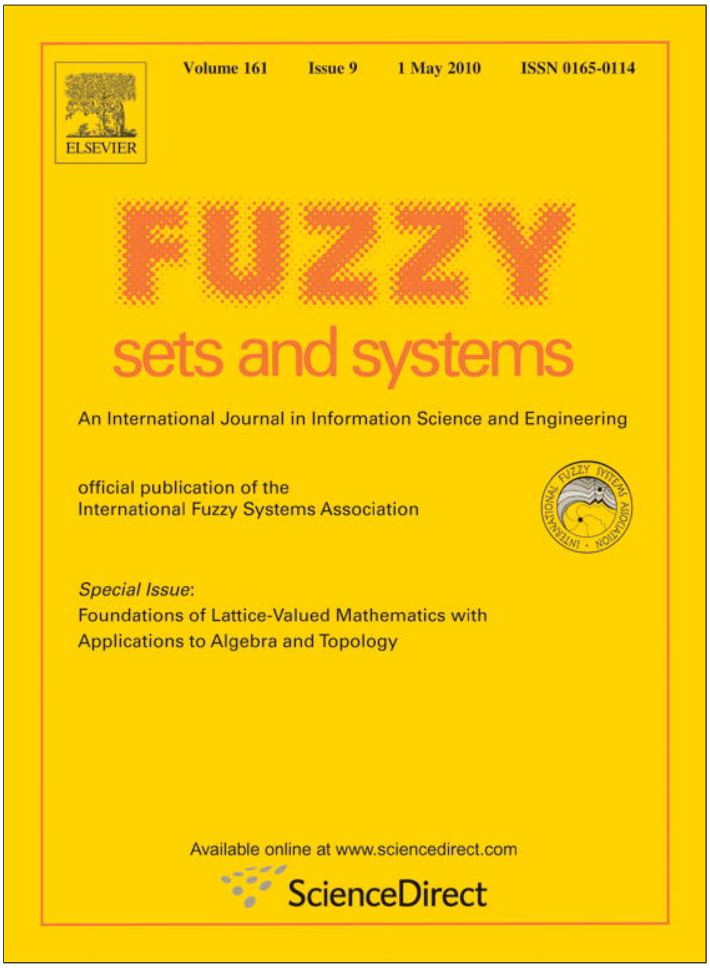
Volume 161 Issue 9 1 May 2010 ISSN 0165-0114
ELSEVIER
FUZZY
sets and systems
An International Journal in Information Science and Engineering
official publication of the
International Fuzzy Systems Association
INTERNATIONAL FUZZY SYSTEMS ASSOCIATION
Special Issue:
Foundations of Lattice-Valued Mathematics with
Applications to Algebra and Topology
Available online at www.sciencedirect.com
ScienceDirect

Available online at www.sciencedirect.com
ScienceDirect
Fuzzy Sets and Systems 161 (2010) 1343–1349
FUZZY sets and systems
www.elsevier.com/locate/fss

Convex normal functions revisited

John Harding, Carol Walker*, Elbert Walker

New Mexico State University, Las Cruces, NM 88003, USA

Available online 1 November 2008

Abstract

The lattice L_u of upper semicontinuous convex normal functions with convolution ordering arises in studies of type-2 fuzzy sets. In 2002, Kawaguchi and Miyakoshi [Extended t-norms as logical connectives of fuzzy truth values, Multiple-Valued Logic 8(1) (2002) 53–69] showed that this lattice is a complete Heyting algebra. Later, Harding et al. [Lattices of convex, normal functions, Fuzzy Sets and Systems 159 (2008) 1061–1071] gave an improved description of this lattice and showed it was a continuous lattice in the sense of Gierz et al. [A Compendium of Continuous Lattices, Springer, Berlin, 1980]. In this note we show the lattice L_u is isomorphic to the lattice of decreasing functions from the real unit interval [0, 1] to the interval [0, 2] under pointwise ordering, modulo equivalence almost everywhere. This allows development of further properties of L_u. It is shown that L_u is completely distributive, is a compact Hausdorff topological lattice whose topology is induced by a metric, and is self-dual via a period two antiautomorphism. We also show the lattice L_u has another realization of natural interest in studies of type-2 fuzzy sets. It is isomorphic to a quotient of the lattice L of all convex normal functions under the convolution ordering. This quotient identifies two convex normal functions if they agree almost everywhere and their intervals of increase and decrease agree almost everywhere.

