Completions of Ordered Algebraic Structures: A Survey

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Summary. Ordered algebraic structures are encountered in many areas of mathematics. One frequently wishes to embed a given ordered algebraic structure into a complete ordered algebraic structure in a manner that preserves some aspects of the algebraic and order theoretic properties of the original. It is the purpose here to survey some recent results in this area.

1 Introduction

An ordered algebraic structure $\mathcal{A}$ consists of an algebra, in the sense commonly used in universal algebra [9], together with a partial ordering on the underlying set of the algebra. We require that the operations of the algebra are compatible with the partial ordering in that they preserve or reverse order in each coordinate. The partial orderings we consider here will almost always be lattice orderings.

Ordered algebraic structures occur in a wide variety of areas. Examples include partially ordered vector spaces, lattice ordered groups, Boolean algebras, Heyting algebras, modal algebras, cylindric algebras, relation algebras, orthomodular posets, and so forth. In applications, the existence of certain infinite joins and meets often play an important role. In analytic applications, certain infinite joins and meets are often related to limit processes; in logical applications, certain infinite joins and meets are often related to existential and universal quantification; and in quantum logic, countable orthogonal joins correspond to experiments built from countable families of mutually exclusive experiments. It is a common task to try to embed a given ordered algebraic structure into one where certain families of joins and meets exist.

Perhaps the best example is the earliest one. In 1858 (published in 1872 [12]) Dedekind used his methods of *cuts* to construct the real numbers $\mathbb{R}$ from the rationals $\mathbb{Q}$. He defined a real number to be a certain type of ordered pair

(A, B) of subsets of the rationals called a cut. Each rational q yields such a cut, and this provides an embedding $\varphi : \mathbb{Q} \to \mathbb{R}$. Dedekind further defines an ordering $\leq$ and operations $+, -, \cdot$ on $\mathbb{R}$. He shows that with these operations $(\mathbb{R}, +, -\cdot, \leq)$ is an ordered field that is conditionally complete, meaning that every non-empty subset of $\mathbb{R}$ that has an upper bound has a least upper bound and every non-empty subset that has a lower bound has a greatest lower bound. Having embedded the rationals into a conditionally complete ordered field, one might ask whether the rationals can even be embedded into a complete ordered field. This is trivially impossible as an ordered field can never have a largest or least element.

The embedding $\varphi : \mathbb{Q} \to \mathbb{R}$ produced above is more than just an order embedding that preserves algebraic structure. The map φ preserves all existing joins and meets in $\mathbb{Q}$, a property we call a regular embedding. Further, each element of $\mathbb{R}$ is both a join and meet of elements of the image of φ, properties called join and meet density. In many instances it may be desirable to find a completion that not only preserves some existing algebraic properties, but also preserves some existing joins and meets. Further, having some sort of density condition, to ensure the resulting completion is somewhat tightly tied to the original, is often desirable.

For a given type of algebraic structure, one can ask a variety of questions regarding the existence of an embedding into a complete ordered structure preserving certain aspects of the algebraic and order theoretic structure. There is a large, mostly scattered, literature on such questions for specific classes of structures. It is not our intent to review this literature in more than an incidental way. Rather, we concentrate on results, mostly in the past 20 years, that seem to form the beginnings of a general theory of such completions.

2 Preliminaries

In this section we review some basic definitions.

Definition 1. *For a poset P, an n-ary operation f on P is called monotone if it preserves or reverses order in each coordinate. An ordered algebraic structure $\mathcal{A} = (A, (f_i)_I, \leq)$ consists of an algebra $(A, (f_i)_I)$, together with a partial ordering $\leq$ on A, such that for each $i \in I$, the operation f_i is monotone.*

Note, this definition allows for Heyting implication $\longrightarrow$ that is order reversing in the first coordinate, and order preserving in the second.

Mostly we will consider here ordered algebraic structures where the underlying ordering is a lattice ordering. Such structures are also known under the name of monotone lattice expansions [14, 15].

