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A REGULAR COMPLETION FOR THE VARIETY GENERATED BY THE THREE-ELEMENT HEYTING ALGEBRA

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Abstract. We show that the variety generated by the three-element Heyting algebra admits a meet dense, regular completion even though it is not closed under MacNeille completions.

1. Introduction

We recall that an embedding of one ordered structure into another is called a regular embedding if it preserves all existing joins and meets, and a meet dense embedding if every element of the co-domain is a meet of elements of the image. It is the purpose of this note to prove the following.

Main Theorem. *Every algebra in the variety $V(\mathbf{3})$ generated by the three-element Heyting algebra can be embedded into a complete algebra in $V(\mathbf{3})$ via an embedding that is both regular and meet dense.*

Our primary interest in this result is that it provides an example of a variety that admits a regular completion, but is not closed under MacNeille completions. While there must surely be other examples, this one involves a particularly simple variety. This result may also be of interest in the study of Heyting algebras, and it yields the completeness with respect to algebraic semantics [8] of the predicate logic associated with the three-element Heyting algebra.

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2. Preliminaries on Stone algebras

A Stone algebra is a pseudocomplemented distributive lattice L that satisfies $x^* \vee x^{**} = 1$. An element $x \in L$ is central if $x = x^{**}$ and dense if $x^* = 0$. The central elements form a subalgebra of L called the center, and the dense elements are a filter of L. The standard representation $x = x^{**} \wedge (x \vee x^*)$ is the unique representation of x as the meet of a central element c and a dense element d with $c^* \leq d$. Further, for b, c central and d, e dense, we have $b \wedge d \leq c \wedge e$ if, and only if, $b \leq c$ and $d \leq c^* \vee e$. The reader should consult [1, 4, 5] for background.

Definition 1. A dual generalized Boolean algebra is a distributive lattice D with top element 1 such that each interval $[d, 1]$ in D is complemented.

For convenience, we include a proof of the following known results.

Proposition 2.1. *The members of $V(\mathbf{3})$ are exactly the Stone algebras whose dense sets are dual generalized Boolean algebras. Further, the Stone algebra homomorphisms between members of $V(\mathbf{3})$ are exactly the Heyting algebra homomorphisms.*

Proof. Let $A \in V(\mathbf{3})$ with dense set D. As $\mathbf{3}$ satisfies $x^* \vee x^{**} = 1$, so does A, so A is a Stone algebra. For f in the interval $[d, 1]$ of D set $g = f \to d$. By basic properties of Heyting algebras g belongs to the interval $[d, 1]$ and $f \wedge g = d$. As $\mathbf{3}$ satisfies the identity $(x \vee x^*) \vee [(x \vee x^*) \to (y \vee y^*)] = 1$, so does A, and as f, d are dense it follows that $f \vee g = f \vee (f \to d) = 1$. So D is a dual generalized Boolean algebra.

Conversely, suppose A is a Stone algebra whose dense set D is a dual generalized Boolean algebra. For $x, y \in A$ with $y \leq x$, let x^y be the complement of $x \vee y^*$ in the interval $[y \vee y^*, 1]$ of D. In [1, pg. 167] it is shown that $x^y \wedge (x^* \vee y^{**})$ is the pseudocomplement of x in $[y, 1]$, hence is $x \to y$. It follows that A is a Heyting algebra. To see A belongs to $V(\mathbf{3})$, suppose S is a subdirectly irreducible Heyting algebra and φ is a Heyting algebra homomorphism from A onto S. As S is a subdirectly irreducible Heyting algebra, the top element 1 of S is join irreducible, and as $x^* \vee x^{**}$ holds in A, it holds in S. So for each $s \in S$ we have that either $s^* = 1$ or $s^{**} = 1$. Suppose s is a non-zero element of S and $a \in A$ is such that $\varphi(a) = s$. As $a = a^{**} \wedge (a \vee a^*)$, we have $s = \varphi(a)^{**} \wedge \varphi(a \vee a^*)$. As s is non-zero and $s \leq \varphi(a)^{**}$ we have $\varphi(a)^{**} = 1$, so $s = \varphi(a \vee a^*)$. So each non-zero element of S is the image of a dense element of A. Therefore the non-zero elements of S are a dual generalized Boolean algebra. But the top element of S is join irreducible, and it follows that S has at most two non-zero elements. Thus S is the one, two or three-element Heyting algebra.