MSC: 06B10; 06B15

Keywords: Type-2 fuzzy set; Uniquely complemented lattice; Complete lattice; Continuous lattice; Metric topology

1. Introduction

The algebra of truth values of type-2 fuzzy sets [8] consists of the set $M = [0, 1]^{[0,1]}$ of all mappings from [0, 1] to [0, 1], together with operations formed as convolutions [7] of the basic operations on the unit interval. In particular, binary operations $\sqcap$, $\sqcup$ on M are defined by setting

$$(f \sqcap g)(x) = \sup\{f(y) \wedge g(z) | y \wedge z = x\},$$

$$(f \sqcup g)(x) = \sup\{f(y) \wedge g(z) | y \vee z = x\}.$$

A unary operation $\neg$ is defined by $\neg f(x) = f(1-x)$ and constants $\overline{0}, \overline{1}$ by $\overline{0}(x) = 1$ if $x = 0$ and $\overline{0}(x) = 0$ otherwise and $\overline{1}(x) = 1$ if $x = 1$ and $\overline{1}(x) = 0$ otherwise.

* Corresponding author.
E-mail address: hardy@nmsu.edu (C. Walker).

0165-0114/$ - see front matter © 2008 Elsevier B.V. All rights reserved.
doi:10.1016/j.fss.2008.10.008

While the algebra $(M, \sqcap, \sqcup, \neg, \overline{0}, \overline{1})$ satisfies some of the properties of bounded lattices, it is not a bounded lattice. In particular, the absorption laws $f \sqcup (f \sqcap g) = f$ and $f \sqcap (f \sqcup g) = f$ do not hold in M. However, there are subalgebras of M of natural interest that do form bounded lattices. It is in several of these that our interest here lies.

We say a function f from [0, 1] to [0, 1] is *normal* if 1 is the supremum of its image, and *strongly normal* if it attains the value 1. We say f is *convex* if it lies above all its chords; that is, if $x \leqslant y \leqslant z$, then $f(y) \geqslant f(x) \vee f(z)$. Let L be the set of convex normal functions, L_1 the set of convex strongly normal functions, and L_u the set of convex strongly normal functions that are *upper semicontinuous*, meaning that the inverse image $f^{-1}[1-\varepsilon, 1-\varepsilon]$ of each closed interval with $0 \leqslant \varepsilon \leqslant 1$, is closed. All three are subalgebras of M, and furthermore all are bounded distributive lattices with the operations $\sqcap, \sqcup$ being meet and join, and $\overline{0}, \overline{1}$ being the lower and upper bounds. The operation $\neg$ makes these into De Morgan algebras [1]. A primary focus in [4] is to show each of these lattices is complete.

As L, L_1, and L_u are lattices, they have an associated partial ordering defined through the meet and join operations $\sqcap$ and $\sqcup$. This ordering, which we call the *convolution ordering*, is not the usual pointwise ordering on functions. For instance, the function $\overline{1}$ is the largest element of L, yet takes the value zero at all points except $x = 1$ where it takes value 1. The convolution ordering can seem quite unnatural and difficult to work with.

A primary contribution of [4] is to realize the lattices L, L_1, and L_u isomorphically as lattices of real-valued functions under the usual pointwise ordering. The idea is very simple. A convex function f from [0, 1] to [0, 1] is increasing on some initial segment, then decreasing on some terminal segment. One *straightens out* the function f to produce a function f^* from [0, 1] to [0, 2] by taking the mirror image about the line $y = 1$ of the increasing portion of f and leaving the remainder of f alone. For instance, the function $\overline{1}$ which takes the value 0 on [0, 1) and takes value 1 at $x = 1$ yields the function $(\overline{1})^*$ taking value 2 on the interval [0, 1) and value 1 at $x = 1$. A complete account is found in [4].

Using this technique, it is shown in [4] that the lattice L is isomorphic to the lattice D of all decreasing functions from [0, 1] to [0, 2] having 1 as an accumulation point when ordered under the usual pointwise ordering. It is also shown that L_1 is isomorphic to the lattice D_1 of all functions in D taking the value 1, and L_u is isomorphic to the lattice D_u of functions in D_1 that are *band semicontinuous*, meaning that the inverse image $f^{-1}[1-\varepsilon, 1+\varepsilon]$ of each band centered at 1 is closed.