Definition 2. *An embedding of an ordered algebraic structures $\mathcal{A}$ into $\mathcal{B}$ is a map $\varphi : A \to B$ that is both a homomorphism and an order embedding. A completion of $\mathcal{A}$ is an embedding $\varphi : \mathcal{A} \to \mathcal{B}$ where the underlying ordering of $\mathcal{B}$ is a complete lattice.*

For an ordered algebraic structure whose underlying ordering is a lattice, one might reasonably argue that an embedding should be required to be a lattice embedding. This is easily accomplished by keeping the current definition and adding the lattice operations as part of the basic algebraic structure.

Definition 3. *An embedding $\varphi : \mathcal{A} \to \mathcal{B}$ is called join dense if for each $b \in B$, we have $b = \bigvee\{\varphi(a) : a \in A \text{ and } \varphi(a) \leq b\}$. We say φ is join regular if for each $S \subseteq A$ that has a join in $\mathcal{A}$, the image $\varphi[S]$ has a join in $\mathcal{B}$, and $\varphi(\bigvee S) = \bigvee \varphi[S]$. Meet dense and meet regular are defined similarly. Finally, call φ regular if it is both join regular and meet regular.*

The following is well known, and easy to prove.

Proposition 1. *If φ is join dense, then it is meet regular, and if φ is meet dense, then it is join regular.*

We next describe various types of ideals an filters that play an important role. We recall that for a subset S of a poset P, that $U(S) = \{p \in P : s \leq p$ for all $s \in S\}$ is the set of upper bounds of S, and $L(S) = \{p \in P : p \leq s$ for all $s \in S\}$ is the set of lower bounds of S.

Definition 4. *For P a poset and $I \subseteq P$ we say*

1. *I is an order ideal if $b \in I$ and $a \leq b \Rightarrow a \in I$.*
2. *I is an ideal if I is an order ideal that is closed under existing finite joins.*
3. *I is a normal ideal if $I = LU(I)$.*

Let $\mathcal{I}_O P$, $\mathcal{I} P$ and $\mathcal{I}_N P$ be the sets of order ideals, ideals, and normal ideals of P, considered as posets under the partial ordering of set inclusion.

Similarly an order filter is a subset $F \subseteq P$ where $a \in F$ and $a \leq b$ implies $b \in F$, a filter is an order filter closed under existing finite meets, and a normal filter is a set F with $F = UL(F)$. We let $\mathcal{F}_O P$, $\mathcal{F} P$ and $\mathcal{F}_N P$ be the sets of order filters, filters, and normal filters partially ordered by reverse set inclusion.

Proposition 2. *S is a normal ideal iff it is the intersection of principal ideals, and S is a normal filter iff it is the intersection of principal filters. Normal ideals are closed under all existing joins, and normal filters are closed under all existing meets.*

While normal ideals are closed under existing joins, in general, there are ideals of a lattice that are closed under existing joins but are not normal. However, for Heyting algebras an ideal is normal iff it is closed under existing joins, but the corresponding result does not hold for normal filters in a Heyting algebra [5].

Definition 5. *For $a \in P$ let $a{\downarrow} = \{p \in P : p \leq a\}$ and $a{\uparrow} = \{p \in P : a \leq p\}$. We call these the principal ideal and principal filter generated by a.*

3 Completion Methods

In this section we collect a number of common completion methods, as well as a general template into which these methods fit. We first discuss matters for posets and lattices, considering additional algebraic operations later.

Proposition 3. *For a poset P, the order ideals $\mathcal{I}_O P$ are a complete lattice and the map $\varphi : P \to \mathcal{I}_O P$ defined by $\varphi(a) = a\!\downarrow$ is a completion of P satisfying*

1. *φ is join dense.*
2. *For $a \in P, S \subseteq P$, if $\varphi(a) \leq \bigvee \varphi[S]$ then $a \leq s$ for some $s \in S$.*

Further, if $\psi : P \to C$ is another completion satisfying these two properties, there is a unique isomorphism $\mu : \mathcal{I}_O P \to C$ with $\mu \circ \varphi = \psi$.

