Lattices of Convex Normal Functions

John Harding, Carol Walker, Elbert Walker

Department of Mathematical Sciences

New Mexico State University

Las Cruces, New Mexico 88003, USA

jharding@nmsu.edu,hardy@nmsu.edu,elbert@nmsu.edu

Abstract

The algebra of truth values of type-2 fuzzy sets is the set of all functions from the unit interval into itself, with operations defined in terms of certain convolutions of these functions with respect to pointwise max and min. This algebra has been studied rather extensively, both from a theoretical and from a practical point of view. It has a number of interesting subalgebras, and this paper is about the subalgebra of all convex normal functions, and closely related ones. These particular algebras are De Morgan algebras, and our concern is principally with their completeness as lattices. A special feature of our treatment is a representation of these algebras as monotone functions with pointwise order, making the operations more intuitive.

Key words: Type-2 fuzzy sets, normal and convex fuzzy sets, complete lattice, De Morgan algebra, continuous lattice

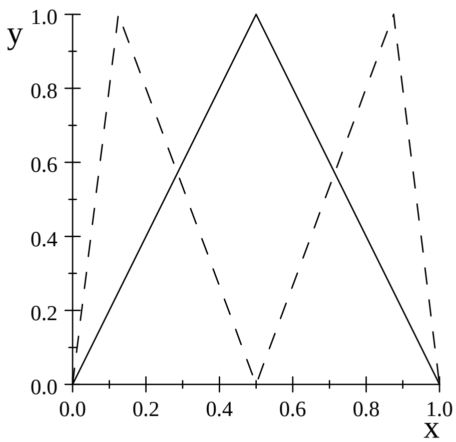
1 Introduction

The subject of this paper is three De Morgan algebras, and our concern is principally with their completeness as lattices and the question of their continuity in the sense of Gierz et al. [3,4]. These algebras arose as subalgebras of the algebra of truth values for fuzzy sets of type-2 [11], the set of all mappings of $[0, 1]$ into $[0, 1]$ with operations certain convolutions of operations on $[0, 1]$, as follows.

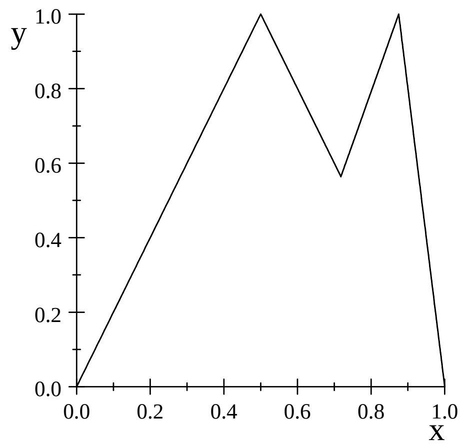
Definition 1 *On $[0,1]^{[0,1]}$, define operations $\sqcup, \sqcap, \neg, \bar{0}, \bar{1}$ as follows:*

(1) $(f \sqcup g)(x) = \sup\{f(y) \wedge g(z) \mid y \vee z = x\}$
(2) $(f \sqcap g)(x) = \sup\{f(y) \wedge g(z) \mid y \wedge z = x\}$
(3) $\neg f(x) = \sup\{f(y) \mid 1 - y = x\} = f(1-x)$
(4) $\bar{1}(x) = \begin{cases} 1 \text{ if } x = 1 \\ 0 \text{ if } x \neq 1 \end{cases}$ *and* $\bar{0}(x) = \begin{cases} 1 \text{ if } x = 0 \\ 0 \text{ if } x \neq 0 \end{cases}$

The algebra $\mathbf{M} = \left([0,1]^{[0,1]}, \sqcup, \sqcap, \neg, \bar{0}, \bar{1}\right)$ is the basic algebra of truth values for type-2 fuzzy sets, and has been studied extensively. See [7–10], for example. Here is one example illustrating the join in $\mathbf{M}$ of two functions f and g.



f (solid line), g (dashed line)



$f \sqcup g$

Determining the properties of the algebra $\mathbf{M}$ is a bit tedious, but is helped by introducing the following auxiliary operations.

Definition 2 *For $f \in \mathbf{M}$, let f^L and f^R be the elements of $\mathbf{M}$ defined by*

$$f^L(x) = \vee_{y \leq x} f(y)$$
$$f^R(x) = \vee_{y \geq x} f(y)$$

One easily sees that f^L and f^R are the pointwise smallest increasing and decreasing functions, respectively, in $\mathbf{M}$ above f. The point of this definition is that the operations $\sqcup$ and $\sqcap$ in $\mathbf{M}$ can be expressed in terms of the pointwise max and min functions and the operations f^L and f^R as follows [9].

Theorem 1 *The following hold for all $f, g \in \mathbf{M}$.*

$$f \sqcup g = \left(f \wedge g^L\right) \vee \left(f^L \wedge g\right) = (f \vee g) \wedge f^L \wedge g^L$$
$$f \sqcap g = \left(f \wedge g^R\right) \vee \left(f^R \wedge g\right) = (f \vee g) \wedge f^R \wedge g^R$$

Using these auxiliary operations, it is fairly routine to verify the following properties of the algebra $\mathbf{M}$. Details may be found in [6,9].

Corollary 2 *Let $f,\ g,\ h \in \mathbf{M}$. The basic properties of $\mathbf{M}$ are these.*

(1) $f \sqcup f = f$; $f \sqcap f = f$
(2) $f \sqcup g = g \sqcup f$; $f \sqcap g = g \sqcap f$
(3) $\bar{1} \sqcap f = f$; $\bar{0} \sqcup f = f$
(4) $f \sqcup (g \sqcup h) = (f \sqcup g) \sqcup h$; $f \sqcap (g \sqcap h) = (f \sqcap g) \sqcap h$
(5) $f \sqcup (f \sqcap g) = f \sqcap (f \sqcup g)$
(6) $\neg\neg f = f$
(7) $\neg (f \sqcup g) = \neg f \sqcap \neg g$; $\neg (f \sqcap g) = \neg f \sqcup \neg g$

The **variety** generated by $\mathbf{M}$ is the collection of all homomorphic images of subalgebras of products of copies of $\mathbf{M}$, or equivalently, the collection of all algebras of the same type (i.e., having two binary operations, one unary operation, and two constants), that satisfy all of the equations satisfied in $\mathbf{M}$. As far as we know, the variety generated by $\mathbf{M}$ has not been studied. It seems not to be known whether or not every equation satisfied by $\mathbf{M}$ is a consequence of those listed in Corollary 2, and whether or not the variety generated by $\mathbf{M}$ is generated by a finite algebra.

A subalgebra of $\mathbf{M}$ is a subset of $\mathbf{M}$ that is closed under all of the operations of $\mathbf{M}$. The algebra $\mathbf{M}$ has a number of interesting subalgebras [9]. Our interest here lies in subalgebras of convex normal functions.

Definition 3 *A function $f \in \mathbf{M}$ is* ***normal*** *if* $\sup\{f(x) : x \in [0,1]\} = 1$*, or equivalently, if $f^L \vee f^R = 1$.*

The set of normal functions forms a subalgebra of $\mathbf{M}$ [9].

