A \to A_1 \times A_2 \times A_3]$ is equal to $[(f_1 \times f_3) \times f_2 : A \to (A_1 \times A_3) \times A_2]$. It is worthwhile to note that for a set K of equivalence classes of binary decompositions, i.e. for K a subset of $BDec\ A$, we now have two notions of compatibility for K. One is the definition provided above that requires members of K to be built from a common decomposition. The other is the usual notion of compatibility in the OMP $BDec\ A$. Fortunately, the results of the previous section show that these notions coincide.

Axioms of an experimental system

DEFINITION 54. An experimental system consists of a leveled set E acted on by $\mathcal{O}$, a set S, and an embedding $D : E \to Dec\ S$ that satisfies the axioms below. For convenience, elements of E are called experiments.

Axiom 1 If e is an n-ary experiment, De is an n-ary decomposition.

Axiom 2 $D(\sigma e) = \sigma(De)$ whenever σe is defined.

Axiom 3 For any $K \subseteq E$, K is compatible if, and only if, $D[K]$ is compatible.

Obviously we could define an experimental system to be a certain type of subalgebra of *Dec* S, and simply eliminate the set E and the embedding D. However, we feel it is worthwhile to retain a distinction between experiments and decompositions, keeping the viewpoint that an experiment induces a decomposition of a set S associated with the system.

The logic of questions

We develop a type of (partial) logic for the *Yes-No* experiments of an experimental system and show that this logic is *locally* Boolean. In large, the mathematics of the results here are presented earlier in Section 6, however our terminology, notation, and context are different, and this can be useful.

DEFINITION 55. Given an experimental system $\mathcal{E} = (E, S, D)$ we let $\mathcal{Q}(\mathcal{E})$ be the set of binary experiments of the system. We call such binary experiments *Yes-No* questions, or simply questions, and refer to outcome one of a question as the *Yes* outcome, and outcome two as the *No* outcome.

Results of the previous section immediately give the following.

LEMMA 56. *If* e, f *are compatible questions, then there is a unique* 4*-ary experiment* g *with* $e = (\{1,2\},\{3,4\})g$ *and* $f = (\{1,3\},\{2,4\})g$*. We call this* g *the standard refinement of* e, f*.*

Now the (partial) logical operations we seek.

DEFINITION 57. For e, f compatible questions and g their standard refinement, we define questions e OR f, e AND f, and NOT e as follows.

1. e OR $f = (\{1,2,3\},\{4\})g$
2. e AND $f = (\{1\},\{2,3,4\})g$
3. NOT $e = (\{2\},\{1\})e$.

Then define a relation IMPLIES on $\mathcal{Q}(\mathcal{E})$ by setting e IMPLIES f if e, f are compatible and (NOT e) OR f is the experiment $(\{1,2,3,4\},\emptyset)g$ we call TRUE.

The point here is that we now have an operational interpretation of the join and meet of compatible questions. Two Yes-No experiments e, f are compatible if there is a 4-ary experiment g so that e is equivalent to placing a Yes detector over outcomes 1, 2 of g and a No detector over outcomes 3, 4 of g; and f is equivalent to placing a Yes detector over outcomes 1, 3 of g and a No detector over outcomes 2, 4 of g. The Yes-No experiment e OR f is equivalent to placing a Yes detector over outcomes 1, 2, 3 of g and calling outcome 4 of g the No outcome of e OR f. The Yes-No experiment e AND f is formed by calling outcome 1 of g the Yes outcome, and placing a single detector over outcomes 2, 3, 4 of g and calling this the No outome.

One easily sees that the No outcome of (NOT e) OR f is outcome 2 of g. So e IMPLIES f holding corresponds to outcome 2 of g being an impossible outcome, as e IMPLIES f holding means (NOT e) OR f is equal to TRUE. If we then form a ternary experiment g' by ignoring this impossible outcome of g, then e corresponds to taking the first outcome of g' as the Yes outcome and the second two outcomes of g' as the No outcome; while f corresponds to taking the first two outcomes of g' as the Yes outcome, and the third outcome of g' as the No outcome. This seems a sane criteria to say e IMPLIES f.

