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κ-Complete Uniquely Complemented Lattices

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Abstract We show that for any infinite cardinal κ, every complete lattice where each element has at most one complement can be regularly embedded into a uniquely complemented κ-complete lattice. This regular embedding preserves all joins and meets, in particular it preserves the bounds of the original lattice. As a corollary, we obtain that every lattice where each element has at most one complement can be embedded into a uniquely complemented κ-complete lattice via an embedding that preserves the bounds of the original lattice.

Keywords Uniquely complemented lattice · Complete

1 Introduction

In the early days of lattice theory, there was a well-publicized debate regarding the basic axiomatics of the subject. This lead Huntington [12] to ask, in 1904, whether every uniquely complemented lattice was distributive. Here, a lattice is called uniquely complemented (abbreviated: UC) if it is bounded and each element has exactly one complement. As described in the articles by Adams [1] and Grätzer [8], this innocuous sounding question had considerable effect on the subsequent development of lattice theory. By 1940, it had been shown that Huntington's question had a positive answer if one placed some further assumptions on the lattice, such as its being modular, or being complete and atomic. These facts can be found in Birkhoff's book [4] on lattice theory, or in Saliĭ's monograph [14]. Apparently, by the 1940's, it was widely believed that Huntington's problem was true in general.

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In 1945 R. P. Dilworth proved the unexpected result that every lattice can be embedded into a UC lattice [7]. The idea behind his proof was simple. Expand the type of lattices to include an extra unary operation symbol $*$ for complementation, then take the free lattice with period two complementation generated by the given lattice. To implement this simple idea, Dilworth gave an involved construction of this free lattice that yielded a solution to its word problem. It should be noted that Dilworth's construction did not preserve the bounds of the original lattice, thus destroyed existing complements.

Chen and Grätzer [5] gave a technically simpler proof of Dilworth's theorem in 1969. A bounded lattice is said to be at most uniquely complemented if each element has at most one complement. Their idea was to use portions of Dilworth's construction to show any at most UC lattice L can be embedded into an at most UC lattice M, via an embedding that preserves the bounds of the original lattice, so that each element of L has a complement in M. One uses this to define recursively a countable chain of lattices beginning with the original lattice, then the union of this chain is a UC lattice containing the original as a bounded sublattice. This result sharpens Dilworth's as existing complements in the original lattice are preserved. Recently, Grätzer and Lakser [8–10] further simplified the proof by using a tranfinite recursion together with a fairly simple construction showing an at most UC L can be embedded into an at most UC M so that a given element of L has a complement in M. Further details on this recent construction are given in Section 2 below.

The study of UC lattices remained an active area of study after Dilworth's result. Many authors proved results stating that under further natural assumptions on L, Huntington's problem has a positive answer. For instance, if L is atomic, or if the complementation on L is order inverting, or if L is complete and continuous, then Huntington's problem has a positive answer [1, 4, 14]. One very natural question remains open, is every complete UC lattice distributive?

Birkhoff alludes to this question about completions [4] when he asks whether the MacNeille completion of a UC is UC. This was shown to not be the case in [11], where the MacNeille completion of the free UC lattice over an unordered set was shown to be a sublattice of its ideal lattice, hence at most UC but not complemented. Saliĭ also raises prominently the question of whether a complete UC lattice need be distributive in his monograph [14]. We cannot answer this question, but show the following.

Main Theorem *For κ an infinite cardinal, every complete at most* UC *lattice can be regularly embedded into a κ-complete* UC *lattice.*

Here the phrase κ-complete means that every subset of cardinality at most κ has a join and meet, and a regular embedding is one that preserves all existing joins and meets. As the ideal lattice of an at most UC lattice is complete and at most UC, we obtain that every at most UC lattice can be embedded, by an embedding that preserves bounds, into a κ-complete UC lattice. Therefore, every lattice can be embedded into a κ-complete UC lattice, but of course this latter embedding cannot be required to preserve bounds.

The proof of the Main Theorem uses transfinite recursion to construct a chain of lattices L_α, where α ranges over all elements of the successor cardinal κ^+. Each L_α is a complete at most UC lattice, and for each $\alpha < \beta$ we have $L_\alpha \leq L_\beta$ is a regular embedding and each element of L_α has a complement in L_β. The union of this chain is the required κ-complete UC lattice extending our original lattice L_0.