Definition 4 *A function $f \in \mathbf{M}$ is* ***convex*** *if for all $x, y, z \in M$ for which $x \leq y \leq z$, we have $f(y) \geq f(x) \wedge f(z)$. Equivalently, f is convex if $f = f^L \wedge f^R$.*

The set of convex functions forms a subalgebra of $\mathbf{M}$ [9].

A **De Morgan algebra** [1] is a bounded distributive lattice with a negation in which the De Morgan laws hold. A subalgebra of $\mathbf{M}$ satisfies all of the properties of Corollary 2. In order for it to be a De Morgan algebra, it must also satisfy the absorption law and distributive laws:

(5') $f \sqcup (f \sqcap g) = f \sqcap (f \sqcup g) = f$
(8) $f \sqcup (g \sqcap h) = (f \sqcup g) \sqcap (f \sqcup g)$ and $f \sqcap (g \sqcup h) = (f \sqcap g) \sqcup (f \sqcap g)$

The following result is found in [6–9].

Theorem 3 *Let* $\mathbf{L}$ *be the subalgebra of convex normal functions. This subalgebra is a De Morgan algebra under the operations* $\sqcup$ *and* $\sqcap$, $\neg$, *and the constants* $\bar{0}$ *and* $\bar{1}$.

A lattice is **complete** if every subset has both a join (least upper bound) and a meet (greatest lower bound). It is a well-known fact of order theory that it is sufficent to show the existence of joins to establish completeness. It is the purpose of this note to investigate questions related to the completeness of $\mathbf{L}$ with respect to the operations $\sqcup$ and $\sqcap$, and of two subalgebras of $\mathbf{L}$. In Section 2, we present our primary tool in this investigation—we show $\mathbf{L}$ is isomorphic to the set $\mathbf{D}$ of all decreasing functions from $[0,1]$ to $[0,2]$ that have 1 as an accumulation point of their image. This allows us to replace the rather complicated definition of the order on $\mathbf{L}$ with the usual pointwise ordering of functions in $\mathbf{D}$.

In Section 3, we use the tools developed in Section 2 to show that $\mathbf{L}$ is a complete distributive lattice. We also show the sublattice $\mathbf{L}_1$ of convex normal functions that attain the value 1 is a complete distributive lattice by similar methods. While both $\mathbf{L}$ and $\mathbf{L}_1$ are complete and distributive, neither satisfies the infinite distributive law $x \wedge (\bigvee y_j) = \bigvee(x \wedge y_j)$, hence neither is a Heyting algebra [1].

In Section 4 we also consider the lattice $\mathbf{L}_u$ of convex normal functions that are upper semicontinuous. This lattice was considered in [5] where it was shown to form a complete Heyting algebra. Using again the techniques from Section 2, we give a proof of the completeness of $\mathbf{L}_u$. In Section 5, we continue to show $\mathbf{L}_u$ is a continuous lattice in the sense of Gierz et al. [4]. This has as a consequence that $\mathbf{L}_u$ is a Heyting algebra.

This paper concludes with a few remarks and questions in Section 6.

2 Another view of the order

As $\mathbf{L}$ is a lattice under the operations $\sqcap, \sqcup$ of $\mathbf{M}$, it has a partial ordering $\sqsubseteq$ defined through this lattice structure. It is the purpose of this section to realize $\sqsubseteq$ in another, more intuitive way. The idea is simply to "straighten out" convex functions to obtain monotone ones, then to consider the ordinary pointwise ordering on these monotone functions. Before beginning this process, we make a simple observation about the ordering $\sqsubseteq$.

Proposition 4 *For $f, g \in \mathbf{L}$, these are equivalent.*

(1) $f \sqsubseteq g$
(2) $g^L \leq f^L$ *and* $f^R \leq g^R$

Proof. Assume that $f \sqsubseteq g$, so that $f \sqcap g = f$ and $f \sqcup g = g$. By Theorem 1, $f \sqcap g = (f \vee g) \wedge g^R \wedge f^R$ and $f \sqcup g = (f \vee g) \wedge f^L \wedge g^L$. As $f \sqcap g = f$, we have $f \leq g^R$, whence $f^R \leq g^R$, and as $f \sqcup g = g$, we have $g \leq f^L$, whence $g^L \leq f^L$.

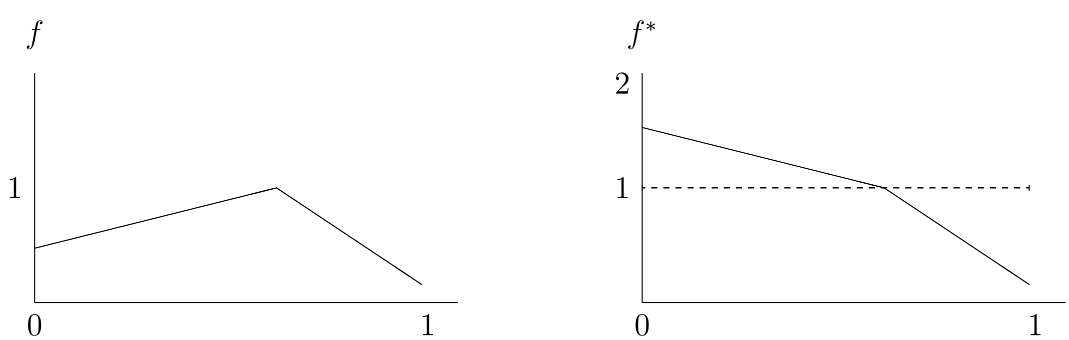
Now assume $g^L \leq f^L$ and $f^R \leq g^R$. As $f \sqcap g = (f \vee g) \wedge g^R \wedge f^R$ and $f^R \leq g^R$, we have, $f \sqcap g = (f \vee g) \wedge f^R$. Clearly $f \leq (f \vee g) \wedge f^R$. But $g \leq g^L \leq f^L$, so $(f \vee g) \wedge f^R \leq f^L \wedge f^R = f$. Thus $f \sqcap g = f$, showing $f \sqsubseteq g$. ■

We recall that I represents the closed unit interval $[0, 1]$ and we shall use I^* for the closed interval $[0, 2]$. The key feature of I^* is that it can be viewed as I with a copy of the dual of I on top, with the top element of I and the bottom element of the dual copy of I identified. We now make precise the idea of "straightening out" a convex function f.

Definition 5 *For $f : I \to I$ define $f^* : I \to I^*$ by setting*

$$f^*(x) = \begin{cases} 2 - f(x) & \text{if } f(x) = f^L(x) \\ f(x) & \text{otherwise} \end{cases}$$

While defined for any function, we only consider f^* in the case that f is convex and normal. Roughly, f^* is produced by taking the mirror image of the increasing portion of f about the line $y = 1$, and leaving the remainder of f alone. The following diagram illustrates the situation.



While we consider the convolution ordering $\sqsubseteq$ on $\mathbf{L}$, we shall consider the ordinary pointwise ordering of functions $\leq$ for functions from I to I^*. Our key result follows below. In its proof, and elsewhere, we use repeatedly two consequences of convexity and normality—for each x in I, at least one of $f^L(x)$ and $f^R(x)$ equals $f(x)$, and at least one of $f^L(x)$ and $f^R(x)$ equals 1.