THEOREM 58. *The partial logic on the questions $\mathcal{Q}(\mathcal{E})$ satisfies the following rules of classical logic. For each question e we have*

$$\text{NOT (NOT } e) = e, \tag{2}$$

$$e \text{ OR (NOT } e) = \text{TRUE}. \tag{3}$$

For each pair e, f of compatible questions we have

$$\text{NOT } (e \text{ OR } f) = (\text{NOT } e) \text{ AND } (\text{NOT } f). \tag{4}$$

And whenever both sides of the following equation are defined we have equality.

$$e \text{ OR } (f \text{ AND } g) = (e \text{ OR } f) \text{ AND } (e \text{ OR } g). \tag{5}$$

Other similar identities are easily obtained from these. We say this partial logic is locally Boolean to express the above properties.

Of course, (2) and (3) are simple consequences of NOT being the orthocomplementation on the OMP $\mathcal{Q}(\mathcal{E})$, and condition (4) then follows as OR and AND give the join and meet of compatible elements. Condition (5) follows as both sides being defined requires any two of e, f, g to be compatible, hence all three are compatible as $\mathcal{Q}(\mathcal{E})$ is regular, so all three lie in a Boolean subalgebra.

Probabilities

DEFINITION 59. A map $p : S \to [0,1]^n$ is called an n-ary probability map on the set S if $\sum_1^n p_i(s)$ is either 0 or 1 for each $s \in S$. *Prob S* denotes the collection of all n-ary probability maps for all $n \geq 1$.

Prob S forms a leveled set where the number associated to $p : S \to [0,1]^n$ is n.

DEFINITION 60. Define an action of $\mathcal{O}$ on the leveled set *Prob S* by setting σp to be the $\|\sigma\|$-ary probability map with $(\sigma p)_i(s) = \sum\{p_j(s) | j \in \sigma(i)\}$.

So if $p : S \to [0,1]^3$ is a ternary probability map we often write p as (p_1, p_2, p_3) where each of p_1, p_2, p_3 are maps from S to $[0,1]$. Then for $\sigma = (\{2\}, \{1,3\})$, by σp we mean the binary probability map $(p_2, p_1 + p_3)$.

DEFINITION 61. An experimental system with probabilities is given by an experimental system $D : E \to \textit{Dec } S$ with a map $P : E \to \textit{Prob } S$ that satisfies

Axiom 4 If e is an n-ary experiment, then Pe is an n-ary probability map.

Axiom 5 $P(\sigma e) = \sigma(Pe)$ for each experiment e and each σ of appropriate size.

Often elements of S are called pure states, or simply states, and $(Pe)_i(s)$ is called the probability of obtaining the i^{th} outcome of the experiment e when the system is in state s.

LEMMA 62. *If s is a state and $\sum_1^{\|e\|}(Pe)_i(s) = 0$ for some experiment e, then $\sum_1^{\|f\|}(Pf)_i(s) = 0$ for all experiments f.*

The simple proof of this result is found in [16, Lemma 4.4]. States as described in this lemma are called null states. They are interpreted as being impossible states of the system.

Observables

Position and momentum are terms used to discuss certain families of compatible experiments. These are customarily called observable quantities, and the particular manner in which numerical values are associated with an observable quantity is called its scaling. Here we define observable quantities and scalings for experimental systems.

DEFINITION 63. An observable quantity is a Boolean subalgebra B of the OMP $\mathcal{Q}(\mathcal{E})$ of questions of the experimental system.

Recall that for a Boolean algebra B, there is a compact Hausdorff space Z that has a basis of sets that are both closed and open (clopen) such that B is isomorphic to the clopen sets of Z. The space Z is called the Stone space of B [2]. The elements of Z are maximal proper filters of B, and the clopen subsets of Z are exactly the ones of the form $e^* = \{F \in Z | e \in F\}$, where $e \in B$. In addition to the topology on Z, there is a σ-algebra of subsets of Z generated by the clopen sets. Measures and measurable functions on Z are understood to be with respect to this σ-algebra.

DEFINITION 64. A scaling of an observable quantity B is a real random variable on the Stone space of B, or in other words, a measurable function from the Stone space of B to the extended reals $[-\infty, \infty]$.