In this note we show that the representation of L_u can be further improved. Let X be the set of all decreasing functions from the interval [0, 1] to [0, 2] under the pointwise ordering. Then for each f in X there is a unique band semicontinuous function agreeing with f almost everywhere (abbreviated: a.e.). So for Θ the relation on X of equivalence a.e., each equivalence class of Θ contains a unique member of D_u. It follows that D_u, and hence L_u, is isomorphic to the lattice X/Θ of decreasing functions from [0, 1] to [0, 2] modulo equivalence a.e. Certainly this lattice X/Θ seems an object of natural interest. Using this representation we are able to establish a number of further properties of the lattice L_u. We show L_u is complete, and completely distributive. From this, the earlier results that L_u is a distributive continuous lattice and a Heyting algebra follow. We also show that L_u is a compact Hausdorff topological lattice under a topology induced by a metric that is simply described using an integral. So the lattice L_u has a most satisfying collection of properties.

It seems natural in the study of type-2 fuzzy sets to consider the notion of equivalence a.e. directly in the context of the lattice L of convex normal functions. One immediately sees that care is required as the bounds $\overline{0}$ and $\overline{1}$ of the lattice L both take value 0 at all but a single point, hence agree a.e. So the relation of equivalence a.e. provides an equivalence relation on L, but not a lattice congruence, and is of limited use. The correct notion seems to come from a strengthening of this relation. We say two convex normal functions f, g agree *convexly almost everywhere* (abbreviated: c.a.e.) if f and g agree a.e. and their intervals of increase and decrease agree a.e. We write $f\Phi g$ if f and g agree c.a.e. For the isomorphism $* : L \to D$ that straightens out convex functions, we have f and g agree c.a.e. if and only if f^* and g^* agree a.e. It follows that Φ is a congruence on L and that L/Φ is isomorphic to D/Θ, which in turn is isomorphic to X/Θ, and hence to L_u. So our lattice L_u is isomorphic also to the lattice of convex normal functions modulo equivalence c.a.e., a lattice that seems natural in considerations of type-2 fuzzy sets.

This paper is organized in the following manner. In the second section we derive some basic properties of the relation Θ of equivalence a.e. on X and show X/Θ is a complete, completely distributive lattice. In the third section we use results of Birkhoff [2] to show the pseudometric $d(f, g) = \int |f(x) - g(x)|\, dx$ on X yields a metric on X/Θ, and that the metric topology on X/Θ is a compact Hausdorff topology making X/Θ a topological lattice. In the fourth, and final, section, we show the lattice X/Θ of decreasing functions modulo equivalence a.e., the lattice L_u of convex upper semicontinuous convex normal functions under convolution order, and the lattice L/Φ of convex normal functions under convolution order modulo equivalence c.a.e. are pairwise isomorphic.

2. The lattice X/Θ

In this section we establish the properties of the lattice of decreasing functions from [0, 1] to [0, 2] modulo equivalence almost everywhere (a.e.). The term decreasing is not meant to imply strictly decreasing, and may be replaced by non-increasing if one desires. Also regarding terminology, the term *countable* refers to a set that is either finite or equipotent with the natural numbers. The key fact used repeatedly in this section is that a decreasing function has only countably many points of discontinuity, and all such discontinuities are jump discontinuities [6].

Definition 2.1. Let X be the set of decreasing functions from [0, 1] to [0, 2].

As the pointwise meet and join of decreasing functions are decreasing, X is a sublattice of the completely distributive lattice $[0,2]^{[0,1]}$ that is closed under arbitrary meets and joins. It follows that X is a complete, completely distributive lattice.

Lemma 2.2. *For $f, g \in X$, these are equivalent.*

(1) *f and g agree a.e.*
(2) *f and g agree on a dense set.*
(3) *f and g agree except at countably many points.*

Proof. $3 \Rightarrow 1 \Rightarrow 2$ is trivial. For $2 \Rightarrow 3$ define $h(x) = |f(x) - g(x)|$ and let C be the set of all points where both f and g are continuous. Then h is continuous at each point of C and $[0,1] \setminus C$ is countable. We claim $h = 0$ on C, which will establish the result. If not, there is $x \in C$ with $h(x) = \varepsilon > 0$. By continuity, there is an interval around x with $h > \varepsilon/2$ on this interval. But then f and g agree at no points of the interval, contrary to their agreeing on a dense set. □

Definition 2.3. Let Θ be the relation on X defined by $f\Theta g$ if $f = g$ a.e.