Proposition 5 *For $f, g \in \mathbf{L}$, $f \sqsubseteq g$ if and only if $f^* \leq g^*$.*

Proof. Assume $f \sqsubseteq g$. Consider possibilities for x. First, suppose $g(x) < g^L(x)$. Then $g^*(x) = g(x) = g^R(x) \geq f^R(x) \geq f(x)$ and the strict inequality $f(x) \leq g(x) < g^L(x) \leq f^L(x)$ implies $f^*(x) = f(x)$. It follows that $f^*(x) \leq g^*(x)$.

Now suppose that $g(x) = g^L(x)$. If $f(x) = f^L(x)$ is also true, then $g(x) \leq f(x)$ so $f^*(x) = 2 - f(x) \leq 2 - g(x) = g^*(x)$. If $f(x) < f^L(x)$, then $f^*(x) = f(x) \leq 1 \leq 2 - g(x) = g^*(x)$.

Now assume $f^* \leq g^*$. To show $f \sqsubseteq g$, by Proposition 4 it is enough to show $g^L \leq f^L$ and $f^R \leq g^R$. Again, consider possibilities for x. First suppose $g(x) < g^L(x)$. Then $g^*(x) = g(x) = g^R(x) < 1$ and $g^L(x) = 1$. Also, $f^*(x) \leq g^*(x) < 1$ implies $f^*(x) = f(x) = f^R(x) < f^L(x)$. Thus $f^R(x) \leq g^R(x)$ and $f^L(x) = g^L(x) = 1$.

Now suppose that $g(x) = g^L(x)$, so $g^*(x) = 2 - g(x)$. If we also have $f(x) = f^L(x)$, then $g^*(x) = 2 - g(x) \geq f^*(x) = 2 - f(x)$ implies $g^L(x) = g(x) \leq f(x) = f^L(x)$. Also, in this case, $f^R(x) = g^R(x) = 1$. Finally, we have the situation that $g(x) = g^L(x) \leq g^R(x) = 1$ and $f(x) < f^L(x) = 1$. Then clearly, $g^L(x) \leq f^L(x) = 1$ and $f^R(x) \leq g^R(x) = 1$. ■

We next describe the functions that arise as f^* for some convex normal f.

Proposition 6 *For $g : I \to I^*$ these are equivalent.*

(1) $g = f^*$ for some $f \in \mathbf{L}$.
(2) g is decreasing and 1 is an accumulation point of the image of g.

Proof. As $f \in \mathbf{L}$, one sees that $J = \left\{x \mid f(x) = f^L(x)\right\}$ is an initial segment of I, that f is increasing on J, and that f is decreasing on $I - J$. As $f^* = 2 - f$ on J, f^* is decreasing on J, and as $f^* = f$ on $I - J$, f^* is decreasing on $I - J$. Then as $f^* \geq 1$ on J and $f^* \leq 1$ on $I - J$, it follows that f^* is decreasing on I. As 1 is the supremum of f, for $\varepsilon > 0$ there is x with $f(x)$ within distance ε of 1. Then both $f(x)$ and $2 - f(x)$ lie within ε of 1, hence $f^*(x)$ lies within ε of 1. So 1 is an accumulation point of the image of f^*.

Now assume that g is decreasing and 1 is an accumulation point of the image of g. Set $f(x) = \min\{g(x), 2 - g(x)\}$. As f is the pointwise meet of a decreasing and increasing function, it is convex. It cannot be that both $g(x)$ and $2-g(x)$ are greater than 1, so $0 \leq f(x) \leq 1$. As 1 is an accumulation point of the image of g, for $\varepsilon > 0$, there is an x with $g(x)$ within ε of 1. Then $g(x)$ and $2-g(x)$ are within ε of 1, hence so is $f(x)$. It follows that f is normal, hence $f \in \mathbf{L}$.

Let $J = \{x \mid 2-g(x) \leq g(x)\}$. Then J is an initial segment of I, $f(x) = 2-g(x)$ on J, and as $2-g(x)$ is increasing, we have $f = f^L$ on J. Thus $f^*(x) = 2-(2-g(x)) = g(x)$ on J. For $x \in I - J$ we have $f(x) = g(x) < 2-g(x)$. In particular, $f(x) < 1$. As f is decreasing on $I-J$ and the supremum of f is 1, there is some $y < x$ with $f(y) > f(x)$. Then $f(x) \neq f^L(x)$, so $f^*(x) = f(x) = g(x)$. ■

Definition 6 *We define sets* $\mathbf{D}$ *and* $\mathbf{D}_1$ *as follows:*

(1) $\mathbf{D} = \{g \mid g : I \to I^*$ *is decreasing and* 1 *is an accumulation point of the image of* $g\}$.

(2) $\mathbf{D}_1 = \{g \mid g : I \to I^*$ *is decreasing and* $g(x) = 1$ *for some* $x \in I\}$.

It is common to consider the collection of all order preserving functions from one poset to another. It is easily seen, and very well known, that the order preserving functions from a poset to a complete lattice form a complete lattice under the pointwise operations. Obviously a similar result holds for order inverting, or decreasing functions. So the sets $\mathbf{D}$ and $\mathbf{D}_1$ form subsets of a rather familiar lattice, the lattice of all decreasing functions from I to I^*.

Proposition 7 *Both* $\mathbf{D}$ *and* $\mathbf{D}_1$ *are sublattices of the lattice of decreasing functions from* I *to* I^*.

Proof. Suppose $f, g \in \mathbf{D}$. We will show $f \wedge g \in \mathbf{D}$ where $\wedge$ is the pointwise meet. This only requires us to show 1 is an accumulation point of the image of $f \wedge g$. Given $\epsilon > 0$ there are x, y with $f(x)$ and $g(y)$ within ϵ of 1. Suppose $x \leq y$. Then $1-\epsilon < f(x) < 1+\epsilon$ and as g is decreasing $1-\epsilon < g(y) \leq g(x)$. It follows that $1-\epsilon < (f \wedge g)(x) < 1+\epsilon$. Similarly, $f \vee g$ also has 1 as an accumulation point of its image. So $\mathbf{D}$ is a sublattice. The argument for $\mathbf{D}_1$ is easier. ■

Theorem 8 $(\mathbf{L}, \sqsubseteq, \sqcap, \sqcup)$ *is isomorphic to* $(\mathbf{D}, \leq, \wedge, \vee)$ *and* $(\mathbf{L}_1, \sqsubseteq, \sqcap, \sqcup)$ *is isomorphic to* $(\mathbf{D}_1, \leq, \wedge, \vee)$.

Proof. In each case, the mapping $f \rightsquigarrow f^*$ provides the required order-isomorphism. ■

3 Completeness of the lattice of convex normal functions

Theorem 8 has the immediate consequence of showing both $\mathbf{L}$ and $\mathbf{L}_1$ are isomorphic to sublattices of the complete lattice $(I^*)^I$ of all functions from I to I^*. We shall show that both $\mathbf{L}$ and $\mathbf{L}_1$ are complete, by showing both $\mathbf{D}$ and $\mathbf{D}_1$ are complete. However, we will see that $\mathbf{D}$ and $\mathbf{D}_1$ are not complete sublattices of $(I^*)^I$ as infinite joins and meets in $\mathbf{D}$ and $\mathbf{D}_1$ are not always pointwise.