For the further remark, suppose $\varphi : A \to B$ is a Stone algebra homomorphism between members of $V(\mathbf{3})$. We must show φ preserves implication. It is sufficient to show this in the case that $y \leq x \in A$. From above, $x \to y = x^y \wedge (x^* \vee y^{**})$. As φ preserves the lattice operations and the pseudocomplement, to show $\varphi(x \to y)$ equals $\varphi(x) \to \varphi(y)$, it is sufficient to show $\varphi(x^y)$ is equal to $\varphi(x)^{\varphi(y)}$. But this follows as x^y is the complement of x in the interval $[y, 1]$, so $\varphi(x^y)$ is the complement of $\varphi(x)$ in the interval $[\varphi(y), 1]$. $\square$

The following result is similar in nature to [5, Theorem 5, pg. 900].

Lemma 2.2. *For a Stone algebra L with center C, dense set D, and $S \subseteq L$, the meet of S in L exists and has standard representation $c \wedge d$ if, and only if, both of*

(1) $c = \max\{b \in C | \text{there exists } e \in D \text{ with } b \wedge e \text{ a lower bound of } S\}$.
(2) $d = \bigwedge_D \{c^* \vee s | s \in S\}$.

We next consider when an embedding of Stone algebras is meet dense and regular.

Lemma 2.3. *Suppose $L \leq L_1$ are Stone algebras with centers $C \leq C_1$ and dense sets $D \leq D_1$ such that*

(1) *$C \leq C_1$ is meet dense.*
(2) *$D \leq D_1$ is meet dense and meet regular.*
(3) *Beneath each non-zero element of L_1 is a non-zero element of L.*

Then $L \leq L_1$ is regular and meet dense.

Proof. Let $x \in L_1$. Then $x = c \wedge d$ for some $c \in C_1$ and $d \in D_1$. As $C \leq C_1$ and $D \leq D_1$ are meet dense $c = \bigwedge_{C_1} S$ for some $S \subseteq C$ and $d = \bigwedge_{D_1} T$ for some $T \subseteq D$. The center is a meet regular sublattice of any Stone algebra, and the dense set is a regular sublattice of any Stone algebra, so $c = \bigwedge_{L_1} S$ and $d = \bigwedge_{L_1} T$. It follows that $c \wedge d = \bigwedge_{L_1} (S \cup T)$, showing that $L \leq L_1$ is meet dense. As any meet dense embedding is join regular, $L \leq L_1$ is join regular.

It remains to show $L \leq L_1$ is meet regular. Suppose $S \subseteq L$ and the meet of S in L exists and has standard representation $c \wedge d$. Set

$$\mathcal{X} = \{b \in C | \text{there exists } e \in D \text{ with } b \wedge e \text{ a lower bound of } S\},$$

$$\mathcal{Y} = \{\beta \in C_1 | \text{there exists } \epsilon \in D_1 \text{ with } \beta \wedge \epsilon \text{ a lower bound of } S\}.$$

By Lemma 2.2 $c = \max \mathcal{X}$ and $d = \bigwedge_D \{c^* \vee s | s \in S\}$. Surely $\mathcal{X} \subseteq \mathcal{Y}$ and as D_1 is a filter of L_1 one sees that $\mathcal{Y}$ is an ideal of C_1. Suppose $\beta \in \mathcal{Y}$ with $c \leq \beta$ and let $\epsilon \in D^*$ be such that $\beta \wedge \epsilon$ is a lower bound of S. If $c < \beta$, then $(\beta \wedge c^*) \wedge \epsilon$ is a non-zero element of L_1. So there is a non-zero element $b \wedge e$ in L lying beneath it. Then $0 < b \leq c^*$. But $b \wedge e$ lies beneath the lower bound $\beta \wedge \epsilon$ of S, so $b \wedge e$ is a lower bound of S, giving $b \in \mathcal{X}$. This contradicts the fact that $c = \max \mathcal{X}$, and it follows that c is the maximum of $\mathcal{Y}$ as well. Then as $d = \bigwedge_D \{c^* \vee s | s \in S\}$ and $D \leq D_1$ is meet regular, we have $d = \bigwedge_{D_1} \{c^* \vee s | s \in S\}$. It follows from Lemma 2.2 that S has a meet in L_1 and that $c \wedge d$ is the standard representation of this meet. □

We recall a few facts about the triple construction of Chen and Grätzer [4, 5].

Definition 2. A triple (C, D, φ) is a Boolean algebra C, a distributive lattice D with largest element, and a bound preserving lattice homomorphism φ from C to the filter lattice of D.

For a Stone algebra L with center C and dense set D we obtain a triple (C, D, φ) by setting $\varphi(c)$ to be the filter of dense elements lying above c^*. The following result of Chen and Grätzer [4] shows all triples arise this way.