It is well-known, and easily seen that Θ is a congruence on the lattice X. We show more that Θ is compatible with arbitrary meets and joins in X.

Lemma 2.4. *If f_i $(i \in I)$ is a family of elements of X, then*

(1) *$(\bigvee f_i)/\Theta$ is the least upper bound of the family f_i/Θ $(i \in I)$.*
(2) *$(\bigwedge f_i)/\Theta$ is the greatest lower bound of the family f_i/Θ $(i \in I)$.*

Thus X/Θ is complete, $(\bigvee f_i)/\Theta = \bigvee(f_i/\Theta)$ and $(\bigwedge f_i)/\Theta = \bigwedge(f_i/\Theta)$.

Proof. Let $f = \bigvee f_i$. For each $i \in I$ we have $f_i \leqslant f$ everywhere, so $f_i/\Theta \leqslant f/\Theta$. Thus f/Θ is an upper bound of this family. As we are in a lattice, to show this element is the least upper bound, it is enough to show that if $g \in X$ is such that $g/\Theta < f/\Theta$, then g/Θ is not an upper bound of this family. For such g, we may assume $g < f$ by considering $g' = f \wedge g$ if necessary. Then as $g/\Theta < f/\Theta$ we cannot have g and f agree on a dense set, so there is an open interval $(x - \varepsilon, x + \varepsilon)$ on which $g < f$. There must be points in this interval where g is continuous, and it does no harm to assume g is continuous at x. As $g < f$ and $f = \bigvee f_i$, there is some $i \in I$ and $\lambda > 0$ with $f_i(x) = g(x) + \lambda$. By continuity, there is some $0 < \varepsilon' < \varepsilon$ with $g(y) < g(x) + \lambda/2$ for all $y \in (x - \varepsilon', x + \varepsilon')$. As f_i is decreasing, for all $y \in (x - \varepsilon', x)$ we have $g(y) < g(x) + \lambda/2 < f_i(x) \leqslant f_i(y)$. So it is not the case that $f_i/\Theta \leqslant g/\Theta$, showing g/Θ is not an upper bound of this family. □

Definition 2.5. A lattice L is completely distributive if whenever we have a set I, and for each $i \in I$ a set J_i, and for each $i \in I$ and $j \in J_i$ an element $a_{ij} \in L$, we have $\bigvee_{i \in I} \bigwedge_{j \in J_i} a_{ij} = \bigwedge_{\alpha \in \prod_I J_i} \bigvee_{i \in I} a_{i\alpha(i)}$ and $\bigwedge_{i \in I} \bigvee_{j \in J_i} a_{ij} = \bigvee_{\alpha \in \prod_I J_i} \bigwedge_{i \in I} a_{i\alpha(i)}$.

Corollary 2.6. *X/Θ is a complete, completely distributive lattice.*

Proof. This is a simple consequence of the previous lemma as the lattice X is completely distributive. □

3. The topology on X/Θ

Following Birkhoff [2], a *valuation* on a lattice L is a map $v : L \to \mathbb{R}$ satisfying $v(x)+v(y) = v(x \vee y)+v(x \wedge y)$. It is called *isotone* if $x \leqslant y$ implies $v(x) \leqslant v(y)$, and *positive* if $x < y$ implies $v(x) < v(y)$.

Definition 3.1. Define $v : X \to \mathbb{R}$ by $v(f) = \int_0^1 f(x)\,\mathrm{d}x$.

A bit of basic analysis provides the following.

Proposition 3.2. *The map v is an isotone valuation on X.*

For $f, g \in X$, set $d(f, g) = v(f \vee g) - v(f \wedge g) = \int_0^1 |f(x) - g(x)|\,\mathrm{d}x$ and let Φ be the relation on X defined by $f \Phi g$ if $d(f, g) = 0$. Then by Birkhoff [2], Φ is a lattice congruence and X/Φ is a metric lattice in the sense of [2]. Further, setting $D(f/\Phi, g/\Phi) = d(f, g)$, we have D is a metric on X/Φ in the sense commonly used in analysis, and under this metric the operations $\wedge$ and $\vee$ are uniformly continuous. But by basic analysis, $\int_0^1 |f(x) - g(x)|\,\mathrm{d}x = 0$ if and only if $f = g$ a.e., and hence $\Theta = \Phi$.