We require the following results from [16, Section 5] to flesh out our treatment of observables, scalings and states.

PROPOSITION 65. *For each state s that is not null, the map $\psi_s : \mathcal{Q}(\mathcal{E}) \to [0, 1]$ defined by*

$$\psi_s(e) = (Pe)_1(s)$$

is a finitely additive state (in the sense of [36]) on the OMP $\mathcal{Q}(\mathcal{E})$*. Further, if B is an observable quantity, there is a unique probability measure μ_s on the Stone space of B satisfying*

$$\mu_s(e^*) = \psi_s(e)$$

for each $e \in B$.

We introduce some terminology that exhibits our intended interpretation.

DEFINITION 66. Suppose f is a scaling of an observable quantity B and s is a state that is not null.

1. For a Borel subset U of the reals, we call $\mu_s(f^{-1}U)$ the probability that a measurement of the observable B will yield a value in the Borel set U under the scaling f when the system is in state s.

2. We call $\int_Z f d\mu_s$ the expected value of the observable B under the scaling f when the system is in state s.

Finally, we develop a calculus of scalings as follows. For any measurable map φ on the reals, define $\varphi(f)$ to be the scaling $\varphi \circ f$.

An easy finite example is of use.

EXAMPLE 67. *Consider the observable quantity* $B = \{\text{FALSE}, \text{TRUE}, e, \text{NOT } e\}$ *where* FALSE, TRUE *are the bounds of* $\mathcal{Q}(\mathcal{E})$. *Note,* B *has two maximal proper filters,* $e\!\uparrow$ *and* NOT $e\!\uparrow$, *hence the Stone space* Z *of* B *is the two-element discrete space* $\{e\!\uparrow, \text{NOT } e\!\uparrow\}$. *For a state* s *that is not null, the measure* μ_s *is is given by* $\mu_s(\{e\!\uparrow\}) = \psi_s e = (Pe)_1(s)$, *the probability of obtaining a Yes outcome to e when the system is in state* s, *and* $\mu_s(\{\text{NOT } e\!\uparrow\})$ *is the probability of obtaining a No outcome. The scaling* $f(e\!\uparrow) = 1.2$ *and* $f(\text{NOT } e\!\uparrow) = 1.7$ *associates the value* 1.2 *to a Yes outcome and the value* 1.7 *to a No outcome. The expected value when the system is in state s is given by* $\int_Z f d\mu_s = 1.2\times$(*the probability of a Yes outcome*)$+$ $1.7 \times$ (*the probability of a No outcome*).

Of course, we should connect our treatment of observables and scalings to what is done in the standard treatment of quantum mechanics, where an observable quantity and its scaling are collectively given by a self-adjoint operator A on the underlying space $\mathcal{H}$ of the system. The key result will be the following.

PROPOSITION 68. *Let* φ *be a Boolean algebra homomorphism from the Borel subsets of the reals to an observable quantity* B. *Then the map* f *from the Stone space of* B *to the extended reals defined by*

$$f(F) = \inf\{\lambda \in \mathbb{R} | \varphi(-\infty, \lambda] \in F\}$$

is both continuous and measurable, so in particular is a scaling.

The standard Hilbert space model

We consider the standard quantum model based on a Hilbert space $\mathcal{H}$, place this in the context of an experimental system with probabilities, and consider observable quantities and scalings both from the perspective of the Hilbert space model, and from the perspective of the experimental system built from this Hilbert space.

DEFINITION 69. For a Hilbert space $\mathcal{H}$, define a set E of experiments, a set S of states, a mapping $D : E \to Dec\ S$ and a mapping $P : E \to Prob\ S$ as follows.

1. E is all finite sequences $(P_1, \ldots, P_n)$ of pairwise orthogonal projection operators that sum to the identity.

2. S is the underlying set of the Hilbert space $\mathcal{H}$.

3. $D(P_1, \ldots, P_n)$ the direct product decomposition of $\mathcal{H}$ as the product of the ranges of the projections $P_1, \ldots, P_n$.

4. $P(P_1, \ldots, P_n)_i(s) = \dfrac{\|P_i s\|^2}{\|s\|^2}$ for each $s \neq 0$.