To produce such a chain, we use the recent construction of Grätzer and Lakser in conjunction with a completion method. In view of the result of Harding above [11] it would be natural to consider the MacNeille completion for this job, however this posed technical difficulties we could not overcome. Instead, we tailor a completion specifically for this job.

2 The Construction of Grätzer and Lakser

In this section we describe the construction of Grätzer and Lakser. This is given in more detail in the survey article of Grätzer [8], and is a special case of results in [9]. The best reference is the manuscript of Grätzer and Lakser [10] found on Grätzer's website. We begin with the general setup.

Definition 2.1 Let K be a bounded lattice and $a \in K$ be an element having no complement in K. Let u be an element not in K, set $Q = K \cup \{u\}$ and extend the partial ordering of K to Q by requiring additionally that $0 \leq u \leq 1$.

One then considers Q to have partially defined binary meet and join operations. These are defined for all elements of K, all comparable elements of Q, and additionally we set $a \wedge u = 0$ and $a \vee u = 1$. Then $F(Q)$ is the lattice freely generated by Q and preserving the binary meets and joins defined in Q. Each element of $F(Q)$ is given by a polynomial A involving elements of Q and the connectives $\wedge$, $\vee$. Different polynomials may give the same elements of $F(Q)$. For polynomials A, B we say $A \leq B$, if there is a comparability between the corresponding elements of $F(Q)$. The first key observation is the following.

Proposition 2.2 *For any polynomial A, there is a largest element A_K of K beneath A and a least element A^K of K above A.*

The elements A_K and A^K were denoted A_* and A^* by Grätzer and Lakser. We need a bit more expressive terminology. We next collect several results established by Grätzer and Lakser. It should be noted that Dean's Theorem [6] on free lattices generated by partially ordered sets plays a key role in the proofs.

Proposition 2.3 *Let K, a, u, and $F(Q)$ be as above.*

1. *K is a sublattice of $F(Q)$ having the same bounds as $F(Q)$.*
2. *If K is at most* UC *so is $F(Q)$.*
3. *a has the complement u in $F(Q)$.*
4. *For polynomials A, B, if $A \vee B \neq 1$, then $(A \vee B)_K = A_K \vee B_K$.*

In the fourth condition, when we write $A \vee B \neq 1$ we mean that the join of the elements to which A and B evaluate is not equal to 1.

3 The Completion Step

In this section we provide the key completions needed for the proof of the Main Theorem. We begin by introducing terminology that is useful in discussing these completions. This terminology is motivated by certain properties of the construction of Grätzer and Lakser described above.

Definition 3.1 A bounded lattice embedding $L \leq M$ is a special embedding if for each $m \in M$ there is a largest element m_L of L beneath m, a least element m^L of L above m, and

$$\text{for all } m, n \in M, \text{ either } m \vee n = 1 \text{ or } (m \vee n)_L = m_L \vee n_L.$$

So the embedding $K \leq F(Q)$ in the previous section is a special embedding. For the following property of special embeddings, we recall that an embedding is regular if it preserves all existing joins and meets.

Proposition 3.2 *If $L \leq M$ is a special embedding, it is a regular embedding.*

Proof Suppose $S \subseteq L$ and a is the least upper bound of S in L. Suppose $m \in M$ is an upper bound of S in M. Then $s \leq m$ for each $s \in S$. As m_L is the largest element of L lying beneath m, we have $s \leq m_L$ for each $s \in S$. As a is the least upper bound of S in L and m_L belongs to L, we have $a \leq m_L$, hence $a \leq m$. So a is the least upper bound of S in M as well. That meets are preserved is dual. □

Proposition 3.3 *If $L \leq M$ and $M \leq P$ are special embeddings, so is $L \leq P$.*