Theorem 2.4. *For a given triple, there is up to isomorphism a unique Stone algebra whose triple is the given one. We call this the Stone algebra for the triple.*

Chen and Grätzer also described homomorphisms in terms of triples. We need the following consequence of their results.

Lemma 2.5. *Suppose (C, D, φ) and (C_1, D_1, φ_1) are triples such that*

(1) *C is a subalgebra of C_1,*
(2) *D is a meet-dense subalgebra of D_1 having the same top element,*
(3) *$\varphi_1(c) \cap D = \varphi(c)$ for each $c \in C$.*

Then the algebra for (C, D, φ) is a subalgebra of that for (C_1, D_1, φ_1).

Proof. This will follow from [4, Theorem 1, pg. 891] if we can show that for each $c \in C$ and $d \in D$ that the least element e in $\varphi(c)$ that lies above d is also the least element in $\varphi_1(c)$ that lies above d. As $\varphi(c) \subseteq \varphi_1(c)$ we have that e is an element of $\varphi_1(c)$ above d. Suppose δ is an element of $\varphi_1(c)$ above d and that f is an upper bound of δ with $f \in D$. Then as $\varphi_1(c)$ is a filter, f belongs to $\varphi_1(c) \cap D = \varphi(c)$, and f lies above d, so $e \leq f$. As $D \leq D_1$ is meet dense and all upper bounds of δ in D lie above e we have $e \leq \delta$. So e is the least element of $\varphi_1(c)$ above d. □

3. Extending a triple

In the following we use the fact that for any distributive lattice D, there is a Boolean algebra called the free Boolean extension of D [1, pg. 97] that contains D as a sublattice, is generated by D, and preserves any existing bounds in D.

Proposition 3.1. *For (C, D, φ) a triple and M the MacNeille completion of the free Boolean extension of D the map*

$$\tilde{\varphi}(c) = \bigwedge\nolimits_M \varphi(c^*)$$

is a homomorphism from C to M with $\varphi(c) = \tilde{\varphi}(c^*)\!\uparrow \cap D$ for each $c \in C$.

Proof. As φ is a bounded homomorphism from C to the filter lattice of D we have $\varphi(1^*) = \varphi(0)$ is the smallest filter $\{1\}$ of D, and $\varphi(0^*) = \varphi(1)$ is the largest filter D of D. Also, $\varphi(b^* \wedge c^*) = \varphi(b^*) \wedge \varphi(c^*)$ and as the meet in the filter lattice is given by intersection, $\varphi(b^* \wedge c^*)$ equals $\{d \vee e | d \in \varphi(b^*), e \in \varphi(c^*)\}$. Similarly $\varphi(b^* \vee c^*) = \varphi(b^*) \vee \varphi(c^*)$, and as the join in the filter lattice is given by the filter generated by the union, $\varphi(b^* \vee c^*)$ equals $\{d \wedge e | d \in \varphi(b^*), e \in \varphi(c^*)\}$.

We first show that $\tilde{\varphi}$ preserves bounds. Note $\tilde{\varphi}(1) = \bigwedge_M \varphi(1^*) = \bigwedge_M \{1\} = 1$. Also, as the meet of D in its free Boolean extension is zero, and the MacNeille completion preserves existing joins and meets, $\tilde{\varphi}(0) = \bigwedge_M \varphi(0^*) = \bigwedge_M D = 0$.

Having established that $\tilde{\varphi}$ is a bound preserving map between Boolean algebras, to show it is a Boolean homomorphism it suffices to show that it preserves finite joins and meets. Suppose $b, c \in C$. Then

$$\begin{aligned}
\tilde{\varphi}(b \wedge c) &= \bigwedge\nolimits_M \varphi(b^* \vee c^*) \\
&= \bigwedge\nolimits_M \{d \wedge e | d \in \varphi(b^*), e \in \varphi(c^*)\} \\
&= \bigwedge\nolimits_M \{d | d \in \varphi(b^*)\} \wedge \bigwedge\nolimits_M \{e | e \in \varphi(c^*)\} \\
&= \tilde{\varphi}(b) \wedge \tilde{\varphi}(c).
\end{aligned}$$

And to show that $\tilde{\varphi}$ preserves finite joins, we make a similar calculation and use the complete distributivity of the complete Boolean algebra M.