Note E is a leveled set where the natural number associated to $(P_1, \ldots, P_n)$ is n, and that there is a natural action of $\mathcal{O}$ on E given by summing projection operators.

One easily obtains the following.

PROPOSITION 70. *For $\mathcal{H}$ a Hilbert space, $\mathcal{E}_{\mathcal{H}} = (E, S, D, P)$ is an experimental system with probabilities whose only null state is the zero vector of $\mathcal{H}$.*

We have seen the following result earlier in Example 34.

PROPOSITION 71. *For $\mathcal{H}$ a Hilbert space, the questions $\mathcal{Q}(\mathcal{E}_{\mathcal{H}})$ of the system $\mathcal{E}_{\mathcal{H}}$ is isomorphic to the* OML *of closed subspaces of $\mathcal{H}$.*

In the standard quantum model, it is assumed that to each observable there corresponds a self-adjoint operator on $\mathcal{H}$ (but not conversely!). If A is such a self-adjoint operator, then its spectral measure φ is a σ-complete homomorphism from the Borel subsets of the reals to a complete Boolean subalgebra B of $Proj\ \mathcal{H}$. Thus B is a Boolean subalgebra of $\mathcal{Q}(\mathcal{E})$, hence an observable quantity of our experimental system $\mathcal{E}$, and Proposition 68 shows that this homomorphism φ gives a scaling of the observable quantity B. So each observable of the standard quantum model gives a self-adjoint operator on $\mathcal{H}$, which in turn gives an observable quantity and scaling of the experimental system $\mathcal{E}$.

physical observables	$\subseteq$	self-adjoint operators on $\mathcal{H}$	$\subseteq$	observable quantities and scalings of $\mathcal{E}_{\mathcal{H}}$

None of the reverse implications will generally hold. Of course, what is crucial is that the standard approach to observables and the experimental system approach give the same results for those observables they do share (or at least for the ones that do correspond to actual physical observables).

THEOREM 72. *Suppose A is a self-adjoint operator on the Hilbert space $\mathcal{H}$ whose spectral measure is φ, and that B is the observable quantity and f the scaling of the experimental system $\mathcal{E}_{\mathcal{H}}$ given by A. Then for any state s that is not null, and any Borel subset U of the reals we have the following.*

1. *The probability that a measurement of the observable yields a result in U when the system is in state s is the same whether computed in the standard way using A and its spectral measure φ, or computed in the experimental system $\mathcal{E}_{\mathcal{H}}$ using the scaling f and the measure μ_s on the Stone space of B.*
2. *The expected value is the same whether computed in the standard way using A and its spectral measure φ, or computed in the experimental system $\mathcal{E}_{\mathcal{H}}$ using the scaling f and the measure μ_s on the Stone space of B.*

A more detailed account of this matter is given in [18], but at heart this is nothing more than a well-developed (but possibly unfashionable) part of the spectral theory of self-adjoint operators [21].

Observables and decompositions

We conclude this section by giving operational motivation for using Stone spaces and Boolean sheaves in the treatment of observable quantities and their scalings.

EXAMPLE 73. *The situation for a measurement of a finitary observable quantity, such as measurement of the spin of a particle that has three possibilities for its spin, is described in Example 67. One conducts a ternary experiment e that tests spin; then the questions built from e form an 8-element Boolean subalgebra B of $\mathcal{Q}(\mathcal{E})$; the Stone space Z of B has three elements that correspond to the outcomes of e; for a state s the measure μ_s on Z is a point charge that gives the probabilities of these outcomes when the system is in state s; and a scaling f is just a way to associate numerical values to these outcomes, such as calling them spin $-\frac{1}{2}, 0, \frac{1}{2}$.*

In the finitary case, the following is a dictionary for an observable quantity B, its Stone space Z, a scaling f and the measure μ_s for a state s.

B	$=$	All Yes-No questions built from a single n-ary experiment e.
Z	$=$	The discrete space whose points correspond to outcomes of e.
f	$=$	A map assigning numerical values to the outcomes of e.
μ_s	$=$	A point charge on Z giving probabilities of the outcomes in state s.