Definition 7 *For $a \in I$ let 1_a and $1'_a$ be functions from I to I^* where 1_a takes value 1 at a and 0 otherwise, while $1'_a$ takes value 1 at a and value 2 otherwise.*

Theorem 9 *Let $(f_j)_J$ be a family in $\mathbf{D}$ and set $a = \sup\{x \mid \bigvee_J f_j(x) \geq 1\}$ and $b = \sup\{x \mid \bigwedge_J f_j(x) \geq 1\}$. Then*

(1) If $\bigvee_J f_j$ belongs to $\mathbf{D}$ this is the join of this family in $\mathbf{D}$. Otherwise $\bigvee_J f_j \vee 1_a$ is the join of this family in $\mathbf{D}$.
(2) If $\bigwedge_J f_j$ belongs to $\mathbf{D}$ this is the meet of this family in $\mathbf{D}$. Otherwise $\bigwedge_J f_j \wedge 1'_b$ is the meet of this family in $\mathbf{D}$.

Proof. Let $f = \bigvee_J f_j$. If $f \in \mathbf{D}$ then it is surely the least upper bound of this family in $\mathbf{D}$. If f does not belong to $\mathbf{D}$, then as f is decreasing, this can only be because 1 is not an accumulation point of the image of f. So there is some $\epsilon > 0$ bounding the image of f away from 1. By the definition of a we have $f(x) \geq 1$ for all $x < a$, and $f(x) < 1$ for all $x > a$. Thus $f(x) \geq 1 + \epsilon$ for all $x < a$ and $f(x) \leq 1 - \epsilon$ for all $x > a$. We claim $f(a) \leq 1 - \epsilon$. Otherwise, $f(a) \geq 1+\epsilon$. So there would be f_j with $f_j(a) \geq 1+\epsilon/2$. But as f_j is decreasing this would yield $f_j(x) \geq 1 + \epsilon/2$ for all $x \leq a$, and as $f_j \leq f$ we would have $f_j(x) \leq 1 - \epsilon$ for all $x > a$. This would contradict 1 being an accumulation point of the image of f_j. Thus $f(a) \leq 1-\epsilon$. It follows that $f \vee 1_a$ is decreasing and takes value 1 at a, hence belongs to $\mathbf{D}$, and is clearly an upper bound of the family $(f_j)_J$. Suppose $g \in \mathbf{D}$ is another upper bound of this family. Then $g \geq f$, so $g(x) \geq 1 + \epsilon$ for all $x < a$. As 1 is an accumulation point of the image of g, it must be an accumulation point of the image of the restriction of g to the interval $[a, 1]$. Then as g is decreasing, we must have $g(a) \geq 1$. Thus $g \geq f \vee 1_a$. This shows $f \vee 1_a$ is the least upper bound of this family in $\mathbf{D}$. The proof of the second statement is similar. ■

Theorem 10 *Let $(f_j)_J$ be a family in $\mathbf{D}_1$ and set $a = \sup\{x \mid \bigvee_J f_j(x) \geq 1\}$ and $b = \sup\{x \mid \bigwedge_J f_j(x) \geq 1\}$. Then*

(1) If $\bigvee_J f_j$ belongs to $\mathbf{D}_1$ this is the join of this family in $\mathbf{D}_1$. Otherwise $\bigvee_J f_j \vee 1_a$ is the join of this family in $\mathbf{D}_1$.
(2) If $\bigwedge_J f_j$ belongs to $\mathbf{D}_1$ this is the meet of this family in $\mathbf{D}_1$. Otherwise $\bigwedge_J f_j \wedge 1'_b$ is the meet of this family in $\mathbf{D}_1$.

Proof. Let $f = \bigvee_J f_j$. If $f \in \mathbf{D}_1$ then it is surely the least upper bound of this family in $\mathbf{D}_1$. If $f \notin \mathbf{D}$, then the previous theorem shows $f \vee 1_a$ is the least upper bound of $(f_j)_J$ in $\mathbf{D}$, and as $f \vee 1_a$ belongs to $\mathbf{D}_1$ it is the least upper bound of this family in $\mathbf{D}_1$ as well. The remaining possibility is that $f \in \mathbf{D}$ but $f \notin \mathbf{D}_1$. In this case we have $f(x) > 1$ for all $x < a$ and $f(x) < 1$ for all $x > a$. We claim $f(a) < 1$. If not, then $f(a) > 1$ would imply there is f_j with $f_j(a) > 1$. Then as f_j is decreasing and $f_j \leq f$ we have $f_j(x) > 1$ for all $x \geq a$ and $f_j(x) < 1$ for all $x > a$. This would mean f_j never takes the value 1 contrary to f_j belonging to $\mathbf{D}_1$. So $f(a) < 1$, and this shows $f \vee 1_a$ is decreasing and takes the value 1, hence belongs to $\mathbf{D}_1$. So $f \vee 1_a$ is an upper bound of this family in $\mathbf{D}_1$. If g is another such upper bound, then $f \leq g$, so $g(x) > 1$ for all $x < a$. Then as g is decreasing and takes the value 1 we must have $g(a) \geq 1$, hence $f \vee 1_a \leq g$. So $f \vee 1_a$ is the least upper bound of this family in $\mathbf{D}_1$. The second statement is similar. ■

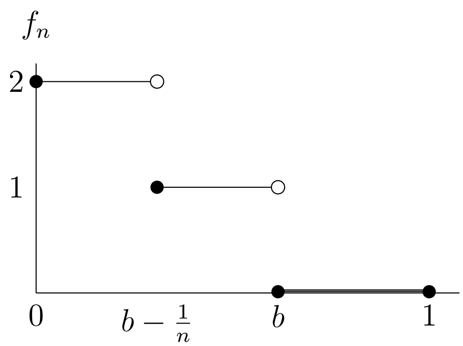
A complete distributive lattice is meet-continuous if $x \wedge \bigvee y_j = \bigvee(x \wedge y_j)$. It is well known that a complete distributive lattice is a Heyting algebra if and only if it is meet-continuous [1]. A distributive lattice that is continuous in the sense of Gierz, et al. (see Section 5) is also meet-continuous [3] (p. 57) or [4] (p. 56). We show that neither the complete lattice $\mathbf{L}$ of convex normal functions, nor the complete lattice $\mathbf{L}_1$ of convex strictly normal functions, is meet-continuous. It follows that neither is a Heyting algebra, nor a continuous lattice. Again, it is more convenient to work through their isomorphic copies $\mathbf{D}$ and $\mathbf{D}_1$.

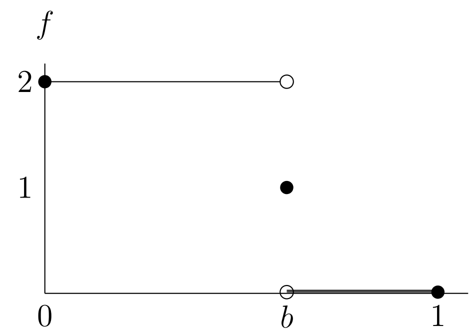
Proposition 11 *Neither* $\mathbf{D}$ *nor* $\mathbf{D}_1$ *satisfies* $x \wedge \bigvee y_j = \bigvee(x \wedge y_j)$.