$$\begin{aligned}
\tilde{\varphi}(b \vee c) &= \bigwedge\nolimits_M \varphi(b^* \wedge c^*) \\
&= \bigwedge\nolimits_M \{d \vee e | d \in \varphi(b^*), e \in \varphi(c^*)\} \\
&= \bigwedge\nolimits_M \{d | d \in \varphi(b^*)\} \vee \bigwedge\nolimits_M \{e | e \in \varphi(c^*)\} \\
&= \tilde{\varphi}(b) \vee \tilde{\varphi}(c).
\end{aligned}$$

It remains to show that $\varphi(c) = \tilde{\varphi}(c^*)\!\uparrow \cap D$ for each $c \in C$. Suppose $a \in \varphi(c)$. Then $\tilde{\varphi}(c^*) = \bigwedge_M \varphi(c) \leq a$, so a belongs to $\tilde{\varphi}(c^*)\!\uparrow \cap D$, giving $\varphi(c) \subseteq \tilde{\varphi}(c^*)\!\uparrow \cap D$. For the converse, let $a \in \tilde{\varphi}(c^*) \cap D$. So a belongs to D and $\bigwedge_M \varphi(c) \leq a$. Note $\varphi(c)$ and $\varphi(c^*)$ are complements in the filter lattice, so the meet of these filters is the smallest filter $\{1\}$. So for any $d \in \varphi(c)$ and $e \in \varphi(c^*)$ we have $d \vee e = 1$, so in M we have $e^* \leq d$. Thus for each $e \in \varphi(c^*)$, we have e^* is a lower bound in M of $\varphi(c)$, hence $e^* \leq \bigwedge_M \varphi(c) \leq a$. As $\varphi(c)$ and $\varphi(c^*)$ are complements in the filter lattice, their join in the filter lattice is the largest filter D, and as $a \in D$, it follows that $a = d \wedge e$ for some $d \in \varphi(c)$ and $e \in \varphi(c^*)$. Then as $e \in \varphi(c^*)$, we have that $e^* \le a \le e$, and this gives $e = 1$. So $a = d \wedge 1 = d$, and as $d \in \varphi(c)$ we have $a \in \varphi(c)$. □

Proposition 3.2. *For C a Boolean algebra, B a bounded distributive lattice, E a filter of B, and $\psi : C \to B$ a bound preserving lattice homomorphism, the map*

$$\hat{\psi}(c) = \psi(c^*)\uparrow \cap E$$

is a bounded lattice homomorphism from C to the filter lattice of E.

Proof. Note that $\hat{\psi}(0) = \psi(1) \uparrow \cap E = \{1\}$ is the smallest filter of E and $\hat{\psi}(1) = \psi(0) \uparrow \cap E = E$ is the largest filter of E, so $\hat{\psi}$ preserves bounds. For $b, c \in C$ we have $\hat{\psi}(b \wedge c) = (\psi(b^*) \vee \psi(c^*))\uparrow \cap E = (\psi(b^*)\uparrow \cap E) \cap (\psi(c^*)\uparrow \cap E)$, so $\hat{\psi}$ preserves finite meets. For finite joins, note $\hat{\psi}(b \vee c) = (\psi(b^*) \wedge \psi(c^*))\uparrow \cap E$ is clearly a filter of E containing both $\psi(b^*)\uparrow \cap E$ and $\psi(c^*)\uparrow \cap E$. We must show it is the least such filter. Suppose e belongs to $(\psi(b^*) \wedge \psi(c^*))\uparrow \cap E$. Then as $\psi(b^*) \wedge \psi(c^*) \le e$ we have $e = (\psi(b^*) \vee e) \wedge (\psi(c^*) \vee e)$, and as E is a filter both $\psi(b^*) \vee e$ and $\psi(c^*) \vee e$ belong to e. Thus e is the meet of an element of $\varphi(b^*)\uparrow \cap E$ and one of $\psi(c^*)\uparrow \cap E$, hence e belongs to any filter containing these. □

Lemma 3.3. *Let (C, D, φ) be a triple and M be the MacNeille completion of the free Boolean extension of D. Suppose further that*

(1) *B is a Boolean algebra containing C as a subalgebra.*
(2) *E is a filter of M containing D.*

Then there is a triple (B, E, φ_1) with $\varphi(c) = \varphi_1(c) \cap D$ for each $c \in C$.