We move to the general case of observable quantities that are not finitary.

EXAMPLE 74. *For an observable quantity such as position, the situation is much different. Even in classical mechanics, the best one can do is test whether a particle lies in a given interval (we deal here with position in one-dimension for convenience). The idea of associating a single real number to position is an ideal* *representing a limiting process of a family of ever finer questions. These ideal questions can never be realized, only approximated.*

Suppose the observable quantity for position is the Boolean subalgebra B of the OMP $\mathcal{Q}(\mathcal{E})$ *of questions of the system. Then these ideal questions for position are maximally consistent families of questions in B. But these maximally consistent sets of questions are exactly the maximal proper filters of B, or in other words, the points of the Stone space of B. A scaling f assigns numerical values to these ideal questions, and for a state s the measure μ_s is a probability distribution on these ideal questions.*

These ideal questions can never be experimentally tested, only approximated. This is reflected in the topology of the Stone space Z by the fact that each ideal question has a neighborhood basis of clopen sets, and it is exactly these clopen sets that correspond to the questions that can be conducted. For such a clopen set K, the measure $\mu_s(K)$ is then the probability that when the system is in state s the question corresponding to K will yield a Yes answer.

In the general case, the following is a dictionary for an observable quantity B, its Stone space Z, a scaling f and the measure μ_s for a state s.

B	=	A Boolean algebra of compatible questions. Operationally the meaning of position is that it is the answer to a certain type of question, namely a question that asks whether a thing is in a certain interval.
Z	=	A topological space whose points are the maximally consistent sets of questions of B, or in other words, ideal questions of the system. The topology on Z has as a basis clopen sets that correspond to the questions which are used to approximate these ideal questions.
f	=	A map assigning numerical values to the ideal questions.
μ_s	=	A probability distribution on the ideal questions so that for any clopen set K corresponding to an actual question e, the measure of K is the probability of obtaining a Yes outcome to e when in state s.

This chapter centers around the idea that a question e will yield a decomposition $S \simeq S_e \times S_{e'}$ of the state space. How then do these ideal questions, built through a limiting process of actual questions, relate to decompositions? The answer is a well-known part of sheaf theory [38]. For an ideal question F, let S_F be the limit (in the precise categorical sense) of the directed family S_e where e ranges over F. Then the sheaf space for the continuously varying decomposition given by the Boolean sheaf for B (see Section 6) is built by topologizing the union of the sets S_F where F ranges over all points in the Stone space of B. Thus, not only are ideal questions for position built through limiting processes, so also is the decomposition of the state space.

PROPOSITION 75. *Suppose B is an observable quantity of an experimental system.*

1. *If B is finitary and given by an n-ary experiment e, then B induces an n-ary direct product decomposition $S \simeq S_1 \times \cdots \times S_n$.*
2. *In general, B induces a decomposition of S as the continuous members of the product $\prod_{F \in Z} S_F$ where $S_F = \lim_{e \in F} S_e$.*

So in the general case, not only are the points F of the indexing set Z constructed through a limiting process, but so also are the stalks S_F of the product.

A concluding example shows that in certain practical situations, this use of Stone spaces can allow more realistic treatments of measurements.

EXAMPLE 76. *Consider a measurement of the energy of a system whose energy can take the values λn^2 where λ is a constant and n ranges over the natural numbers. Surely a single experiment can determine only whether the energy takes on one of finitely many specified values. It does not seem possible to construct an experiment to determine whether the energy belongs to $\{\lambda n^2 | n \text{ is odd }\}$. The Boolean algebra of questions for such energy measurements will be isomorphic to the set of all finite subsets of the natural numbers and their complements. Its Stone space is the one-point compactification of the natural numbers, with this extra point being an ideal outcome ∞ of infinite energy. The clopen sets of this Stone space are exactly the ones that are either finite and do not contain ∞, or infinite and do contain ∞, corresponding exactly to our conception of the possible questions. For any state s, the measure μ_s will vanish on $\{\infty\}$, reflecting the fact that this ideal outcome is not possible.*

To conclude this section, note that the standard Hilbert space treatment of the experiment in the above example forces us to deal with questions such as the one asking if energy lies in $\{\lambda n^2 | n \text{ is odd }\}$. Likely this causes no harm, but we should be aware that in many respects this Hilbert space model is not such a tight fit.