Proof. Suppose $0 < b < 1$ and define a family of functions f_n by setting

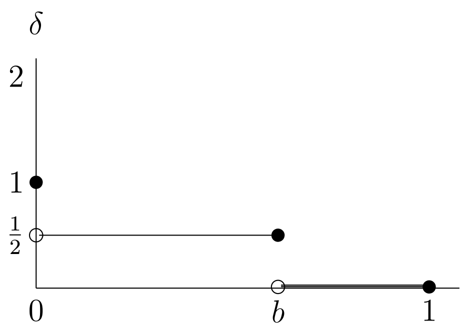
$$f_n(x) = \begin{cases} 2 \text{ if } x < b - \frac{1}{n} \\ 1 \text{ if } b - \frac{1}{n} \leq x < b \\ 0 \text{ if } b \leq x \end{cases}$$

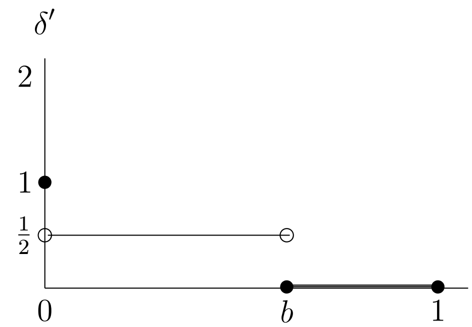
Here we only define f_n for n large enough to ensure $b - \frac{1}{n} > 0$. Note that these functions are decreasing and take the value 1, hence belong to $\mathbf{D}_1$ and therefore also to $\mathbf{D}$. As the pointwise join $\bigvee f_n$ takes only the values 0 and 2 it does not belong to $\mathbf{D}$. Therefore $f = \bigvee f_n \vee 1_b$ is the join of the family f_n in both $\mathbf{D}$ and $\mathbf{D}_1$. These functions are shown below.





Consider the functions δ and δ' shown below. Surely both are decreasing and take the value 1, hence belong to both $\mathbf{D}$ and $\mathbf{D}_1$.





It is clear that $\delta \leq f$, hence $\delta \wedge f = \delta$, while $\delta \wedge f_n = \delta'$ for each n. Writing $\bigcup f_n$ for the join of the family f_n in either $\mathbf{D}$ or $\mathbf{D}_1$, we have $\delta \wedge \bigcup f_n = \delta > \delta' = \bigcup \delta \wedge f_n$ in both $\mathbf{D}$ and $\mathbf{D}_1$. Thus neither lattice satisfies the indicated distributive law. ∎

4 Completeness and semicontinuity

A function f is **lower semicontinuous** if the set $\{x \mid f(x) \leq \alpha\}$ is closed for each real α, and **upper semicontinuous** if each set $\{x \mid f(x) \geq \alpha\}$ is closed. These are abbreviated as LSC and USC respectively. It is well known [3,4] that the lower semicontinuous functions from I to I, or for that matter from I to I^*, form a very special type of complete distributive lattice known as continuous distributive lattice. In particular, this lattice of lower semicontinuous functions is a complete Heyting algebra. Dually, the upper semicontinuous functions from I to itself form a dual continuous lattice, hence a complete dual Heyting algebra. In this section we consider semicontinuity in the setting of convex normal functions.

Definition 8 *Let $\mathbf{L}_U$ be the set of convex, normal, upper semicontinuous functions from I to I.*

This collection of functions has been studied in [5] where it was shown to form a complete Heyting algebra. Here we consider it from the perspective of decreasing functions from I to I^* and establish again that $\mathbf{L}_U$ forms a complete Heyting algebra. In the following section will use these results to show further that $\mathbf{L}_U$ forms a continuous lattice.

Proposition 12 *For a function $g \in \mathbf{D}$ these are equivalent.*

(1) $g = f^*$ for some $f \in \mathbf{L}_U$.
(2) $g^{-1}[\alpha, 2-\alpha]$ is closed for each $0 \leq \alpha \leq 1$.
(3) $g \vee \mathbf{1}$ is LSC *and $g \wedge \mathbf{1}$ is* USC, *where $\mathbf{1}$ is the constant function with value 1.*

Further, these conditions imply that g attains the value 1.

Proof. Let $g = f^*$ where f is upper semicontinuous. So $\{x \mid \alpha \leq f(x)\}$ is closed. We show this set equals $g^{-1}[\alpha, 2-\alpha]$. If $f(x) = f^L(x)$ then as $1 \leq 2 - f(x) = f^*(x)$ we have $\alpha \leq f(x)$ if and only if $\alpha \leq f^*(x) \leq 2 - \alpha$. If $f(x) \neq f^L(x)$ then as $f(x) = f^*(x)$ and $1 \leq 2 - \alpha$ we have $\alpha \leq f(x)$ if and only if $\alpha \leq f^*(x) \leq 2 - \alpha$. Thus (1) implies (2).

Now assume $g^{-1}[\alpha, 2-\alpha]$ is closed for each $0 \leq \alpha \leq 1$. To show $g \vee \mathbf{1}$ is LSC it is enough to show $(g \vee \mathbf{1})^{-1}[0, 2-\alpha]$ is closed for each $0 \leq \alpha \leq 1$. As $g \in \mathbf{D}$, we have $\mathbf{1}$ is an accumulation point of the image of g, so $X = g^{-1}[\alpha, 2-\alpha]$ is non-empty. Say $x \in X$. Then as $g \vee \mathbf{1}$ is decreasing, we have $X = g^{-1}[\alpha, 2-\alpha] \cup [x, 1]$, hence is closed. The argument to show $g \wedge \mathbf{1}$ is USC is similar. Thus (2) implies (3).

Finally, assume that $g \vee \mathbf{1}$ is LSC and $g \wedge \mathbf{1}$ is USC. As $g \in \mathbf{D}$, we have shown in Proposition 6 the function f defined by $f(x) = \min\{g(x), 2 - g(x)\}$ belongs to $\mathbf{L}$ and satisfies $g = f^*$. But for $0 \leq \alpha \leq 1$ we have $\alpha \leq f(x)$ if and only if $\alpha \leq g(x)$ and $\alpha \leq 2 - g(x)$, which is equivalent to $\alpha \leq g(x) \leq 2 - \alpha$. Thus $f^{-1}[\alpha, 1] = g^{-1}[\alpha, 2-\alpha]$, hence is closed. So f is USC so belongs to $\mathbf{L}_U$. Thus (3) implies (1).

To see that g attains the value 1, note that for $0 \leq \alpha < 1$ the sets $g^{-1}[\alpha, 2-\alpha]$ form a decreasing family of closed sets. As 1 is an accumulation point of the image of g, we have that each of these sets is non-empty. As I is compact, the intersection of this family of sets is non-empty, providing some x with $g(x) = 1$. ■

Definition 9 *We call a function $g \in \mathbf{D}$ that satisfies the the conditions of Proposition 12 a* ***band semicontinuous*** *function, and let $\mathbf{D}_U$ denote the* *collection of all such g.*

Proposition 13 *$(\mathbf{L}_U, \sqsubseteq)$ is isomorphic to $(\mathbf{D}_U, \leq)$.*

Proof. $\mathbf{D}_U$ is the image of $\mathbf{L}_U$ under the order embedding $f \rightsquigarrow f^*$. $\blacksquare$

Our next task is to explore the lattice properties of $\mathbf{D}_U$, and hence of $\mathbf{L}_U$. As the union and intersection of two closed sets is closed, it follows that the pointwise join and meet of two LSC functions is LSC and the pointwise join and meet of two USC functions is USC. Further, as the intersection of any family of closed sets is closed, the pointwise join of any family of LSC functions is LSC, and the pointwise meet of any family of USC functions is USC. It follows from this that for any function f there is a largest LSC function f^- beneath f, the pointwise join of all LSC functions beneath f, and there is a smallest USC function f^+ above f.