Proof Clearly L, M, P are bounded lattices having the same bounds. For each $p \in P$, we have p_M is the largest element of M beneath p, and for this element p_M, we have $(p_M)_L$ is the largest element of L beneath it. This element $(p_M)_L$ is an element of L beneath p. If a is an element of L beneath p, then a is an element of M beneath p, hence $a \leq p_M$, so $a \leq (p_M)_L$. Thus $(p_M)_L$ is the largest element of L beneath p, hence $(p_M)_L = p_L$. The dual argument shows there is a least element of L above p. Suppose $p, q \in P$ with $p \vee q \neq 1$. As $M \leq P$ is special, $(p \vee q)_M = p_M \vee q_M$. As $L \leq M$ is special and $p_M \vee q_M \neq 1$, we have $(p_M \vee q_M)_L = (p_M)_L \vee (q_M)_L$. As we have seen $(p_M)_L = p_L$, we have $(p \vee q)_L = ((p \vee q)_M)_L = (p_M \vee q_M)_L = (p_M)_L \vee (q_M)_L = p_L \vee q_L$. Thus $L \leq Q$ is special. □

We now come to the first of our completions. We remark that for a bounded lattice M, we consider ideals of M to be by definition non-empty. So the ideal lattice $\mathcal{I}(M)$ of M is a complete lattice and there is an obvious embedding $M \leq \mathcal{I}(M)$ sending each $m \in M$ to the principal ideal $m\downarrow$ it generates. This embedding clearly preserves bounds. Unfortunately, the ideal lattice completion is not suitable for our purposes as it is not regular, and we must modify the notion to suit our needs.

Definition 3.4 For $L \leq M$ a bounded lattice embedding, let $\mathcal{I}_L(M)$ be the collection of all ideals I of M with $I \cap L$ a normal ideal of L.

As the intersection of a family of normal ideals of L is again a normal ideal of L, it follows that $\mathcal{I}_L(M)$ is a collection of subsets of M that is closed under intersections, hence forms a complete lattice under set inclusion. There is an obvious embedding of L into $\mathcal{I}_L(M)$ sending an element $a \in L$ to the principal ideal $a\downarrow$ of M it generates. We slightly abuse notation and consider L as a sublattice of $\mathcal{I}_M(L)$. A technically correct approach would be to define a new lattice $\mathcal{I}'_L(M)$ formed from $\mathcal{I}_L(M)$ by replacing the image of L in this lattice with L itself.

Proposition 3.5 *Suppose $L \leq M$ is a special embedding and L is complete.*

1. *$\mathcal{I}_L(M)$ is a bounded sublattice of the ideal lattice of M.*
2. *$L \leq \mathcal{I}_L(M)$ is a special embedding.*
3. *$\mathcal{I}_L(M)$ contains all principal ideals of M.*

Proof Clearly $1\!\downarrow$ is the largest element of $\mathcal{I}_L(M)$ and $0\!\downarrow$ is the least. So the embedding $L \leq \mathcal{I}_L(M)$ preserves bounds and these bounds agree with those of the ideal lattice of M.

Suppose $I \in \mathcal{I}_L(M)$. As L is complete, the normal ideals of L are exactly the principal ideals. As $I \cap L$ is normal, it is principal, so there is a largest element a in $I \cap L$. Then $a\!\downarrow$ is the largest element of the image of L contained in I, so is I_L. Let U be the set of upper bounds of I in L and b be the meet of U in L. As $L \leq M$ is a special embedding, it is a regular embedding, so b is also the meet of U in M. Then b is an upper bound of I in L, hence $b\downarrow$ is the least element of the image of L containing I, and therefore is I^L.

Suppose $I, J \in \mathcal{I}_L(M)$ and let $I \vee J$ be their join in the ideal lattice of M. We claim $I \vee J$ belongs to $\mathcal{I}_L(M)$, hence is the join of I, J in $\mathcal{I}_L(M)$ as well. This is clear if $I \vee J = 1\!\downarrow$, so suppose $1 \notin I \vee J$. If $I_L = a\!\downarrow$ and $J_L = b\downarrow$, we then have $a \in I \cap L$ and $b \in J \cap L$, so $a \vee b \in (I \vee J) \cap L$. Suppose $c \in (I \vee J) \cap L$. Then as $c \in I \vee J$ there are $x \in I$ and $y \in J$ with $c \leq x \vee y$. As x_L is the largest element of L under x we have $x_L \in I \cap L$, hence $x_L \leq a$, and similarly $y_L \leq b$. As $L \leq M$ is special and $x \vee y \neq 1$ (as $1 \notin I \vee J$) we have $c \leq (x \vee y)_L = x_L \vee y_L \leq a \vee b$. So $(I \vee J) \cap L = (a \vee b)\!\downarrow$. This shows $I \vee J$ belongs to $\mathcal{I}_L(M)$ and that if $I \vee J \neq 1\!\downarrow$, then $(I \vee J)_L = I_L \vee J_L$.