8 FURTHER RESULTS AND OPEN PROBLEMS

Here we briefly describe some additional results on decompositions, and present a few open problems whose solution would advance the study of the subject.

Which OMPs arise as Fact $\mathcal{A}$?

One might hope that every OMP arises as *Fact* $\mathcal{A}$ for some structure $\mathcal{A}$, or at least that every OMP can be embedded into *Fact* A for some set A. Here an OMP-embedding is a one-one map that preserves orthocomplementation and finite orthogonal joins. The following shows this hope is not realized, but the situation may in fact be more interesting.

THEOREM 77. *There is a finite* OMP *that can't be embedded in Fact A for any set A.*

One such OMP is presented in [14] where it was further shown that this same OMP provided the first example of an OMP that cannot be embedded into an OML. In unpublished work, further methods of constructing OMPs not embeddable in *Fact A* are given, and again none of these can be embedded in an OML.

OPEN PROBLEM 78. For an OMP P are these equivalent?

1. P can be embedded into *Fact A* for some set A.
2. P can be embedded into an OML.

The following result from [17] points to some of the difficulties in this problem.

THEOREM 79. *There is a finite* OMP *that cannot be embedded into Fact A for any finite set A, but can be embedded into Fact A for an infinite set A.*

This result is established by finding a stateless OMP P that can be embedded into *Fact A* for A infinite, and noting that for every finite set A that *Fact A* has at least one state. One additional problem on this topic is prompted by the earlier result that each OMP *Fact A* is regular.

OPEN PROBLEM 80. Characterize those OMPs that can be embedded into a regular OMP. In particular, can every OMP be embedded into a regular OMP?

Fact A and BDec A in a more general setting

A relation algebra $(R, \circ, -1, \Delta)$ is a type of algebra modeled after the algebra $(Pow\ X \times X, \circ, -1, \Delta)$ of all binary relations on a set X. Relation algebras were introduced by Tarski [39] and have been extensively studied since. We define $x \in R$ to be an equivalence element if $\Delta \leq x = x^{-1} = x \circ x$, and call an ordered pair of equivalence elements (x, x') a factor pair if $x \wedge x' = \Delta$ and $x \circ x' = 1$. We then set $R^{(2)}$ to be the set of all factor pairs of R and define $\leq$ and $*$ on $R^{(2)}$ in a way obviously similar to the way these are defined on *Fact A* in Definition 18.

THEOREM 81. *For a relation algebra R, the structure* $(R^{(2)}, \leq, *)$ *is a regular* OMP.

This clearly generalizes the construction of *Fact A*. For another modification, consider the situation for *Fact* $\mathcal{V}$ where $\mathcal{V}$ is a vector space. The congruences of $\mathcal{V}$ correspond to subspaces of $\mathcal{V}$, and as all congruences of $\mathcal{V}$ permute, the elements of *Fact* $\mathcal{V}$ correspond to ordered pairs of complementary elements of the modular lattice of subspaces of $\mathcal{V}$. This can be extended as follows.

THEOREM 82. *For a bounded modular lattice M, let* $M^{(2)}$ *be the set of ordered pairs of complementary elements of M. Define* $(x, x')^* = (x', x)$ *and set* $(x, x') \leq (y, y')$ *if* $x \leq y$ *and* $y' \leq x'$*. Then* $(M^{(2)}, \leq, *)$ *is a regular* OMP.

As it stands, the OMPs arising as $M^{(2)}$ for some modular lattice M can be shown [14] to be a special case of those of the form $R^{(2)}$ for some relation algebra

R. However, Mushtari [32] independently discovered the above result in a more general setting — the lattice M needn't be modular, only M-symmetric and M^*-symmetric, provided one considers those complementary pairs that are modular and dual modular pairs.

OPEN PROBLEM 83. Compare the generality of the following constructions.

1. $Fact\ A$ for a set A.
2. $R^{(2)}$ for a relation algebra R.
3. $M^{(2)}$ for an M-symmetric and M^*-symmetric lattice M.