The order ideal completion preserves all existing meets as it is a join dense completion, but destroys all existing joins except those of subsets with a maximum element. Also of interest is that $\mathcal{I}_O P$ is a completely distributive lattice.

Proposition 4. *For a poset P, the ideals $\mathcal{I}P$ are a complete lattice and the map $\varphi : P \to \mathcal{I}P$ defined by $\varphi(a) = a\!\downarrow$ is a completion of P satisfying*

1. *φ is join dense.*
2. *For $a \in P, S \subseteq P$, if $\varphi(a) \leq \bigvee \varphi[S]$ then $\varphi(a) \leq \bigvee \varphi[S']$ for some finite $S' \subseteq S$.*

Further, if $\psi : P \to C$ is another completion satisfying these two properties, there is a unique isomorphism $\mu : \mathcal{I}_O P \to C$ with $\mu \circ \varphi = \psi$.

The ideal completion preserves all existing meets, and all existing finite joins, however it destroys all existing joins that are not essentially finite. In the next section when we consider preservation of identities, we see some of the main advantages of the ideal completion.

Proposition 5. *For a poset P, the normal ideals $\mathcal{I}_N P$ are a complete lattice and the map $\varphi : P \to \mathcal{I}P$ defined by $\varphi(a) = a\!\downarrow$ is a completion of P satisfying*

1. *φ is join dense.*
2. *φ is meet dense.*

Further, if $\psi : P \to C$ is another completion satisfying these two properties, there is a unique isomorphism $\mu : \mathcal{I}_O P \to C$ with $\mu \circ \varphi = \psi$.

The normal ideal completion is often called the MacNeille completion, or the completion by cuts. It was introduced by MacNeille [28] in the poset setting as an extension of the method used by Dedekind to construct the reals from the rationals. It preserves all existing joins and meets, so is a regular completion. The above characterization is due to Banaschewski and Schmidt

[2, 31]. It provides a minimal completion of P in that for any completion $f : P \to C$ there is an order embedding $\mu : \mathcal{I}_N P \to C$ with $\mu \circ \varphi = f$, and this can be used to show that MacNeille completions provide strictly injective essential extensions in the category of posets [3].

One similarly obtains order filter, filter, and filter completions of a poset P using the embedding $\psi(a) = a\!\uparrow$. They have similar properties to the above, except that the roles of joins and meets are interchanged. As the roles of join and meet are symmetric for the normal ideal completion, $\mathcal{I}_N P$ and $\mathcal{F}_N P$ are isomorphic, and the maps U, L provide mutually inverse isomorphisms.

One can create additional types of completions by taking other families of order ideals or order filters that are closed under intersections. For instance, the family of all order ideals closed under existing countable joins provides a join dense completion that preserves all existing meets, and existing countable joins. We soon generalize this observation, but first we consider one more completion.

Proposition 6. *For a bounded lattice L, there is a completion $\varphi : L \to C$ satisfying*

1. *Each $c \in C$ is both a join of meets and a meet of joins of elements from the image of L.*
2. *For $S, T \subseteq L$, $\bigwedge \varphi[S] \leq \bigvee \varphi[T]$ iff $\bigwedge S' \leq \bigvee T'$ for some finite $S' \subseteq S, T' \subseteq T$.*

Further, if $\varphi' : L \to C'$ is another completion satisfying these two properties, there is a unique isomorphism $\mu : C \to C'$ with $\mu \circ \varphi = \varphi'$.

Proof. We provide a sketch, details are found in [14]. Let $\mathcal{I}$ and $\mathcal{F}$ be the sets of ideals and filters of L and define a binary relation R from $\mathcal{F}$ to $\mathcal{I}$ by setting $F\,R\,I$ iff $F \cap I \neq \emptyset$. Then the polars of R [8] give a Galois connection between the power set of $\mathcal{F}$ and the power set of $\mathcal{I}$. The Galois closed elements of the power set of $\mathcal{F}$ form a complete lattice C, and the map $\varphi : L \to C$ defined by $\varphi(a) = \{F \in \mathcal{F} : a \in F\}$ is the required embedding. This gives existence, uniqueness is not difficult. ∎