We have shown $L \leq \mathcal{I}_L(M)$ is special and that binary joins in $\mathcal{I}_L(M)$ agree with those in the ideal lattice. As we remarked above, arbitrary meets in $\mathcal{I}_L(M)$ are given by intersections, as are those in the ideal lattice. So $\mathcal{I}_L(M)$ is a sublattice of the ideal lattice and the bounds of $\mathcal{I}_L(M)$ agree with those in the ideal lattice. The final condition, that $\mathcal{I}_L(M)$ contains all principal ideals of M, is a direct consequence of $L \leq M$ being special. □

We shall require a second completion method closely related to the first. The second completion method allows us to deal not with a single special embedding, but rather with a certain type of chain of special embeddings.

Definition 3.6 Let λ be an ordinal and L_α $(\alpha \in \lambda)$ be a family of lattices.

1. This family is a chain if $L_\alpha \leq L_\beta$ for each $\alpha \leq \beta$.
2. It is a special chain if $L_\alpha \leq L_\beta$ is a special embedding for each $\alpha \leq \beta$.

It is well known that if L_α $(\alpha \in \lambda)$ is a chain, then $\bigcup_{\beta\in\lambda} L_\beta$ is a lattice containing each L_α as a sublattice. In fancier terms, this union is the direct limit of the family.

Proposition 3.7 *If L_α $(\alpha \in \lambda)$ is a special chain, each $L_\alpha \leq \bigcup_{\beta\in\lambda} L_\beta$ is special.*

Proof Let $M = \bigcup_{\beta\in\lambda} L_\beta$ and $\alpha \in \lambda$. As all the L_β have the same bounds, these are surely the bounds of M as well. Suppose $x \in M$. Then there is $\beta \in \lambda$ with $x \in L_\beta$, and this β may be chosen so that $\alpha \leq \beta$. Then as $L_\alpha \leq L_\beta$ is special, there is a largest element x_{L_α} of L_α beneath x and a least element x^{L_α} of L_α above x. We note that x_{L_α} and x^{L_α} do not depend on the choice β. Suppose $x, y \in M$ with $x \vee y \neq 1$. Then there is β with $x, y \in L_\beta$, and this β may be chosen so that $\alpha \leq \beta$. Then as $L_\alpha \leq L_\beta$ is special, we have $(x \vee y)_{L_\alpha} = x_{L_\alpha} \vee y_{L_\alpha}$. So $L_\alpha \leq M$ is special. □

Definition 3.8 For L_α $(\alpha \in \lambda)$ a special chain and $M = \bigcup_{\beta\in\lambda} L_\beta$, let $\mathcal{I}_\lambda(M)$ be the collection of all ideals I of M with $I \cap L_\alpha$ a normal ideal of L_α for each $\alpha \in \lambda$.

As the intersection of normal ideals is normal, $\mathcal{I}_\lambda(M)$ is a set of subsets of M closed under intersections, so is a complete lattice under set inclusion. For $m \in M$ and $\alpha \in \lambda$, as $L_\alpha \leq M$ is special, there is a largest element m_{L_α} in L_α under m. Then $m\downarrow \cap L_\alpha$ is the principal ideal of L_α generated by m_{L_α}, so is normal. Thus $m\downarrow \in \mathcal{I}_\lambda(M)$, giving an embedding of M into $\mathcal{I}_\lambda(M)$, hence an embedding of each L_α into $\mathcal{I}_\lambda(M)$. We abuse notation and consider M to be a sublattice of $\mathcal{I}_\lambda(M)$.