In particular, are there OMPs that can be embedded into one produced by one of these constructions and not another?

Looking at our definition of $BDec\ A$, one sees easily that it is an essentially categorical definition at heart. In [18] we define the notion of an honest category as one that has finite products and for certain ternary products $A \times B \times C$, the obvious diagram built from $A \times B \times C$, $A \times B$, $B \times C$, B is a pushout.

THEOREM 84. *For an object A in an honest category $\mathcal{C}$, the structure $BDec\ A$ is a generalization of an* OMP *known as an orthoalgebra.*

While this result gives the first steps toward developing a categorical treatment of decompositions, it seems highly likely that improvement is possible. In particular, if the following can be achieved, it would seem worthwhile.

OPEN PROBLEM 85. Develop a categorical treatment of $BDec\ A$ that includes sets, algebras, topological spaces, as well as the constructions $R^{(2)}$ and $M^{(2)}$ described above, and produces regular OMPs as a result.

Fact $\mathcal{A}$ for specific structures $\mathcal{A}$

Consider $Fact\ \mathcal{L}$ for a bounded lattice $\mathcal{L}$. It is easily seen that the decompositions of $\mathcal{L}$ correspond to central elements of $\mathcal{L}$, so $Fact\ \mathcal{L}$ is isomorphic to the center of $\mathcal{L}$ and is therefore Boolean. This obviously extends to any algebra with a lattice reduct, or more generally, to any algebra with a distributive congruence lattice. Similarly, for a ring $\mathcal{R}$ we have $Fact\ \mathcal{R}$ is isomorphic to the central idempotents of $\mathcal{R}$, and therefore is Boolean. Deep results for binary relational structures [29, Theorem 5.18] show that any connected poset $\mathcal{P}$ has the strict refinement property [29, pg. 312], and therefore $Fact\ \mathcal{P}$ is Boolean. So there are a good number of classes of structures where $Fact\ \mathcal{A}$ is Boolean, and for these structures the behaviour of $Fact\ \mathcal{A}$ is obviously well understood. However, there are many classes of structures, including many common ones, where $Fact\ \mathcal{A}$ need not be Boolean, and here here there is very little known about the structures $Fact\ \mathcal{A}$.

OPEN PROBLEM 86. Further develop the theory of $Fact\ \mathcal{A}$ as it applies to familiar structures such as groups, vector spaces, graphs, and topological spaces.

One type of structure with a non-trivial theory of decompositions has been studied in some detail. These are normed groups with operators whose basic theory is developed in [14]. These consist of groups $\mathcal{G}$, perhaps with additional operators in the sense of van der Waerden [40], with a map $\|\cdot\| : \mathcal{G} \to [0,\infty)$ that satisfies (i) $\|x\| = 0$ if, and only if, $x = 0$, (ii) $\|x\| = \|-x\|$, and (iii) $\|x+y\| \leq \|x\| + \|y\|$. We have written the group structure additively, but these groups need not be abelian. These structures admit an algebraic and topological theory similar to that of Hilbert spaces, and have ties to experimental systems with probabilities.

OPEN PROBLEM 87. Develop further the theory of decompositions of normed groups with operators.

Experimental systems

There is likely much more that can be done in the development of the notion of an experimental system with probabilities, especially if one begins restricting somewhat the types of examples we have in mind.

OPEN PROBLEM 88. Develop further the theory of experimental systems with probabilities. In particular, examine the possibilities for a dynamics, and a treatment for compound systems.

9 CONCLUSIONS

Perhaps it is poor form, but I will present a few of my own opinions. First, OMPs are a generalization of the structures *Fact A* much the way that modular lattices are generalizations of lattices of permuting equivalence relations. To me, this connection to such a basic principal as direct product decompositions goes far in explaining why the orthomodular law leads to such interesting mathematics.

Second, I believe decompositions are a primitive notion, and a development of their properties is overdue.

Finally, I seem to periodically change my opinion whether orthomodularity has anything substantial to do with quantum mechanics. But I have grown to believe that if such a connection exists, it is due to an underlying connection between direct product decompositions and quantum mechanics.

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