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Hausdorff dimension of self-similar sets with overlaps

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Abstract We provide a simple formula to compute the Hausdorff dimension of the attractor of an overlapping iterated function system of contractive similarities satisfying a certain collection of assumptions. This formula is obtained by associating a non-overlapping infinite iterated function system to an iterated function system satisfying our assumptions and using the results of Moran to compute the Hausdorff dimension of the attractor of this infinite iterated function system, thus showing that the Hausdorff dimension of the attractor of this infinite iterated function system agrees with that of the attractor of the original iterated function system. Our methods are applicable to some iterated function systems that do not satisfy the finite type condition recently introduced by Ngai and Wang.

Keywords: Hausdorff dimension, iterated function system, self-similarity

MSC(2000): 28A78, 28A80

1 Introduction

Consider an iterated function system (abbreviated: IFS) consisting of similarities $S_1, \dots, S_m : \mathbb{R}^d \to \mathbb{R}^d$ where

$$\|S_i(x) - S_i(y)\| = \rho_i \|x - y\| \quad \text{for all } x, y \in \mathbb{R}^d,$$

and $0 < \rho_i < 1$ for $i = 1, \dots, m$.

Let K be the attractor of this system and let $\mathcal{H}^s(K)$ and $\dim K$ be the s-dimensional Hausdorff measure and Hausdorff dimension of K, respectively. The computation of $\dim K$ is well known when the system satisfies the open set condition. Without the open set condition, computation of $\dim K$ can only be achieved in certain instances[1−5]. Recently, Ngai and Wang[6] introduced the notion of an IFS of finite type and gave a formula for computing $\dim K$ for an attractor K generated by an overlapping IFS of finite type. Their results include many previous results as special cases.

In this paper we provide yet another method to compute $\dim K$ for an attractor generated by an overlapping IFS satisfying certain assumptions. This method can be applied in some instances where the IFS is not of finite type. The basic idea is to decompose the overlapping finite IFS into an infinite IFS satisfying the open set condition whose attractor has the same

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Hausdorff dimension as the original. Thus the computation of $\dim K$ can be done through an open set condition argument.

The idea of decomposing an overlapping IFS into an infinite IFS satisfying the open set condition is not new. Moran[7] has shown that any overlapping IFS of contractive similarities can be decomposed into an infinite IFS satisfying the open set condition whose attractor has the same Hausdorff dimension as that of the original. Our contribution is to give conditions on the overlapping IFS sufficient to provide an explicit description of such an associated infinite IFS, and thus to provide a simple formula for the Hausdorff dimension of the attractor of the original overlapping IFS.

It is shown in [7] that if a self-similar set is generated by an infinite IFS, then its Hausdorff measure may not be positive even if the system satisfies the open set condition, this is in contrast to the case when the attractor is generated by a finite IFS. In order to obtain a positive Hausdorff measure for the attractor, we further show that, under our conditions, the given IFS satisfies the weak separation condition (WSC) introduced in [8, 9], and by a theorem of Zerner[9], it implies that the attractor has a positive Hausdorff measure.

This paper is organized in the following manner. We review some preliminaries in the second section. The main result providing a formula for the Hausdorff dimension of the attractor of an IFS satisfying certain conditions is given in the third section. In the final section we provide several examples illustrating the use of our result.

2 Preliminaries

We briefly review some results about finite and infinite iterated function systems, and introduce our notation and terminology for these matters. For infinite iterated function systems, we follow [7], but have made several simplifications to his presentation as we do not need the full generality of his results. The reader should also consult [10, 11] for further details.

An infinite iterated function system over a compact set $X \subset \mathbb{R}^d$ is a countable family $\mathcal{F} = \{f_i : i \in \mathcal{I}\}$ such that each $f_i : X \to X$ is a similarity with contraction ratio ρ_i and there is an upper bound $\rho < 1$ with $\rho_i < \rho$ for each $i \in I$.

For $\mathcal{F} = \{f_i : i \in \mathcal{I}\}$ an infinite IFS over the compact set X we let $\mathcal{F}^\infty$ be the set of all infinite sequences $\{f_{i_k}\}$ of members of $\mathcal{F}$, and set

$$K_{\mathcal{F}} = \bigcup \left\{ \bigcap_{n=1}^{\infty} f_{i_1} f_{i_2} \cdots f_{i_n}(X) | \{f_{i_k}\} \in \mathcal{F}^\infty \right\}.$$

The set $K_{\mathcal{F}}$ is called the invariant set, or attractor, of $\mathcal{F}$. We will denote $K_{\mathcal{F}}$ by K if there is no confusion.