Proof. By Proposition 3.1 there is a Boolean algebra homomorphism $\tilde{\varphi} : C \to M$ with $\varphi(c) = \tilde{\varphi}(c^*) \uparrow \cap D$ for each $c \in C$. As M is a complete Boolean algebra, hence injective [1, pg. 113], and C is a subalgebra of B, there is a homomorphism $\gamma : B \to M$ with $\gamma(c) = \tilde{\varphi}(c)$ for each $c \in C$. As E is a filter in M, Proposition 3.2 shows that the map φ_1 from B to the filter lattice of E defined by $\varphi_1(b) = \gamma(b^*)\uparrow \cap E$ is a bounded lattice homomorphism from B to the filter lattice of E. So (B, E, φ_1) is a triple, and for $c \in C$ we have $\varphi(c) = \tilde{\varphi}(c^*)\uparrow \cap D = \gamma(c^*)\uparrow \cap D = \gamma(c^*)\uparrow \cap E \cap D = \varphi_1(c) \cap D$. □

We recall that a lattice is called conditionally complete if every non-empty subset that has an upper bound has a least upper bound, and every non-empty subset that has a lower bound has a greatest lower bound.

Theorem 3.4. *Each algebra in $V(\mathbf{3})$ can be regularly and meet densely embedded into an algebra in $V(\mathbf{3})$ with a complete center and a conditionally complete dense set.*

PROOF. Suppose L belongs to $V(\mathbf{3})$ has triple (C, D, φ). Let B be the MacNeille completion of C and note that $C \leq B$ is meet dense. As L belongs to $V(\mathbf{3})$ its dense set D is a dual generalized Boolean algebra, and therefore a filter in its free Boolean extension [1, pg. 101]. Let E be the filter of the MacNeille completion M of the free Boolean extension of D generated by D. Then E is conditionally complete and E contains D. As the free Boolean extension of D is join and meet dense in M and D is a filter in its free Boolean extension, D is join and meet dense in E, hence D is meet dense and meet regular in E.

By Lemma 3.3 there is a triple (B, E, φ_1) with $\varphi(c) = \varphi_1(c) \cap D$ for each $c \in C$. Let L_1 be the algebra for this triple. Then as D is meet dense in E, Lemma 2.5 gives that L is a subalgebra of L_1. As C is meet dense in its MacNeille completion B and D is meet dense and meet regular in E, our result will follow from Lemma 2.3 if we can show that beneath each non-zero element of L_1 lies a non-zero element of L. Suppose $b \wedge e$ is a non-zero element of L_1. Then $b \neq 0$, and as B is the MacNeille completion of C there is a non-zero element c of C with $c \leq b$. Also, from the definition of E, there is some $d \in D$ with $d \leq e$. Then $c \wedge d$ is the required non-zero element of L beneath $b \wedge d$. □

4. The Main Theorem

We are to show that for an algebra L in $V(\mathbf{3})$, with center C and dense set D, that L can be regularly and meet densely embedded into a complete algebra in $V(\mathbf{3})$. Theorem 3.4 shows that we may assume, without loss of generality, that C is complete and D is conditionally complete. Throughout this section we shall make these assumptions, and we further let M be the MacNeille completion of the free Boolean extension of D. We begin with the following simple observation.

Lemma 4.1. *For $S \subseteq D$, $c \in C$ and $d \in D$ these are equivalent.*

(1) *$c \wedge d$ is a lower bound of S.*
(2) *d is a lower bound of $\{c^* \vee s | s \in S\}$.*

Recall that we have assumed D is conditionally complete, so any subset of D that has a lower bound in D has a greatest lower bound in D. We make use of this in the second part of the following definition.

Definition 3. For $S \subseteq D$ define

(1) $\mathcal{X}(S) = \{c \in C | \text{there exists } d \in D \text{ with } c \wedge d \text{ a lower bound of } S\}$.
(2) $d(S, c) = \bigwedge_D \{c^* \vee s | s \in S\}$ for each $c \in \mathcal{X}(S)$.

Remark. For those familiar with Pierce sheaves or Boolean products [3, 6] the above definitions can be profitably considered in these terms. The stalks of the

Pierce sheaf of a Stone algebra L are directly irreducible, hence have all non-zero elements dense. For S a subset of the dense elements of L, each member of S gives a global section that is never zero. The set $\mathcal{X}(S)$ is an ideal of the center of L, hence gives an open subset of the Stone space of L. This open set is the largest one on which S locally has a dense lower bound. For each $c \in \mathcal{X}(S)$ the dense element $d(S,c)$ then is the choice function taking value 1 outside of the clopen set for c, and taking the greatest lower bound of D as its value on c.

Lemma 4.2. *If $S, T \subseteq D$ and $\bigwedge_M S = \bigwedge_M T$ then*

(1) $\mathcal{X}(S) = \mathcal{X}(T)$.
(2) $d(S,c) = d(T,c)$ *for each* $c \in \mathcal{X}(S)$.