Proposition 14 *$\mathbf{D}_U$ is a lattice with finite joins and meets being pointwise.*

Proof. It is enough to show that if $f, g \in \mathbf{D}_U$ then $f \vee g$ and $f \wedge g$ are in $\mathbf{D}_U$. We have seen that both are elements of $\mathbf{D}$. Note $(f \vee g) \vee \mathbf{1} = (f \vee \mathbf{1}) \vee (g \vee \mathbf{1})$ and $(f \vee g) \wedge \mathbf{1} = (f \wedge \mathbf{1}) \vee (g \wedge \mathbf{1})$. By part 3 of Proposition 12 it follows that $(f \vee g) \vee \mathbf{1}$ is the join of two LSC functions, hence is LSC, and $(f \vee g) \wedge \mathbf{1}$ is the join of two USC functions, hence is USC. By Proposition 12 it follows that $f \vee g$ belongs to $\mathbf{D}_U$. The argument showing $f \wedge g$ belongs to $\mathbf{D}_U$ is similar. $\blacksquare$

Corollary 15 *$\mathbf{D}_U$ and $\mathbf{L}_U$ are distributive lattices.*

We next consider the matter of infinite joins and meets in $\mathbf{D}_U$. In the following proposition, f^+ denotes the pointwise meet of all USC functions above f, and f^- denotes the pointwise join of all LSC functions beneath f, as in the discussion before Proposition 14.

Proposition 16 *Let $(f_j)_J$ be a family in $\mathbf{D}_U$ and set $f = \bigvee_J f_j$ and $g = \bigwedge_J f_j$.*

(1) The join of this family in $\mathbf{D}_U$ is $f \vee (f \wedge \mathbf{1})^+$.
(2) The meet of this family in $\mathbf{D}_U$ is $(g \vee \mathbf{1})^- \wedge g$.

Proof. Let $k = f \vee (f \wedge \mathbf{1})^+$ and note k is decreasing. To see that 1 is an accumulation point of its image, suppose that the image of k is disjoint from $(1-\epsilon, 1+\epsilon)$. Let $X = k^{-1}(1+\epsilon/2, 2]$. Note that X is an initial segment of I. Further, as each $f_j(1) \leq 1$ we have $k(1) \leq 1$, so X is a proper segment. As $f \vee \mathbf{1}$ is the join of the LSC functions $f_j \vee \mathbf{1}$, it is LSC, and as $X = (f \vee \mathbf{1})^{-1}(1+\epsilon/2, 2]$ we have X is open. For $x \notin X$ we have $k(x) \leq 1-\epsilon$, so $(f \wedge \mathbf{1})^+(x) \leq 1-\epsilon$. Thus the inverse image of $[1-\epsilon/2, 2]$ under $(f \wedge \mathbf{1})^+$ is the open set X, contrary to $(f \wedge \mathbf{1})^+$ being USC. This shows $k \in \mathbf{D}$.

As $(f \wedge \mathbf{1})^+ \leq \mathbf{1}$ we have $(f \vee (f \wedge \mathbf{1})^+) \vee \mathbf{1} = f \vee \mathbf{1}$ and $(f \vee (f \wedge \mathbf{1})^+) \wedge \mathbf{1} = (f \wedge \mathbf{1}) \vee (f \wedge \mathbf{1})^+ = (f \wedge \mathbf{1})^+$. Thus $k \vee \mathbf{1} = f \vee \mathbf{1}$ and $k \wedge \mathbf{1} = (f \wedge \mathbf{1})^+$. As noted above $f \vee 1$ is LSC and by definition $(f \wedge \mathbf{1})^+$ is USC, it follows by Proposition 12 that k belongs to $\mathbf{D}_U$.

As $f \leq k$ we have that k is an upper bound of the family $(f_j)_J$ in $\mathbf{D}_U$. If $h \in \mathbf{D}_U$ is another such upper bound, then $f_j \leq h$ for each $J \in J$. Thus $f \leq h$ in the pointwise order. This then gives $f \wedge \mathbf{1} \leq h \wedge \mathbf{1}$, hence $(f \wedge \mathbf{1})^+ \leq h \wedge \mathbf{1}$ as $h \wedge \mathbf{1}$ is USC. This then shows $k = f \vee (f \wedge \mathbf{1})^+ \leq h$. The first statement is then established, and the second follows similarly. ■

Definition 10 *We use $\prod$ and $\sum$ for infinite meets and joins in $\mathbf{D}_U$. Since finite meets and joins in $\mathbf{D}_U$ are given pointwise, we use $\wedge$ and $\vee$ for these.*

Lemma 17 *If f, g are decreasing maps from I to I^*, then $(f \wedge g)^+ = f^+ \wedge g^+$.*

Proof. As f is decreasing, its discontinuities are simple jump discontinuities and f^+ is defined by adjusting the value at these discontinuities so that the result is continuous from the left. Thus, for each $x > 0$ we have $f^+(x) = \inf\{f(y) \mid y < x\}$ and $f^+(0) = f(0)$. Obviously $(f \wedge g)^+(0) = f^+(0) \wedge g^+(0)$. For $x > 0$ we then have that $(f \wedge g)^+(x) = \inf\{(f \wedge g)(y) \mid y < x\} = \inf\{f(y) \mid y < x\} \wedge \inf\{g(y) \mid y < x\}$ and this is equal to $f^+(x) \wedge g^+(x)$. ■

Proposition 18 *$\mathbf{D}_U$ satisfies the infinite distributive law $g \wedge \sum_J f_j = \sum_J (g \wedge f_j)$, hence is a complete Heyting algebra.*

Proof. Suppose g and $(f_j)_J$ belong to $\mathbf{D}_U$ and set $f = \bigvee_J f_j$, the pointwise join. Then using the above description of joins in $\mathbf{D}_U$ we have $\sum_J f_j = f \vee (f \wedge \mathbf{1})^+$ and clearly $\bigvee_J (g \wedge f_j) = g \wedge f$. Thus

$$
\begin{aligned}
g \wedge \sum_J f_j &= (g \wedge f) \vee (g \wedge (f \wedge \mathbf{1})^+) \\
\sum_J (g \wedge f_j) &= (g \wedge f) \vee (g \wedge f \wedge \mathbf{1})^+
\end{aligned}
$$

Note $(f \wedge \mathbf{1})^+ \leq \mathbf{1}$, so $g \wedge (f \wedge \mathbf{1})^+ = (g \wedge \mathbf{1}) \wedge (f \wedge \mathbf{1})^+$. As g belongs to $\mathbf{D}_U$ we have $g \wedge \mathbf{1}$ is USC, so the above lemma yields $(g \wedge \mathbf{1}) \wedge (f \wedge \mathbf{1})^+ = (g \wedge \mathbf{1} \wedge f \wedge \mathbf{1})^+$. Thus the two equations above are equal. ■

To conclude this section we address the obvious question of why one considers band semicontinuous functions in $\mathbf{D}$ rather than upper or lower semicontinuous ones. The simple answer is that the USC functions in $\mathbf{D}$ do not form a complete lattice, nor do the LSC ones. One sees this by taking a family of continuous functions taking the value 2 from 0 to $1/2-\epsilon$, dropping rapidly, then taking the value 0 from 1/2 to 1. The pointwise join of these functions takes value 2 on the interval $[0, 1/2)$ and value 0 on $[1/2, 1]$. One can surely find a least decreasing USC function above this, but there is no least decreasing USC function, having 1 as an accumulation point of its image, above it. Similar comments hold for LSC functions.