This completion is called the canonical completion. The embedding φ preserves all finite joins and meets, so is a lattice embedding, but destroys all existing essentially infinite joins and meets. Canonical completions have their origins in Stone duality. For a Boolean algebra B the canonical completion is the natural embedding of B into the power set of its Stone space, for a distributive lattice it is given by the upsets of the Priestley space, and for general lattices it is given by the stable subsets of the Urquhart space [32]. An abstract characterization similar to that above was given in the Boolean case by Jónsson and Tarski [24], and in the distributive case by Gehrke and Jónsson [16]. The above abstract characterization in the lattice setting was given by Gehrke and Harding [14].

3.1 A General Template for Completions

The technique used in constructing the canonical completion can be adapted to create a range of completions. Let P be a poset, $\mathcal{I}$ be some set of order ideals of P containing all principal ideals, and $\mathcal{F}$ some set of order filters containing all principal filters. Define a relation R from $\mathcal{F}$ to $\mathcal{I}$ by $F\,R\,I$ iff $F \cap I \neq \emptyset$. Then the polars of R give a Galois connection, and the Galois closed subsets of $\mathcal{F}$ form a complete lattice $\mathcal{G}(\mathcal{F}, \mathcal{I})$. The map $\alpha : P \to \mathcal{G}(\mathcal{F}, \mathcal{I})$ defined by $\alpha(a) = \{F \in \mathcal{F} : a \in F\}$ is an embedding.

For sets $\mathcal{I}$ and $\mathcal{F}$ of order ideals and order filters, the completion $\alpha : P \to \mathcal{G}(\mathcal{F}, \mathcal{I})$ has the property that each element of $\mathcal{G}(\mathcal{F}, \mathcal{I})$ is both a join of meets and a meet of joins of elements of the image of P. Further, the embedding α preserves all existing joins in P under which each member of $\mathcal{I}$ is closed, and all existing meets in P under which each member of $\mathcal{F}$ is closed. It destroys all other joins and meets.

A number of common completions arise this way. The order ideal completion arises by choosing $\mathcal{I}$ to be all order ideals of P and $\mathcal{F}$ to be all principal filters of P; the ideal completion by choosing $\mathcal{I}$ to be all ideals, $\mathcal{F}$ to be all principal filters; the MacNeille completion by choosing $\mathcal{I}$ to be all normal ideals, $\mathcal{F}$ to be all principal filters, or alternately, by choosing $\mathcal{I}$ to be all principal ideals and $\mathcal{F}$ all principal filters; and the canonical completion by choosing $\mathcal{I}$ to be all ideals, and $\mathcal{F}$ to be all filters. Clearly others are possible as well.

3.2 Extending Additional Operations

Suppose P is a poset, $\alpha : P \to C$ is a completion of P, and f is a monotone n-ary operation on P. We recall monotone means that f preserves or reverses order in each coordinate. For convenience we write $\overline{a}$ for an n-tuple of elements $(a_1, \ldots, a_n)$ of P, $\overline{c}$ for an n-tuple of elements $(c_1, \ldots, c_n)$ of C, and $\alpha(\overline{a})$ for $(\alpha(a_1), \ldots, \alpha(a_n))$.

Definition 6. *Let $\leq_f$ be the ordering on C^n defined by $\overline{c} \leq \overline{d}$ if $c_i \leq d_i$ for each i with f order preserving in the i^{th} coordinate, and $d_i \leq c_i$ for all other i.*

We now describe two ways to lift the operation f on P to an operation on C.

Definition 7. *Let f^- and f^+ be the n-ary operations on C defined by*

$$f^-(\overline{c}) = \bigvee\{\alpha(f(\overline{a})) : \alpha(\overline{a}) \leq_f \overline{c}\}.$$

$$f^+(\overline{c}) = \bigwedge\{\alpha(f(\overline{a})) : \overline{c} \leq_f \alpha(\overline{a})\}.$$

We call f^- and f^+ the lower an upper extensions of f.