Proposition 3.9 *Let L_α $(\alpha \in \lambda)$ be a special chain and $M = \bigcup_{\beta\in\lambda} L_\beta$. If each L_α is complete, then*

1. *$\mathcal{I}_\lambda(M)$ is a bounded sublattice of the ideal lattice of M.*
2. *$L_\alpha \leq \mathcal{I}_\lambda(M)$ is a special embedding for each $\alpha \in \lambda$.*

Proof For each $\alpha \in \lambda$ Proposition 3.7 shows $L_\alpha \leq M$ is special. As we have assumed L_α is complete, Proposition 3.5 yields $\mathcal{I}_{L_\alpha}(M)$ is a bounded sublattice of the ideal lattice of M. Working directly from the definitions of $\mathcal{I}_{L_\alpha}(M)$ and $\mathcal{I}_\lambda(M)$ it follows that $\mathcal{I}_\lambda(M) = \bigcap_{\alpha\in\lambda} \mathcal{I}_{L_\alpha}(M)$. Then as $\mathcal{I}_\lambda(M)$ is the intersection of a family of bounded sublattices of the ideal lattice of M, it is also bounded sublattice of the ideal lattice. The proof of the second statement is like that of Proposition 3.5. For $I \in \mathcal{I}_\lambda(M)$, the largest element I_{L_α} in the image of L_α contained in I is $a\downarrow$ where a is the largest element of the principal ideal $I \cap L_\alpha$, and the least element I^{L_α} in the image of L_α containing I is $b\downarrow$ where b is the meet in L_α of the set of upper bounds of I in L_α. Finally, for $I, J \in \mathcal{I}_\lambda(M)$ with $I \vee J \neq 1\downarrow$, showing $(I \vee J)_{L_\alpha} = I_{L_\alpha} \vee J_{L_\alpha}$ follows as in Proposition 3.5 without substantial modification. □

4 Proof of the Main Theorem

In this section, we prove the Main Theorem as stated in the introduction.

Proposition 4.1 *If M is an at most* UC *lattice, then the ideal lattice of M is bounded and the complemented elements of this lattice are exactly the principal ideals $a\downarrow$ generated by complemented elements of M.*

Proof Clearly $1\downarrow$ is the largest ideal of M and $0\downarrow$ is the least. If a, b are complements in M, then $a\downarrow \vee b\downarrow = 1\downarrow$ and $a\downarrow \wedge b\downarrow = 0\downarrow$. Suppose I, J are ideals of M that are complements in the ideal lattice. As $I \vee J = 1\downarrow$ there are $a \in I$ and $b \in J$ with

$a \vee b = 1$. Then as $I \wedge J = 0{\downarrow}$ and $a \wedge b \in I \wedge J$, we have $a \wedge b = 0$. As $a \in I$ we have $a{\downarrow} \subseteq I$. If $c \in I$, then $a \vee c \in I$. Surely $(a \vee c) \vee b = 1$ and as $I \wedge J = 0{\downarrow}$ we have $(a \vee c) \wedge b = 0$. So $a \vee c$ is a complement of b. As a is also a complement of b and M is at most UC we have $a \vee c = a$, hence $c \leq a$. So $a{\downarrow} = I$. □

Proposition 4.2 *If L is complete and at most* UC, *there is a special $L \leq L^*$ where*

1. *L^* is complete and at most* UC.
2. *Each element of L has a complement in L^*.*

Proof Suppose L has cardinality λ and a_α $(\alpha < \lambda)$ is an enumeration of L. We define recursively a family of lattices L_α $(\alpha \leq \lambda)$ with $L_0 = L$ such that

1. For each $\alpha \leq \lambda$, L_α is complete and at most UC.
2. For each $\alpha < \lambda$, the element a_α has a complement in $L_{\alpha+1}$.
3. For each $\alpha \leq \lambda$, the family L_β $(\beta \leq \alpha)$ is a special chain.

Set $L_0 = L$. Assume $\alpha < \lambda$ and L_α is defined. If a_α has a complement in L_α, set $L_{\alpha+1} = L_\alpha$. If a_α has no complement in L_α, apply the construction of Grätzer and Lakser with $K = L_\alpha$ and $a = a_\alpha$. Then by Proposition 2.3 $L_\alpha \leq F(Q)$ is special, as L_α is at most UC so is $F(Q)$, and a_α has a complement in $F(Q)$. Set $L_{\alpha+1} = \mathcal{I}_{L_\alpha}(F(Q))$ and note that $L_{\alpha+1}$ is complete. As L_α is complete, Proposition 3.5 gives $L_\alpha \leq L_{\alpha+1}$ is special and $L_{\alpha+1}$ is a sublattice of the ideal lattice of $F(Q)$. As the complements in the ideal lattice of $F(Q)$ are exactly the principal ideals generated by the complemented elements of $F(Q)$, we have $L_{\alpha+1}$ is at most UC and a_α has a complement in $L_{\alpha+1}$. Further, as we assumed the chain L_β $(\beta \leq \alpha)$ is special, and $L_\alpha \leq L_{\alpha+1}$ is special, it follows from Proposition 3.3 that L_β $(\beta \leq \alpha + 1)$ is special.