Proof. It is enough to show that if $c \in \mathcal{X}(S)$, then $c \in \mathcal{X}(T)$, as symmetry then gives $\mathcal{X}(S) = \mathcal{X}(T)$; and that $d(S,c) \leq d(T,c)$, as symmetry then gives equality. So it is sufficient to show that if $c \in \mathcal{X}(S)$ then $d(S,c) \leq c^* \vee t$ for each $t \in T$ as this shows $c \wedge d(S,c)$ is a lower bound of T, hence $c \in \mathcal{X}(T)$, and that $d(S,c) \leq d(T,c)$.

The key point is that D is a sublattice of both L and M, so for two elements of x, y of D we may write $x \leq_L y$, or $x \leq_D y$ or $x \leq_M y$, and that all these mean the same. Similar comments hold for $x \vee_L y$, $x \vee_D y$ and $x \vee_M y$.

Suppose $c \in \mathcal{X}(S)$ and $t \in T$. We must show $d(S,c) \leq c^* \vee_L t$, or equivalently, that $z \leq c^* \vee_L t$ where $z = (c^* \vee_L t) \vee_D d(S,c)$. Note that for each $s \in S$ we have $d(S,c) \leq_D c^* \vee_L s$, so $z \leq_M (c^* \vee_L t) \vee_D (c^* \vee_L s) = (c^* \vee_L t) \vee_D s$. It then follows that $z \leq_M \bigwedge_M \{(c^* \vee_L t) \vee_M s | s \in S\}$, and using the complete distributivity of M we have $z \leq_M (c^* \vee_L t) \vee_M \bigwedge_M S$. As $\bigwedge_M S = \bigwedge_M T$ we have $\bigwedge_M S \leq_M t \leq_M (c^* \vee_L t)$. So $z \leq_M (c^* \vee_L t)$ as required. □

Lemma 4.3. *D is meet dense in M.*

Proof. Let $x \in M$. As noted earlier $0 = \bigwedge_M D$, so $x = x \vee \bigwedge_M D$ and using complete distributivity, $x = \bigwedge_M \{x \vee d | d \in D\}$. But D is a filter in its free Boolean extension and each element of M is a meet of elements of the free Boolean extension, therefore $x \vee d$ is a meet of elements of D for each $d \in D$. □

Definition 4. Let $x \in M$ and suppose S is a subset of D with $x = \bigwedge_M S$. Define

(1) $\mathcal{X}(x) = \mathcal{X}(S)$.
(2) $d(x,c) = d(S,c)$ for each $c \in \mathcal{X}(x)$.

Note that Lemmata 4.2 and 4.3 ensure this is a proper definition.

To construct the completion we desire we use again the results of the previous section on extending triples. We are happy with the center of our Stone algebra, but must modify the dense set. To do so, we require a filter of M to build a new triple.

Definition 5. Define $E = \{x \in M | \bigvee_C \mathcal{X}(x) = 1\}$.

Lemma 4.4. *E is a filter of M that contains D.*

Proof. Let $x, y \in M$ and set $S = \{s \in D | x \leq s\}$ and $T = \{t \in D | y \leq t\}$. Then $x = \bigwedge_M S$ and $y = \bigwedge_M T$ so $\mathcal{X}(x) = \mathcal{X}(S)$ and $\mathcal{X}(y) = \mathcal{X}(T)$. Suppose $x \in E$ and $x \leq y$. As $x \leq y$ we have $T \subseteq S$, and it follows that $\mathcal{X}(S) \subseteq \mathcal{X}(T)$. As $\bigvee_C \mathcal{X}(S) = 1$ we then have $\bigvee_M \mathcal{X}(y) = 1$, so $y \in E$. Suppose $x, y \in E$. Then as $\bigwedge_M (S \cup T) = x \wedge y$, we have $\mathcal{X}(x \wedge y) = \mathcal{X}(S \cup T)$. Suppose $b \in \mathcal{X}(S)$ and $c \in \mathcal{X}(T)$. Then there are $d, e \in D$ with $b \wedge d$ a lower bound of S and $c \wedge e$ a lower bound of T. Then $(b \wedge c) \wedge (d \wedge e)$ is a lower bound of $S \cup T$, so $b \wedge c \in \mathcal{X}(S \cup T)$. It follows that $\mathcal{X}(x \wedge y)$ contains $\{b \wedge c | b \in \mathcal{X}(x), c \in \mathcal{X}(y)\}$. Using complete distributivity, we have $1 = \bigvee_M \mathcal{X}(x) \wedge \bigvee_M \mathcal{X}(y) = \bigvee_M \{b \wedge c | b \in \mathcal{X}(x), c \in \mathcal{X}(y)\} \leq \bigvee_M \mathcal{X}(x \wedge y)$. Thus $x \wedge y \in E$. This shows E is a filter of M. To see that E contains D, suppose $d \in D$. Then as $1 \wedge d \leq d$ we have $1 \in \mathcal{X}(d)$, so $\bigvee_M \mathcal{X}(d) = 1$, giving $d \in E$. □