Proposition 7. *Both f^- and f^+ are monotone maps and with repect to either extension, α is a homomorphism.*

For a join dense completion each $c \in C$ is given by $c = \bigvee\{\alpha(a) : \alpha(a) \leq c\}$. For f unary and order preserving, $f^-(c) = \bigvee\{\alpha(a) : \alpha(a) \leq c\}$. Clearly this is a natural choice of extension. Similarly, for a meet dense completion, f^+ is a natural choice. So for MacNeille completions, both are reasonable choices. In particular instances one may be preferable to the other. For Heyting algebras, extending the Heyting implication $\rightarrow$ using the upper extension yields a Heyting algebra, while the lower extension does not.

Canonical completions have neither join nor meet density, however, every element of is a join of meets and a meet of joins of elements of the image. We use this to define extensions of monotone maps suited to this type of completion. Let K be all elements that are meets of elements of the image and O be all elements that are joins of elements of the image. Then $c = \bigvee\{\bigwedge\{\alpha(a) : k \leq a\} : k \leq c \text{ and } k \in K\}$ for each c in the canonical completion, with a similar expression involving a meet of joins and all $c \leq o$ with $o \in O$.

Definition 8. *For f monotone and unary define f^σ and f^π by*

$$f^\sigma(c) = \bigvee\{\bigwedge\{\alpha(f(a)) : k \leq a\} : k \leq c \text{ and } k \in K\}.$$

$$f^\pi(c) = \bigwedge\{\bigvee\{\alpha(f(a)) : a \leq o\} : c \leq o \text{ and } o \in O\}.$$

We call f^σ and f^π the lower and upper canonical extensions of f.

This definition extends in a natural way to monotone n-ary operations, but one must use a mixture of open and closed elements depending on whether the coordinate of f preserves of reverses order. In this generality we have the following.

Proposition 8. *Both f^σ and f^π are monotone maps and with respect to either extension, α is a homomorphism.*

For a completion $\alpha : P \rightarrow C$ and a family of monotone operations $(f_i)_I$ on P, a map $\beta : I \rightarrow \{-, +, \sigma, \pi\}$ can be used to indicate which extension method to apply to each operation f_i.

Definition 9. *For an ordered structure $(A, (f_i)_I, \leq)$ and map $\beta : I \rightarrow \{-, +, \sigma, \pi\}$, define the β-ideal completion, β-MacNeille completion, and β-canonical completion to be the corresponding completion applied to the underlying ordered structure with operations extended in the indicated way.*

This by no means exhausts the range of possible completions, but it does include many of those commonly encountered. Generally one tends to use $-$ extensions for ideal completions, $+$ extensions for filter completions, $-, +$ for MacNeille completions, and σ, π for canonical completions to take advantage of various density properties.

4 Preservation of Identities

We consider the question of when an identity holding in an ordered structure $\mathcal{A}$ holds in a certain type of completion of $\mathcal{A}$. In the case of lattice ordered structures, a natural question becomes when a variety of lattice ordered structures is closed under a certain type of completion. There has been considerable progress in this area in the past twenty five years, but one of the more useful results is still one of the oldest.

Definition 10. *If $\mathcal{A}$ is a lattice with a family of operations that are order preserving in each coordinate, then every identity valid in $\mathcal{A}$ is valid in the ideal lattice completion of $\mathcal{A}$ where the operations are extended by the $-$ extension.*

Proof. As in [10] one shows that for a term $t(x_1, \ldots, x_n)$ and ideals $I_i, \ldots, I_n$ of $\mathcal{A}$, that $t(I_1, \ldots, I_n) = \{b \in A : b \leq t(a_1, \ldots, a_n)$ for some $a_1 \in I_1, \ldots, a_n \in I_n\}$. ∎

While ideal completions work very well with order preserving operations, they work very poorly when an operation has a coordinate where it is order reversing. For the basic case of Boolean algebras, the ideal lattice is hopeless as the ideal completion of a Boolean algebra is Boolean only if it is finite.