Assume $\alpha \leq \lambda$ is a limit ordinal and L_β is defined for all $\beta < \alpha$. Let $M = \bigcup_{\beta<\alpha} L_\beta$ and set $L_\alpha = \mathcal{I}_\alpha(M)$. Note that L_α is complete. By Proposition 3.9 we have $L_\beta \leq L_\alpha$ is special for each $\beta < \alpha$ and L_α is a sublattice of the ideal lattice of M. As M is the union of a chain of at most UC lattices, it is at most UC, and as L_α is a sublattice of the ideal lattice of M, we have L_α is at most UC as well.

Having constructed the special chain L_α $(\alpha \leq \lambda)$, we set $L^* = L_\lambda$. Then L^* is complete and at most UC. As this chain is special, we have $L_0 \leq L_\lambda$ is special, hence $L \leq L^*$ is special. For any $a \in L$ we have $a = a_\alpha$ for some $\alpha < \lambda$. Then $\alpha + 1 \leq \lambda$. As $a = a_\alpha$ has a complement in $L_{\alpha+1}$ and $L_{\alpha+1} \leq L_\lambda$ we have a has a complement in $L_\lambda = L^*$. Thus each element of L has a complement in L^*. □

We come to our Main Theorem which we restate below.

Main Theorem *For κ an infinite cardinal, every complete at most* UC *lattice can be regularly embedded into a κ-complete* UC *lattice.*

Proof Let κ^+ be the successor cardinal to κ. It is well known that κ^+ has cofinality κ^+, which means that any subset of κ^+ of cardinality less than κ^+ is bounded above by some $\alpha < \kappa^+$ [13]. Given a complete at most UC lattice L, we define recursively a family of lattices L_α $(\alpha < \kappa^+)$ with $L_0 = L$ such that

1. For each $\alpha < \kappa^+$, L_α is complete and at most UC.
2. For each $\alpha < \kappa^+$, every element in L_α has a complement in $L_{\alpha+1}$.
3. For each $\alpha < \kappa^+$, the family L_β $(\beta \leq \alpha)$ is a special chain.

Set $L_0 = L$. Assuming $\alpha < \kappa^+$ and L_α is defined, let $L_{\alpha+1} = L_\alpha^*$ the lattice provided by the above proposition. Then $L_\alpha \leq L_{\alpha+1}$ is special, $L_{\alpha+1}$ is complete and at most UC, and every element of L_α has a complement in $L_{\alpha+1}$. Further, as the chain L_β $(\beta \leq \alpha)$ is special, Proposition 3.3 gives L_β $(\beta \leq \alpha + 1)$ is special.

Assume $\alpha < \kappa^+$ is a limit ordinal and L_β is defined for all $\beta < \alpha$. As in the previous proposition, let $M = \bigcup_{\beta<\alpha} L_\beta$ and set $L_\alpha = \mathcal{I}_\alpha(M)$. Then L_α is complete. By Proposition 3.9 we have $L_\beta \leq L_\alpha$ is special for each $\beta < \alpha$, and L_α is a sublattice of the ideal lattice of the at most UC lattice M, so L_α is at most UC.

Set $C = \bigcup_{\beta<\kappa^+} L_\beta$. As the chain L_α $(\alpha < \kappa^+)$ is special, by Proposition 3.7 $L_\alpha \leq C$ is special for each $\alpha < \kappa^+$. So by Proposition 3.2 $L_\alpha \leq C$ is regular for each $\alpha < \kappa^+$. In particular, as $L_0 = L$, we have $L \leq C$ is regular. As the bounds of C agree with those of each L_α, and each L_α is at most UC, we have C is at most UC. If $a \in C$, then $a \in L_\alpha$ for some $\alpha < \kappa^+$, so a has a complement in $L_{\alpha+1}$, showing a has a complement in C. Thus C is uniquely complemented.