Corollary 3.3. *X/Θ is a metric space under the metric*

$$D(f/\Theta, g/\Theta) = \int_0^1 |f(x) - g(x)|\,\mathrm{d}x$$

and under this metric, meet and join are uniformly continuous.

We will show this topology on X/Θ is compact, but first a lemma.

Lemma 3.4. *For $f \in X$ and $\varepsilon > 0$ there is a natural number n and $\delta > 0$ so that for any $g \in X$ with $|g(i/n) - f(i/n)| < \delta$ for each $i = 0, \ldots, n$, we have $d(f, g) < \varepsilon$.*

Proof. Choose n so that $1/n < \varepsilon/4$ and let $\delta = \varepsilon/2$. For $i = 0, \ldots, n$ let $x_i = i/n$ and $y_i = f(x_i)$, and for $i = 1, \ldots, n$ let J_i be the interval $[x_{i-1}, x_i]$. Consider the behavior of f and g on the interval J_i. As f is decreasing we have $y_i \leqslant f \leqslant y_{i-1}$ on J_i. As $g(x_{i-1})$ is within δ of y_{i-1}, $g(x_i)$ is within δ of y_i, and g is decreasing, we have $y_i - \delta < g < y_{i-1} + \delta$ on J_i. So $|f - g| < y_{i-1} - y_i + \delta$ on J_i. Thus

$$\int_0^1 |f - g|\,\mathrm{d}x < \frac{1}{n}(y_0 - y_1 + \delta) + \frac{1}{n}(y_1 - y_2 + \delta) + \cdots + \frac{1}{n}(y_{n-1} - y_n + \delta).$$

So $d(f, g) < (1/n)(y_0 - y_n) + \delta$, and as y_0, y_n lie between 0 and 2, $d(f, g) < \varepsilon$. □

Proposition 3.5. *The metric topology on X/Θ is compact.*

Proof. With the usual topology, [0, 2] is compact, so $T = [0, 2]^{[0,1]}$ is compact in the product topology. We first show X is a closed subspace of T. Suppose $f \notin X$. Then there are $x < y$ with $f(x) < f(y)$, so $f(y) = f(x) + \varepsilon$ for some $\varepsilon > 0$. The set of all $g \in T$ lying within $\varepsilon/2$ of f in both the x and y coordinates is an open cylinder in T that contains f but does not contain any decreasing function. So X is closed in T, hence is compact under the subspace topology.

We next show that the canonical quotient map $\kappa : X \to X/\Theta$ is continuous with respect to the subspace topology on X and the metric topology on X/Θ. For $f \in X$ and $\varepsilon > 0$ we seek an open neighborhood of f in X mapped by κ into the ball in X/Θ of radius ε centered at f/Θ. This is precisely what is provided by Lemma 3.4. Then as X is compact, and κ is onto and continuous, it follows that X/Θ is compact. □

4. The isomorphisms between X/Θ, L/Φ, and L_u

In [4] we showed that the lattice L_u of convex strictly normal upper semicontinuous functions is isomorphic to the lattice D_u of decreasing functions f from [0, 1] to [0, 2] that take value 1 and are band semicontinuous, meaning that $f^{-1}[1-\varepsilon, 1+\varepsilon]$ is closed for each $\varepsilon > 0$. Our first task is to show X/Θ is isomorphic to D_u, and hence is also isomorphic to L_u.

Definition 4.1. For $f \in X$ and $a \in [0,1]$ let

(1) $f(a^-) = \lim_{x\to a^-} f(x)$;
(2) $f(a^+) = \lim_{x\to a^+} f(x)$.

We next provide a result that describes when a function $f \in X$ belongs to D_u. Informally, it says that at each jump discontinuity, the value f attains must be the one as close as possible to the line $y = 1$. The proof is similar to that of the well known fact that a decreasing function is upper semicontinuous if and only if it is continuous from the left, and we omit it.