The preferred method to complete a Boolean algebra with additional operations is the canonical completion, usually using the σ extension of maps. Here there is an extensive literature, beginning with the work of Jónsson and Tarski [24, 25] in the 1950's, and continuing with the use of Kripke semantics in modal logic (see [7] for a complete account). Primary concern is Boolean algebras with additional operations that preserve finite joins in each coordinate. Such operations are called operators.

Theorem 1. *(Jónsson-Tarski) The canonical completion of a Boolean algebra with operators preserves all identities that do not use the Boolean negation.*

In the 1970's, Sahlqvist [30] generalized this result to apply to equations in which negation occurs, provided they are of a certain form. Usually, these equations are called Sahlqvist equations. While we don't describe the exact form here, we do remark that Sahlqvist terms are the ones that correspond to first order properties of the associated Kripke frame.

Theorem 2. *(Sahlqvist) The canonical completion of a Boolean algebra with operators preserves all Sahlqvist equations.*

Sahlqvist's result was set and proved via Kripke frames, which tied it to the Boolean algebra with operator setting. Jónsson [26] gave an algebraic proof that seems more portable. Gehrke, Nagahashi and Venema [17] used Jónsson's method to extend Sahlqvist's theorem to distributive modal logics, but it remains an open problem to see what portions of this result can be extended to canonical completions in more general settings. We remark that Jónsson and Tarski's original result extends nicely to this setting as described below.

Theorem 3. *(Gehrke-Harding) The canonical completion of a bounded lattice with additional monotone operations preserves all identities involving only operators.*

Note that join is an operator on any lattice, but meet being an operator is equivalent to distributivity. Of course, canonical completions of such lattices with operations also preserve some identities involving order inverting operations, such as those for orthocomplementations, and this provides an advantage for them over ideal completions in such settings. To illustrate the utility of canonical completions in the general setting we have the following.

Theorem 4. *(Gehrke-Harding) Let $\mathcal{K}$ be a class of bounded lattices with additional monotone operations. If $\mathcal{K}$ is closed under ultraproducts and β-canonical completions, where β uses only the extensions σ, π, then the variety generated by $\mathcal{K}$ is closed under β-canonical completions.*

In particular, the variety generated by a single finite lattice with monotone operations is closed under β-canonical completions. The proof of this result requires showing canonical completions work well with homomorphic images, subalgebras and Boolean products [14].

Turning to MacNeille completions, there are a good number of results from different areas stating that a particular variety of interest is closed under MacNeille completions. For instance, the varieties of lattices, Boolean algebras, Heyting algebras, ortholattices, closure algebras, and post algebras are closed under MacNeille completions, although one must be careful about choosing the $+, -$ extension of maps in certain cases.

The first general study of preservation of identities under MacNeille completions was conducted by Monk [29], who showed an analogous theorem to 1 holds for MacNeille completions of Boolean algebras with operators provided the operators preserve all existing joins in each coordinate. Givant and Venema [18] used Jónsson's technique to extend this result and obtain a type of Sahlqvist theorem for preservation of identities for MacNeille completions of Boolean algebras with operators. The key point in their work is the notion of a conjugated map, which plays the role for order preserving operations similar to that of residuation for order inverting ones. Among their results is the following which refers to the $-$ extensions of maps.

Theorem 5. *(Givant-Venema) The MacNeille completion of a Boolean algebra with a family of additional conjugated operators preserves Sahlqvist identities.*

I am not aware of a version of 4 for MacNeille completions, but in the setting of Boolean algebras with operators, a type of converse holds [15]. The following uses the idea from Kripke semantics that a set with a family of relations produces a Boolean algebra with operators consisting of the power set of the set and the operators defined using relational image.

Theorem 6. *(Gehrke-Harding-Venema) If a variety of Boolean algebras with operators is closed under MacNeille completions using the $-$ extensions of maps, then the variety is generated by an elementary class of relational structures.*

There is also an interesting connection between Boolean products and MacNeille completions. In [11] it was shown that if a lattice ordered algebraic structure has a well behaved Boolean product representation, then its MacNeille completion lies in the variety generated by the original. This result was used to show any variety of orthomodular lattices that is generated by its finite height members is closed under MacNeille completions. It can also be used to show that Post algebras are closed under MacNeille completions. We note that these results for orthomodular lattices have implications also for Boolean algebras with operators as every variety of ortholattices can be interpreted in a certain variety of modal algebras [21].