Suppose $S \subseteq C$ and S has cardinality at most κ. As κ^+ has cofinality κ^+, there is some $\alpha < \kappa^+$ with $S \subseteq L_\alpha$. As L_α is complete, S has a join and a meet in L_α. As $L_\alpha \leq C$ is regular, this join and meet are the join and meet of S in C as well. So C is κ-complete. □

We conclude with several simple consequences of this result.

Corollary 4.3 *Let κ be an infinite cardinal. Then every at most* UC *lattice can be embedded into a κ-complete* UC *lattice by an embedding that preserves the bounds of the given lattice.*

Proof Given an at most UC lattice L, by Proposition 4.1 the ideal lattice $\mathcal{I}(L)$ is at most UC and the embedding $L \leq \mathcal{I}(L)$ preserves bounds. We then apply the Main Theorem to embed $\mathcal{I}(L)$ into a κ-complete uniquely complemented lattice. □

Corollary 4.4 *Let κ be an infinite cardinal and L be a lattice. Then L can be embedded into a κ-complete* UC *lattice by an embedding that preserves all existing joins and all existing non-empty meets of elements of L.*

Proof Take the MacNeille completion of L then add to this a new top element to form M. Then M is complete and at most UC, and the embedding $L \leq M$ preserves all existing joins and all existing non-empty meets of elements of L. Then apply the Main Theorem to M. □

5 Conclusions

We conclude with a few remarks and questions. First, the embedding in Corollary 4.4 cannot be chosen to preserve bounds as the lattice we are given may have elements with more than one complement. In the proof of Corollary 4.3 the embedding provided is not a regular embedding. It is natural to ask whether this embedding can be chosen to be regular, I suspect it cannot. A more technical question that seems to be of some interest is whether the use of these somewhat more exotic completions can be replaced by a suitable use of MacNeille completions. Finally, the key question remains whether a complete UC lattice must be Boolean.

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References

1. Adams, M.E.: Uniquely complemented lattices. In: Bogart, K., Freese, R. Kung, J. (eds.) The Dilworth Theorems: Selected Papers of Robert P. Dilworth, pp. 79–84. Birkhäuser, Boston (1990)
2. Adams, M.E., Sichler, J.: Cover set lattices. Can. J. Math. **32**, 1177–1205 (1980)
3. Adams, M.E., Sichler, J.: Lattices with unique complementation. Pac. J. Math. **92**, 1–13 (1981)
4. Birkhoff, G.: Lattice Theory. Amer. Math. Soc. Coll. Publ. XXV, 3rd edn. American Mathematical Society, Providence (1967)
5. Chen, C.C., Grätzer, G.: On the construction of complemented lattices. J. Algebra **11**, 56–63 (1969)
6. Dean, R.A.: Free lattices generated by partially ordered sets and preserving bounds. Can. J. Math. **16**, 136–148 (1964)
7. Dilworth, R.P.: Lattices with unique complements. Trans. Am. Math. Soc. **57**, 123–154 (1945)
8. Grätzer, G.: Two problems that shaped a century of lattice theory. Not. Am. Math. Soc. **54**(6), 696–707 (2007)
9. Grätzer, G., Lakser, H.: Freely adjoining a relative complement to a lattice. Algebra Univers. **53**(2), 189–210 (2005)
10. Grätzer, G., Lakser, H.: Freely adjoining a complement to a lattice, manuscript. http://www.maths.umanitoba.ca/homepages/gratzer
11. Harding, J.: The MacNeille completion of a uniquely complemented lattice. Can. Math. Bull. **37**(2), 222–227 (1994)
12. Huntington, E.V.: Sets of independent postulates for the algebra of logic. Trans. Am. Math. Soc. **79**, 288–309 (1904)
13. Kunen, K.: Set Theory. An Introduction to Independence Proofs. Studies in Logic and the Foundations of Mathematics, 102. North-Holland, Amsterdam (1980)
14. Saliĭ, V.N.: Lattices with Unique Complements. Translations of the Amer. Math. Soc. American Mathematical Society, Providence (1988)