Lemma 4.2. *For $f \in X$ we have $f \in D_u$ if and only if the following hold.*

(1) *If $f(0^+)\geqslant 1$, then $f(0) = f(0^+)$, otherwise $f(0) = 1$.*
(2) *If $f(1^-)\leqslant 1$, then $f(1) = f(1^-)$, otherwise $f(1) = 1$.*
(3) *If $f(a^-)\leqslant 1$, then $f(a) = f(a^-)$.*
(4) *If $f(a^+)\geqslant 1$, then $f(a) = f(a^+)$.*
(5) *If $f(a^-) > 1$ and $f(a^+) < 1$, then $f(a) = 1$.*

Proposition 4.3. *For each $f \in X$ there is a unique $f^\dagger \in D_u$ that agrees with f a.e. Further, the mapping $\dagger : X \to D_u$ is an idempotent lattice endomorphism, i.e. a retraction.*

Proof. To produce such $f^\dagger$ one modifies the values of f at 0, 1 and any of the countably many jump discontinuities to comply with the conditions of the above lemma. Namely, if $f(0^+)\geqslant 1$, we set $f^\dagger(0) = f(0^+)$ and otherwise set $f^\dagger(0) = 1$, and so forth. The resulting $f^\dagger$ is seen to be decreasing. So $f^\dagger(0^+)$, $f^\dagger(1^-)$, $f^\dagger(a^-)$, and $f^\dagger(a^+)$ exist for all $0 < a < 1$. As $f^\dagger$ agrees with f at all but countably many points, these values agree with $f(0^+)$, $f(1^-)$, $f(a^-)$, and $f(a^+)$ for all $0 < a < 1$. It follows that $f^\dagger$ satisfies the conditions of the above lemma, hence belongs to D_u, and by construction $f^\dagger$ agrees with f a.e.

For uniqueness, suppose g is a function in D_u that agrees with f a.e. Then by Lemma 2.2 we have f and g agree on a dense set, and this implies $f(0^+) = g(0^+)$, $f(1^-) = g(1^-)$, $f(a^-) = g(a^-)$ and $f(a^+) = g(a^+)$ for each $0 < a < 1$. As $g \in D_u$, by the above lemma its values are determined by the values of the $g(0^+)$, $g(1^-)$, $g(a^-)$, and $g(a^+)$, hence g is determined by f.

For the further comments, idempotence is obvious as $f^\dagger$ is a member of D_u that agrees with itself a.e., and hence $f^{\dagger\dagger} = f^\dagger$. To see that $\dagger$ preserves finite meets, note first that finite meets in D_u are given componentwise [4]. So for $f, g \in X$ we have $f^\dagger \wedge g^\dagger$ belongs to D_u, and as $f^\dagger$ agrees with f a.e. and $g^\dagger$ agrees with g a.e., we have $f^\dagger \wedge g^\dagger$ agrees with $f \wedge g$ a.e. Thus $(f \wedge g)^\dagger = f^\dagger \wedge g^\dagger$. That $\dagger$ preserves finite joins follows from symmetry. □

Theorem 4.4. *The lattice D_u, and therefore also L_u, is isomorphic to X/Θ.*

Proof. This follows immediately from Proposition 4.3 as $\dagger : X \to D_u$ is a surjective homomorphism whose kernel is Θ. □

Before considering the next isomorphic realization of L_u, we recall a few facts about the lattice L of convex normal functions. For a function $f \in L$ and a point $x \in [0,1]$, we say x is a point of increase of f if $f(x)\geqslant f(y)$ for all $y\leqslant x$, and x is a point of decrease of f if $f(x)\geqslant f(y)$ for all $x\leqslant y$. As f is convex, each point is either a point of increase or a point of decrease; and as f is normal, a point x is both a point of increase and a point of decrease if, and only if, $f(x) = 1$. Clearly the points of increase form an interval containing 0, and the points of decrease form an interval containing 1.

In [4] we defined a function $* : L \to D$ to *straighten out* convex normal functions by setting

$$f^*(x) = \begin{cases} 2 - f(x) & \text{if } x \text{ is a point of increase,} \\ f(x) & \text{if } x \text{ is a point of decrease.} \end{cases}$$

Definition 4.5. For functions $f, g \in L$ we say f, g agree *convexly almost everywhere* (c.a.e.) if f and g agree a.e. and their intervals of increase and decrease agree a.e. Let Φ be the relation on L given by $f \Phi g$ if f and g agree c.a.e.