5 Comparing Completions

Here there is unfortunately little known. The main result is found in [15].

Theorem 7. *(Gehrke-Harding-Venema) For a bounded lattice L with additional monotone operations, the canonical completion of L is isomorphic to a sublattice of the MacNeille completion of an ultrapower of L. Here any mixture of σ and π extensions of maps can be used for the canonical completion provided the MacNeille completion uses the corresponding $-$ and $+$ extensions of these maps.*

Proof. The key point in the proof [15] is that every ideal of a sufficiently saturated ultrapower of L is a normal ideal. ∎

This is vaguely reminiscent of a result of Baker and Hales [1] showing that the ideal lattice of a lattice L is isomorphic to a subalgebra of an ultrapower of L. Perhaps other such relationships can be found among various completions.

6 Exploring the Boundaries

In this section, we look at a number of results that point to what may, and what may not, be possible. To begin, it is not the case that every ordered algebraic structure can be embedded into one that is complete and satisfies the same identities as the original. The rationals $\mathbb{Q}$ provide an example of a structure without such a completion as no lattice ordered group can have a largest or least element. For a simple example where even a conditional completion is impossible, consider the variety V of diagonalizable algebras [6]. These modal algebras have an order preserving unary operation f and for each member in V we have $x \leq f(x)$ implies $x = 0$. One then finds some $\mathcal{A} \in V$ with a family $a_1 \leq a_2 \leq \cdots$ where $a_n \leq f(a_{n+1})$. Then in any completion of $\mathcal{A}$ for $x = \bigvee a_n$ we have $x \leq f(x)$. Kowalski and Litak [27] provide a number of varieties sourced in logic that admit no completion.

Modular ortholattices provide another example of a variety admitting no completion, but the only known proof of this relies on Kaplansky's result that every complete modular ortholattice is a continuous geometry, and von Neumann's result that a continuous geometry has a dimension function, two of the deepest results in lattice theory. In contrast to this, it is known that every complemented modular lattice can be embedded into a complete complemented modular lattice, via a method known as the Frink embedding [10] which is a modification of the ideal lattice of the filter lattice. A related question, that remains open, is whether every orthomodular lattice admits a completion.

Moving to the topic of how specific completion methods behave, we first consider the canonical completion. Here it had long been conjectured that every variety of Boolean algebras with operators that is closed under canonical completions is generated by an elementary class of frames. Hodkinson and Venema [19] showed this is not the case, but their counterexample is not finitely based. The question remains open in the finitely based setting. On a related note, the matter of determining whether a finitely based variety of Boolean algebras with operators is closed under canonical completions is an undecidable problem [33]. A simpler problem, but also open, is to determine which varieties of lattices are closed under canonical completions. Here we know every finitely generated variety of lattices is closed, but the variety of modular lattices is not closed under canonical completions.

There are a number of results showing that subvarieties of familiar varieties are not closed under MacNeille completions. The only varieties of lattices closed under MacNeille completions are the trivial variety and the variety of all lattices [20]; for Heyting algebras only the trivial variety, the variety of Boolean algebras, and the variety of all Heyting algebras are closed [5]; and [6] describes the situation for some varieties of closure algebras and derivative algebras. Belardinelli, Jipsen, and Ono have shown [4] that in a certain setting, cut elimination for a logic implies the closure of a corresponding variety under MacNeille completions. So the above result for Heyting algebras explains why so few superintuitionistic logics have cut elimination.

It can be a non-trivial task to determine when the variety generated by a given finite ordered structure $\mathcal{A}$ is closed under β-MacNeille completions for some given method β of extending the operations. Indeed, this is not trivial even for $\mathcal{A}$ being the two-element lattice, or the three-element Heyting algebra. It would be desirable to have a decision process for this problem, if indeed it is even decidable.