The following result can be found in [11].

Proposition 1. *For an infinite IFS $\mathcal{F} = \{f_i : i \in \mathcal{I}\}$ the invariant set $K_{\mathcal{F}}$ satisfies $K_{\mathcal{F}} = \bigcup_{i \in \mathcal{I}} f_i(K_{\mathcal{F}})$.*

In the case of a finite IFS, it is well known that this invariant set $K_{\mathcal{F}}$ is compact, and is in fact the unique non-empty compact set satisfying the above property.

A finite or infinite IFS $\mathcal{F} = \{f_i : i \in \mathcal{I}\}$ over the compact set X is said to satisfy the open set condition (abbreviated: OSC) if there is a non-empty open set $V \subseteq X$ such that

(1) $f_i(V) \subseteq V$ for each $i \in \mathcal{I}$, and

(2) $f_i(V) \cap f_j(V) = \emptyset$ for each $i \neq j$.

The following result can be found in [7].

Proposition 2. *Suppose that $\mathcal{F} = \{f_i : i \in \mathcal{I}\}$ is a relatively compact (in the topology of the uniform convergence over compact sets) infinite IFS satisfying the OSC, then*

$$\dim K_{\mathcal{F}} = \inf \left\{ t \in R : \sum_{i \in I} \rho_i^t \leqslant 1 \right\},$$

where ρ_i is the contraction ratio of f_i.

In the following section, we place conditions on an IFS sufficient to allow us to give an explicit definition of an infinite IFS that satisfies the open set condition and whose attractor has the same Hausdorff dimension as the original one. The above formula then gives us an explicit formula for the Hausdorff dimension of the attractor.

3 The main theorem

Throughout $\{S_i\}_{i=1}^m$ is an IFS on $\mathbb{R}^d$ with contraction ratios $\rho_1, \ldots, \rho_m$ and is considered to be defined over a large bounded closed ball X. Further, k is an integer with $1 \leqslant k < m$, and we set $\Sigma_1 = \{1, \ldots, k\}$, $\Sigma_2 = \{k+1, \ldots, m\}$ and $\Sigma = \{1, \ldots, m\}$.

For any $A \subseteq \Sigma$ let A^* be the set of all finite sequences whose members belong to A. The sequence of length 0 is denoted by $\emptyset$. For sequences $I = i_1 \cdots i_n$ and $J = j_1 \cdots j_r$ in Σ^* define $IJ = i_1 \cdots i_n j_1 \cdots j_r$, $S_I = S_{i_1} \circ S_{i_2} \cdots \circ S_{i_n}$, and $\rho_I = \rho_{i_1} \rho_{i_2} \cdots \rho_{i_n}$. Note that S_I is a contraction and ρ_I is its contraction ratio. If $I = \emptyset$ is the empty sequence, we let $S_\emptyset$ be the identity map and $\rho_\emptyset = 1$.

The following assumptions on $\{S_i\}_{i=1}^m$ hold throughout this section.

Assumption $\mathscr{H}$: Assume V is a bounded open subset of X so that

(1) $\{S_i\}_{i=1}^k$ and $\{S_i\}_{i=k+1}^m$ satisfy the OSC with respect to V,

(2) for each $i \in \Sigma_1$ there is some $j \in \Sigma_2$ with $S_i(V) \cap S_j(V) \neq \emptyset$,

(3) for each $i \in \Sigma_1$ and $j \in \Sigma_2$ with $S_i(V) \cap S_j(V) \neq \emptyset$, there is $n_{ij} \in \Sigma_2$ and $I_{ij} \in \Sigma_2^*$ with

 (a) $S_i(V) \cap S_j(V) \subseteq S_i \circ S_{n_{ij}}(V)$,

 (b) $S_i \circ S_{n_{ij}} = S_j \circ S_{I_{ij}}$.

It is convenient to introduce the following notations:

For the IFS $\{S_i\}_{i=1}^m$ for each $i \in \Sigma_1$ set

(i) $B_i = \{j \in \Sigma_2 : S_i(V) \cap S_j(V) \neq \emptyset\}$,

(ii) $C_i = \{n_{ij} : j \in B_i\}$,

(iii) $D_i = \Sigma_2 \setminus C_i$,

(iv) $\mathcal{F}_1 = \{S_i : i \in \Sigma_1\}$,

(v) $\mathcal{F}_2 = \{S_i : i \in \Sigma_2\}$,

(vi) $\mathcal{F}_3 = \{S_I \circ S_i \circ S_j : I \in \Sigma_1^*, i \in \Sigma_1, j \in D_i\}$.