Theorem 4.5. *If $L \in V(\mathbf{3})$ has a complete center and conditionally complete dense set, L can be regularly and meet densely embedded into a complete algebra in $V(\mathbf{3})$.*

Proof. Suppose (C, D, φ) is the triple for L and let E be the filter in the MacNeille completion M of the free Boolean extension of D constructed in Definition 5 above. By Lemma 3.3 there is a triple (C, E, φ_1) with $\varphi(c) = \varphi_1(c) \cap D$ for each $c \in C$. Let L_1 be the algebra for this triple.

Surely C is meet dense in itself, and by its construction D is meet dense in E. As D is a generalized Boolean algebra, it is a filter in its free Boolean extension, hence is a regular sublattice of its free Boolean extension, and therefore is also a regular sublattice of M, and therefore of E as well. So if we can show that beneath each non-zero element of L_1 there is a non-zero element of L, it will follow from Lemma 2.3 that $L \leq L_1$ is regular and meet dense.

Claim. If $x \in E$ and $c \in \mathcal{X}(x)$, then $c^* \vee_{L_1} x = d(x, c)$.

By the definition of E we have $x = \bigwedge_M S$ for $S = \{s \in D | x \leq s\}$, hence $x = \bigwedge_E S$ and as the dense elements E are a regular sublattice of L_1 we have $x = \bigwedge_{L_1} S$. By the definition of $d(x, c)$ we have $d(x, c) = \bigwedge_D \{c^* \vee s | s \in S\}$ and as D is a regular sublattice of E and hence of L_1 we have $d(x, c) = \bigwedge_{L_1} \{c^* \vee s | s \in S\}$. As c^* is central we have $c^* \vee \bigwedge_{L_1} S = \bigwedge_{L_1} \{c^* \vee s | s \in S\}$ as this is true in any bounded distributive lattice. Thus $c^* \vee_{L_1} x = d(x, c)$.

Suppose $b \wedge x$ is a non-zero element of L_1 where $b \in C$ and $x \in E$. As b is non-zero, $\mathcal{X}(x)$ is an ideal, and as $\bigvee_C \mathcal{X}(x) = 1$, there is some non-zero c in $\mathcal{X}(x)$ with $c \leq b$. The above claim shows $c^* \vee_{L_1} x = d(x,c)$, hence $c \wedge d(x,c) = c \wedge x \leq b \wedge x$. But $c \wedge d(x,c)$ is an element of L and is non-zero as c is non-zero. We have shown that $L \leq L_1$ is meet dense and regular.

It remains to show L_1 is complete. It suffices to show each subset of L_1 has a meet, and as each element is the meet of a central element and a dense element, it suffices to show each subset of the center and each subset of the dense elements has a meet in L_1. As the center C is complete and the center of any Stone algebra is a meet regular sublattice, every subset of the center has a meet in L_1. We need only show that every subset of E has a meet in L_1, and as D is meet dense in E, it is enough to show each subset of D has a meet in L_1. Suppose $S \subseteq D$.

Claim. $\bigwedge_{L_1} S = c \wedge x$ where $c = \bigvee_C \mathcal{X}(S)$ and $x = \bigwedge_M \{c^* \vee s | s \in S\}$.

Set $T = \{c^* \vee s | s \in S\}$. Then $c^* \in \mathcal{X}(T)$ and $\mathcal{X}(S) \subseteq \mathcal{X}(T)$, so $\bigvee_C \mathcal{X}(T) = 1$. Then as $x = \bigwedge_M T$ we have $x \in E$. Since $x \leq c^* \vee s$ for each $s \in S$, we have $c \wedge x \leq s$ for each $s \in S$, so $c \wedge x$ is a lower bound of S in L_1. To show $c \wedge x$ is the greatest lower bound, it is sufficient to show there is no strictly greater lower bound. Suppose $b \wedge y$ is a lower bound of S where $b \in C$ and $y \in E$, and that $c \wedge x \leq b \wedge y$. We must show $c \wedge x = b \wedge y$.