One might also ask whether a variety is closed under MacNeille completions in the sense that for each $\mathcal{A} \in V$, the operations on $\mathcal{A}$ can be extended in some manner to the MacNeille completion of $\mathcal{A}$ to produce an algebra in V. Here there are further complications. An example in [6] gives a variety V

generated by a four-element modal algebra that is not closed under MacNeille completions using either the $-, +$ extensions of maps, but whose closure under MacNeille completions in this more general sense is equivalent to some weak form of the axiom of choice.

Rather than focusing on MacNeille completions, one may ask more generally whether a variety admits some type of regular completion. In many applications it is the regularity that is of primary interest anyway. In some instances, such as for orthomodular lattices, a variety admitting a regular completion is equivalent to closure under MacNeille completions as any regular completion factors through the MacNeille completion [21]. However, this is not generally the case. The variety generated by the three-element Heyting algebra is not closed under MacNeille completions, but does admit a regular completion [22]. This is the only example of this phenomenon that I know. It would be natural to see if there are other varieties of lattices or Heyting algebras that admit regular completions.

7 Conclusions and Discussions

We described several common completions of ordered sets, including the ideal completion, the MacNeille completion, and the canonical completion. We also discussed a template to produce completions, the $\mathcal{I}, \mathcal{F}$-method, that includes these completions as special cases. We discussed a number of methods to extend operations from an ordered algebraic structure to a completion, including the $-, +$ and σ, π extensions, obtaining the notion of β-ideal, β-MacNeille, and β-canonical completions.

A good deal of information was given about preservation of identities by various completion methods. The MacNeille and canonical completions were then compared, and it was shown that any variety of bounded lattice ordered algebraic structures that is closed under MacNeille completions is also closed under canonical completions.

A number of limiting results were given. These included a discussion of varieties of ordered algebraic structures that admit no completion whatever, as well as results relating to closure under particular completion methods. We also pointed out two examples that show our current understanding of matters is limited. The first showed that a variety can be closed under MacNeille completions with some unusual method of extending operations, but not closed under any standard method of extending the operations. The second showed a variety can admit a regular completion but not be closed under any type of MacNeille completion.

The topic of completing ordered structures is a broad one with a long history. The results mentioned here focus on one area of this topic, preservation of identities, and deal only with a fragment of the array of completion methods available. Still, there is reason to believe these results form a basis around which a unified theory can be built, and that this theory addresses questions of concern in many areas of mathematics.

We have a fairly general template for completions, the $\mathcal{I}, \mathcal{F}$-method, that includes many of the completions commonly encountered. This method also points a way to tailor completions to specific need. We have also various methods of extending operations to such completions, and Jónsson's approach to Sahlqvist's theorem may give a portable tool to address preservation of certain types of identities by various types of completions.

We have also seen several techniques occurring repeatedly in our work. These include the use of ultraproducts, which occurs when considering the preservation of identities by canonical extensions, and also in the proof that closure of a variety under MacNeille completions implies closure under canonical extensions. Ultraproducts are also related to the formation of the ideal lattice. Boolean products are another recurring theme. They occur in connection with preservation of identities by canonical extensions, when considering preservation of identities by MacNeille completions, and are also key in constructing a variety that admits a regular completion but is not closed under MacNeille completions.

The use of relational structures or Kripke frames has also been closely tied to our considerations of completions of Boolean algebras with operators, both in the case of canonical completions and MacNeille completions. It would desirable to extend this to the more general setting of bounded lattices with additional operations. There are more general notions of frames in this setting, see for example Gehrke [13]. Also, Harding [23] has a notion of frames involving a set X with additional binary relation P and family of relations R_i satisfying certain conditions, where the relations R_i are used to form operations on the Galois closed subsets of X under the polarity induced by P. No matter which method one uses to create frames, it would seem worthwhile to see the extent to which results can be lifted from the Boolean setting to completions of more general structures.

In sum, it seems an exciting time to be working in this area of completions. A sufficient groundwork has been laid to map out a direction of research, and a number of the tools have been identified. To be sure, there remains much work to be done, with likely more than a few surprises, but one hopes to see considerable progress in the near future.

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