5 Continuous lattices and semicontinuity

As we mentioned earlier, it is well known that the LSC functions on I form a special type of lattice called a continuous lattice. In this section we show that the lattice $\mathbf{D}_U$, hence also the lattice $\mathbf{L}_U$, is a distributive, continuous lattice. We recall a few basics. For more detail the reader should consult [3] or [4].

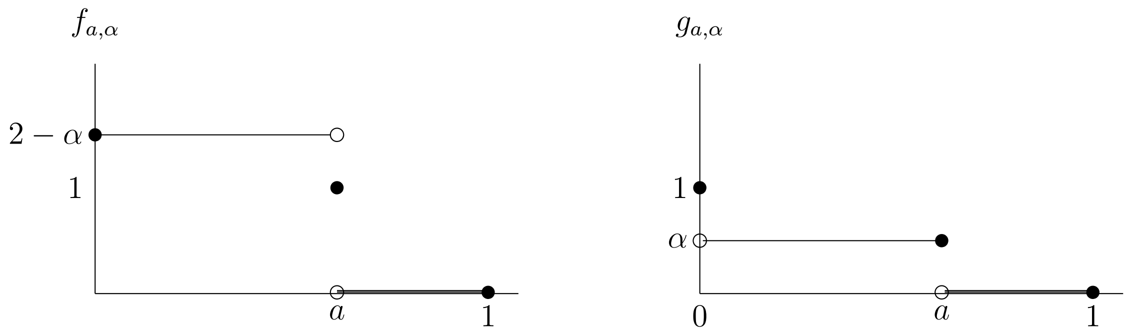
Definition 11 *Let L be a complete lattice. For elements $x, y \in L$ we say x is **way below** y and write $x \ll y$, if whenever $S \subseteq L$ and $y \leq \bigvee S$ there is a finite $S' \subseteq S$ with $x \leq \bigvee S'$. The lattice L is called a **continuous lattice** if every element of L is the join of the elements that are way below it.*

A simple example of a continuous lattice is provided by the unit interval I. Here we have $x \ll y$ if and only if $x < y$. It is well known [3,4] that a distributive continuous lattice satisfies the infinite distributive law $x \wedge \bigvee y_j = \bigvee (x \wedge y_j)$ hence is a complete Heyting algebra. Thus, neither of the complete distributive lattices $\mathbf{L}$ nor $\mathbf{L}_1$ is continuous.

Definition 12 *For $0 < a \leq 1$ and $0 \leq \alpha \leq 1$ define functions $f_{a,\alpha}$ and $g_{a,\alpha}$ by*

$$f_{a,\alpha}(x) = \begin{cases} 2-\alpha & \text{if } x < a \\ 1 & \text{if } x = a \\ 0 & \text{if } a < x \end{cases} \qquad \text{and} \qquad g_{a,\alpha}(x) = \begin{cases} 1 & \text{if } x = 0 \\ \alpha & \text{if } 0 < x \leq a \\ 0 & \text{if } a < x \end{cases}$$

Diagrams of these functions are given below.



Lemma 19 *Each $f_{a,\alpha}$ and $g_{a,\alpha}$ belong to $\mathbf{D}_U$ and every element of $\mathbf{D}_U$ is a join of such elements.*

Proof. Clearly each $f_{a,\alpha}$ and $g_{a,\alpha}$ is decreasing and takes the value 1. One easily sees they are band semi-continuous, so belong to $\mathbf{D}_U$. Suppose that $h \in \mathbf{D}_U$, let Φ be the collection of all $f_{a,\alpha}$ and $g_{a,\alpha}$ that lie beneath h, and set $h' = \sum \Phi$. Surely $h' \leq h$, we claim equality.

Consider first the case of $x \in I$ with $h(x) = \alpha$ being strictly greater than 1. Note, as $h(x) > 1$ we have $x < 1$. Suppose $0 < \epsilon < \alpha - 1$. As $h(x) = \alpha$ we have x belongs to $(h \vee \mathbf{1})^{-1}(\alpha - \epsilon, 2]$. As $h \vee \mathbf{1}$ is LSC this inverse image is open, so there is some y in this set with $x < y$. As $(h \vee \mathbf{1})(y) > \alpha - \epsilon > 1$, we have $h(y) > \alpha - \epsilon$. As h is decreasing we have $f_{y,\alpha-\epsilon} \leq h$, giving that this function belongs to Φ. Thus $f_{y,\alpha-\epsilon} \leq h'$. As $x < y$, we then have $\alpha - \epsilon \leq h'(x)$. As this holds for all $0 < \epsilon < \alpha - 1$, it follows that $\alpha \leq h'(x)$, hence $h'(x) = h(x)$.

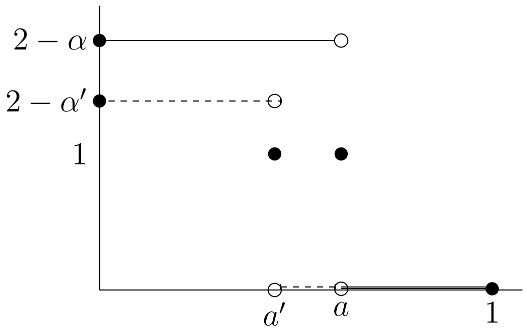
Consider next the case where $x \in I$ with $h(x) = \alpha$ where $\alpha \leq 1$. If $x \neq 0$, then as h is decreasing we have $g_{x,\alpha} \leq h$. So $g_{x,\alpha} \leq h'$, and as $g_{x,\alpha}(x) = \alpha$, it follows that $h'(x) = h(x)$. If $x = 0$, then as $h(x) \leq 1$ we must have $h(x) = 1$ as $h \in \mathbf{D}_U$. Then $g_{1,0}$ is the function taking value 1 at 0 and 0 elsewhere, so $g_{1,0} \leq h$, hence $g_{1,0} \leq h'$, and this gives $h'(1) = 1$. ■

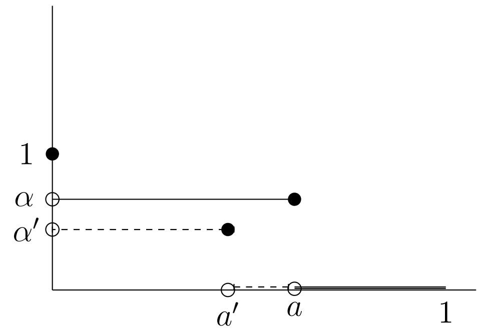
Lemma 20 *Suppose $0 < a \leq 1$ and $0 \leq \alpha < 1$.*

(1) If $0 < a' < a$ and $\alpha < \alpha' < 1$ then $f_{a',\alpha'} \ll f_{a,\alpha}$.
(2) $f_{a,\alpha} = \sum\{f_{a',\alpha'} \mid 0 < a' < a$ and $\alpha < \alpha' < 1\}$.