Proposition 4.6. *For $f, g \in L$, $f \Phi g$ if and only if $f^* \Theta g^*$.*

Proof. If $f \Phi g$, then for almost all x, we have $f(x) = g(x)$ and x is a point of increase of f if, and only if, it is a point of increase of g. It follows that f^* agrees with g^* a.e. For the converse, suppose f^*, g^* agree a.e. Note f is increasing at x if, and only if, $f^*(x) \geqslant 1$ and f is decreasing at x if, and only if, $f(x) \leqslant 1$. It follows that the intervals of increase and decrease of f, g agree a.e. and that f, g agree a.e. □

Theorem 4.7. *L/Φ is isomorphic to D/Θ, and hence also to X/Θ.*

Proof. From [4] the map $* : L \to D$ is a lattice isomorphism. So by the previous result L/Φ is isomorphic to D/Θ. It was also shown in [4] that D is a sublattice of X that contains D_u. It then follows from Proposition 4.3 that $\dagger : D \to D_u$ is a surjective homomorphism whose kernel is Θ. Thus D/Θ is isomorphic to D_u and hence to X/Θ. □

Corollary 4.8. *Each of the isomorphic lattices L_u, X/Θ, and L/Φ is complete, and completely distributive. Therefore each is a complete Heyting algebra as well as a continuous lattice. Each of these lattices has a natural metric that makes it a compact Hausdorff topological lattice, and this compact Hausdorff topology agrees with its Lawson topology.*

Proof. As these lattices are isomorphic, it is enough to establish these results for any one of them. In Section 2 we showed X/Θ is complete and completely distributive. This obviously shows it is a Heyting algebra, and by [3, p. 85, Cor I-2.9] this also implies it is a continuous lattice. In Section 3 we showed there is a metric on X/Θ giving a compact Hausdorff topology under which the lattice operations $\wedge, \vee$ are even uniformly continuous. By [3, p. 85, Cor I-2.9], on any continuous lattice the Lawson topology is the unique compact Hausdorff topology making $\wedge$ continuous, therefore the metric topology and the Lawson topology agree. □

Throughout, we have considered our structures only in terms of the bounded lattice operations. One can, however, equip each lattice with an additional operation $\neg$ called negation. On X we define $(\neg f)(x) = 2 - f(1-x)$. One easily sees that $\neg$ is order inverting and of period two, thus an antiautomorphism of X. So with this operation X is a De Morgan algebra [1]. Clearly this operation is compatible with the congruence Θ of agreement a.e., so yields a De Morgan negation also on X/Θ. It is simple to see $d(f, g) = d(\neg f, \neg g)$, so the De Morgan negation is uniformly continuous with respect to the natural metric on X/Θ. Summarizing, we have the following.

Theorem 4.9. *Each of the lattices L_u, X/Θ and L/Φ has a De Morgan negation that is uniformly continuous with respect to the metric topology. In particular, each of these lattices is self-dual.*

Finally, we remark that the lattice X/Θ of decreasing functions modulo equivalence a.e. is a natural object of study. Additionally, it has a large number of very attractive order theoretic and topological properties. It would be of interest to see if there is some abstract characterization of this lattice, perhaps in terms of some kind of universal property.

Acknowledgments

We thank Ted Stanford for helpful discussions on topological matters, and Sándor Jenei, who all along thought quotients of convex normal functions were of interest.

References

[1] R. Balbes, Ph. Dwinger, Distributive Lattices, University of Missouri Press, 1974.
[2] G. Birkhoff, Lattice Theory, vol. 25, American Mathematical Society Colloquium Publications, 1995 (First edition, 1940).
[3] G. Gierz, K.H. Hofmann, K. Keimal, J.D. Lawson, M.W. Mislove, D.S. Scott, A Compendium of Continuous Lattices, Springer, Berlin, 1980.
[4] J. Harding, C. Walker, E. Walker, Lattices of convex, normal functions, Fuzzy Sets and Systems 159 (2008) 1061–1071.
[5] M. Kawaguchi, M. Miyakoshi, Extended t-norms as logical connectives of fuzzy truth values, Multiple-Valued Logic 8 (1) (2002) 53–69.
[6] W. Rudin, Principles of Mathematical Analysis, International Series in Pure and Applied Mathematics, McGraw-Hill, New York, 1976.
[7] C. Walker, E. Walker, The algebra of fuzzy truth values, Fuzzy Sets and Systems 149 (2005) 309–347.
[8] L. Zadeh, The concept of a linguistic variable and its application to approximate reasoning, Inform Sci. 8 (1975) 199–249.