We first show $b = c$. If not, then as $c \wedge x \leq b \wedge y$ we have $c \leq b$, so $c < b$, giving $b \wedge c^* \neq 0$. As $\mathcal{X}(y)$ is an ideal with $\bigvee_C \mathcal{X}(y) = 1$ there is some non-zero a in $\mathcal{X}(y)$ with $a \leq b \wedge c^*$. By the previous claim $a^* \vee y = d(y,a)$, so $a \wedge d(y,a) = a \wedge y \leq b \wedge y$ giving $a \wedge y$ is a lower bound of S. This yields $a \in \mathcal{X}(S)$, so $a \leq \bigvee_C \mathcal{X}(S) = c$, a contradiction. So $b = c$, hence $b \wedge y = c \wedge y$, and as $b \wedge y$ is a lower bound of S we have $c \wedge y \leq s$ for each $s \in S$. Then $y \leq c^* \vee s$ for each $s \in S$. But $y \in E$ and $x = \bigwedge_E \{c^* \vee s | s \in S\}$, hence $y \leq x$. □

Main Theorem. *Every algebra in the variety $V(\mathbf{3})$ generated by the three-element Heyting algebra can be embedded into a complete algebra in $V(\mathbf{3})$ via an embedding that is both regular and meet-dense.*

Proof. Theorem 3.4 shows that every algebra in $V(\mathbf{3})$ can be regularly and meet densely embedded into an algebra in $V(\mathbf{3})$ that has a complete center and a conditionally complete dense set, and Theorem 4.5 shows that every algebra in $V(\mathbf{3})$ that has a complete center and conditionally complete dense set can be regularly and meet densely embedded into a complete algebra in $V(\mathbf{3})$. □

5. Concluding remarks

One cannot be too ambitious in generalizing the above result as the following remark shows that variety of all Stone algebras admits no regular completion.

Remark. It is well known [1, pg. 233] that there is a bounded distributive lattice D that cannot be regularly embedded into any complete distributive lattice. Taking the Stone algebra L for the triple $(\mathbf{2}, D, \varphi)$ where $\mathbf{2}$ is the two-element Boolean algebra and φ is the obvious bound preserving map into the filter lattice of D, the dense set for L is D. Any regular embedding of L into a complete Stone algebra would yield a regular completion of D, an impossibility.

One possible generalization of our results is described below.

Remark. A key step in our proof is that an algebra in $V(\mathbf{3})$ is a Stone algebra whose dense set D is a dual generalized Boolean algebra. This means that every interval $[d, 1]$ in D belongs to $V(\mathbf{2})$. It is true that for $\mathbf{n}$ the n-element chain, any algebra in $V(\mathbf{n}+\mathbf{1})$ is a Stone algebra whose dense set is a dual generalized algebra from $V(\mathbf{n})$, meaning that each interval $[d, 1]$ belongs to $V(\mathbf{n})$. Perhaps there is an inductive proof that each variety $V(\mathbf{n})$ admits a meet dense and regular completion. A useful step may be to show that each generalized $V(\mathbf{n})$ algebra can be regularly embedded into a filter of an algebra in $V(\mathbf{n})$.

The variety of linear Heyting algebras is the one generated by all chains.

Question. Does the variety of linear Heyting algebras admit a regular completion?

The following remark shows a direct construction of our completion, similar to the construction of the MacNeille completion via normal ideals, is problematic.

Remark. In proving the Main Theorem we use a weak version of the axiom of choice by invoking Sikorski's Theorem on injective Boolean algebras. This is unavoidable. For an infinite set X, consider the triple consisting of the Boolean algebra C of finite and cofinite subsets of X, and the homomorphism from C to $\mathbf{2}$ mapping all finite sets to zero. Let L be the algebra for this triple and suppose $L \leq L_1$ is a meet dense and regular completion. Each dense element of L_1 is the meet of dense elements of L, so there are two dense elements of L_1. It follows that the center of L is meet dense in the center of L_1, so the center of L_1 is the MacNeille completion of C, hence is the power set of X. The triple for L_1 consists of a homomorphism from the power set of X to $\mathbf{2}$ mapping all finite sets to zero. The existence of such a homomorphism for each infinite set X is equivalent to the Boolean ultrafilter theorem [7, page 328]. See [2, Theorem 6.18] for a related result.

Question. Can one prove the variety $V(\mathbf{3})$ admits a regular completion (that is not necessarily meet dense) without any form of the axiom of choice?

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