Proof. For the first statement suppose that $a' < a'' < a$. If $f_{a,\alpha} \leq \sum g_j$ then as $f_{a,\alpha}(a'') = 2 - \alpha > 1$ we have $\sum g_j(a'') \geq 2 - \alpha > 1$. From the description of joins given in Proposition 16 we have $\sum g_j(a'') = \bigvee g_j(a'')$. Thus there is some g_j with $g_j(a'') \geq 2 - \alpha'$. Then $f_{a',\alpha'} \leq g_j$. For the second statement one sees that $f_{a,\alpha}$ is even the pointwise join of this family. ■

The diagram below at left illustrates Lemma 20, the one at right is for Lemma 21.





Lemma 21 *Suppose* $0 < a \leq 1$ *and* $0 < \alpha \leq 1$.

(1) If $0 < a' < a$ *and* $0 < \alpha' < \alpha$ *then* $g_{a',\alpha'} \ll g_{a,\alpha}$.
(2) $g_{a,\alpha} = \sum\{g_{a',\alpha'} \mid 0 < a' < a$ *and* $0 < \alpha' < \alpha\}$.

Proof. For the first statement suppose that $a' < a'' < a$. If $g_{a,\alpha} \leq \sum f_j$ then we claim there is some f_j with $f_j(a'') \geq \alpha'$. If not, then for the function h defined to take value 2 on $[0, a'')$, value 1 at a'', and value α' on $(a'', 1]$, we have $h \in \mathbf{D}_U$ and each $f_j \leq h$. This would give $\sum f_j \leq h$, hence $g_{a,\alpha} \leq h$. But $g_{a,\alpha}(a) = \alpha$ and $h(a) \leq h(a'') = \alpha'$, a contradiction. Suppose then that f_j is such that $f_j(a'') \geq \alpha'$. As $f_j(0) \geq 1$ and f_j is decreasing, it follows that $g_{a',\alpha'} \leq f_j$. For the second statement, let S be the set $\{g_{a',\alpha'} \mid 0 < a' < a$ and $0 < \alpha' < \alpha\}$. For $g = \bigvee S$ we have g agrees with $g_{a,\alpha}$ except at a where $g(a) = 0$ and $g_{a,\alpha}(a) = \alpha$. By Proposition 16 we have $\sum S = g \vee (g \wedge \mathbf{1})^+$. As $(g \wedge \mathbf{1})^+$ is USC and decreasing it is continuous from the left, so takes value α at a. Thus $\sum S = g_{a,\alpha}$. ∎

Theorem 22 $\mathbf{D}_U$ *is a continuous lattice.*

Proof. As every member of $\mathbf{D}_U$ is a join of functions of the form $f_{a,\alpha}$ and $g_{a,\alpha}$, it is enough to show that each such function is the join of some family of functions way below it. Lemma 20 provides this for all $f_{a,\alpha}$ except for those where $\alpha = 1$, and Lemma 21 provides this for all $g_{a,\alpha}$ except for those where $\alpha = 0$. But $f_{a,1} = g_{a,1}$, so each of these is the join of ones way below it, and $g_{a,0}$ is the least element of the lattice $\mathbf{D}_U$, hence is way below itself. ∎

6 Conclusion

The lattice $\mathbf{D}_U$ of band semicontinuous decreasing functions, and its alter-ego $\mathbf{L}_U$ of convex normal upper semicontinuous functions, have a number of desirable properties. These lattices are distributive continuous lattices. As a consequence of this, they are complete Heyting algebras. But there is more. The unary operation $\neg$ on $\mathbf{L}$ defined by $\neg f(x) = f(1-x)$ clearly restricts to a unary operation on $\mathbf{L}_U$. Thus $\mathbf{L}_U$ is a De Morgan algebra. Hence it is also a dual continuous lattice, and a dual Heyting algebra. It might be of interest to determine the dual space of this lattice. We have not done so.

As the lattice $\mathbf{D}_U$ is a continuous lattice, its Lawson topology [3,4] is a compact and Hausdorff topology on $\mathbf{D}_U$ making the meet operation $\wedge$ continuous. It may be of interest to examine properties of this topology further. For instance, if we take all functions $f_{a,\alpha}$ and $g_{a,\alpha}$ as in Definition 12, but for a and α rational, we obtain a countable subset of D_U. Taking finite joins of members of this family gives a countable basis of D_U in the sense of [3] (p. 168). So by [3] (p. 172) the Lawson topology on D_U is metrizable. It may be of interest to find a metric for this topology.

As a final comment, we note that many of our results may be adaptable to a wider setting. Indeed, rather than considering maps from the unit interval I to itself, one can consider maps from any chain to another, or more generally from any poset to another. While generalizing the convolution ordering directly may have its problems, it seems a simpler matter to work through the method of decreasing functions. Indeed, for a posets P and Q one can construct Q^* to be the poset Q with a dual copy of Q placed on top. One then considers the decreasing functions from P to Q^* in an analogous way to what was done with I and I^*. It would perhaps be of some interest to make a general study of this construction.

References

[1] R. Balbes and Ph. Dwinger, *Distributive Lattices*, University of Missouri Press (1974).

[2] M. Gehrke, C. Walker, and E. Walker, "Some Comments on Interval-Valued Fuzzy Sets," *International Journal of Intelligent Systems* **11** (1996) 751-759.

[3] G. Gierz, K. H. Hofmann, K. Keimal, J. D. Lawson, M. W. Mislove, and D. S. Scott, *A Compendium of Continuous Lattices*, Springer-Verlag (1980).

[4] G. Gierz, K. H. Hofmann, K. Keimal, J. D. Lawson, M. W. Mislove, and D. S. Scott, *Continuous Lattices and Domains*, Cambridge University Press (2003).

[5] M. Kawaguchi and M. Miyakoshi, "Extended t-Norms as Logical Connectives of Fuzzy Truth Values," *Multi. Val. Logic* **8**(1) (2002) 53-69.

[6] M. Mizumoto and K. Tanaka, "Some Properties of Fuzzy Sets of Type-2," *Information and Control* 31 (1976) 312-340.

[7] M. Mukaidono, "Algebraic Structures of Truth Values in Fuzzy Logic," *Fuzzy Logic and Fuzzy Control*, *Lecture Notes in Artificial Intelligence*, Springer-Verlag **833** (1994)15-21.

[8] M. Emoto and M. Mukaidono, "Necessary and Sufficient Conditions for Fuzzy Truth Values to Form a De Morgan Algebra," *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems* **7(4)** (1999) 309-316.

[9] C. Walker and E. Walker, "The Algebra of Fuzzy Truth Values," *Fuzzy Sets and Systems*, **149** (2005) 309-347.

[10] C. Walker and E. Walker, "Automorphisms of the Algebra of Fuzzy Truth Values," *International Journal of Uncertainty, Fuzziness and Knowledge-based Systems*, **14**(6) (2006), 711-732.

[11] L. Zadeh, "The Concept of a Linguistic Variable and Its Application to Approximate Reasoning," *Inform Sci.* **8** (1975) 199-249.