We let K be the attractor of the IFS $\{S_i\}_{i=1}^m$ and let K_1 and K_2 be the attractors of $\mathcal{F}_1$ and $\mathcal{F}_2$ respectively.

The following two lemmas (Lemmas 3 and 5) are the keys in decomposing an overlapping IFS into an infinite IFS satisfying the OSC. The crucial step is to repeatedly make use of

Assumption $\mathscr{H}$ (3) to replace $S_I \circ S_j$, where $I \in \Sigma^*$ and $j \in \Sigma_2$, by a composition of functions in $\mathcal{F}_2 \cup \mathcal{F}_3$ and show that the IFS $\mathcal{F}_2 \cup \mathcal{F}_3$ satisies the OSC.

Lemma 3. *The IFS $\mathcal{F}_2 \cup \mathcal{F}_3$ satisfies the OSC with respect to V.*

Proof. Assumption $\mathscr{H}$ (1) states that $\mathcal{F}_1$ and $\mathcal{F}_2$ satisfy the OSC with respect to V, so $S_i(V) \subseteq V$ for each $i \in \Sigma$. It follows that $f(V) \subseteq V$ for each $f \in \mathcal{F}_2 \cup \mathcal{F}_3$. It remains to show $f(V) \cap g(V) = \emptyset$ for any distinct $f,\ g \in \mathcal{F}_2 \cup \mathcal{F}_3$. We consider several cases.

Case 1. $f,\ g \in \mathcal{F}_2$.

The result follows as $\mathcal{F}_2$ satisfies the OSC with respect to V.

Case 2. $f \in \mathcal{F}_2$ and $g \in \mathcal{F}_3$.

Suppose $g = S_I \circ S_i \circ S_l$ where $I \in \Sigma_1^*$, $i \in \Sigma_1$ and $l \in D_i$. We show $f(V) \cap g(V) = \emptyset$ for each $f \in \mathcal{F}_2$ by induction on the length of I. If I has length 0 then $g = S_i \circ S_l$ and as $f \in \mathcal{F}_2$ we have $f = S_j$ for some $j \in \Sigma_2$. As $g(V) \subseteq S_i(V)$ the result is trivial if $S_i(V) \cap S_j(V) = \emptyset$. Assume that this intersection is non-empty and consider n_{ij} given by Assumption $\mathscr{H}$ (3). As $l \in D_i$ the definition of D_i shows $l \neq n_{ij}$. Then as n_{ij} and l belong to Σ_2 and $\mathcal{F}_2$ satisfies the OSC, we have $S_{n_{ij}}(V) \cap S_l(V) = \emptyset$. Assumption $\mathscr{H}$ (3) then gives

$$\begin{aligned}
f(V) \cap g(V) &= S_j(V) \cap S_i \circ S_l(V) \\
&= S_j(V) \cap S_i(V) \cap S_i \circ S_l(V) \\
&\subseteq S_i \circ S_{n_{ij}}(V) \cap S_i \circ S_l(V) \\
&= S_i(S_{n_{ij}}(V) \cap S_l(V)) \\
&= \emptyset.
\end{aligned}$$

Suppose that I has the length greater than 0 and let r be the first term of I and J be the remainder of the sequence. Then $g = S_r \circ S_J \circ S_i \circ S_l$. As $g(V) \subseteq S_r(V)$ our result is trivial if $S_r(V) \cap S_j(V) = \emptyset$, so we assume that this intersection is non-empty and consider n_{rj}. A calculation similar to the one above shows $f(V) \cap g(V) \subseteq S_r(S_{n_{rj}}(V) \cap S_J \circ S_i \circ S_l(V))$.

The inductive hypothesis gives $S_{n_{rj}}(V) \cap S_J \circ S_i \circ S_l(V) = \emptyset$, and our result follows.

Case 3. $f, g \in \mathcal{F}_3$.

As $f \in \mathcal{F}_3$ there is a sequence $J = j_1 \cdots j_s \in \Sigma_1^*$, some $j_{s+1} \in \Sigma_1$ and $r \in D_{j_{s+1}}$ with $f = S_J \circ S_{j_{s+1}} \circ S_r$, or equivalently $f = S_{j_1 \cdots j_{s+1}} \circ S_r$. Similarly $g = S_{i_1 \cdots i_{n+1}} \circ S_l$ for some $i_1, \ldots, i_{n+1} \in \Sigma_1$ and $l \in D_{i_{n+1}}$. We may assume $s+1 \leqslant n+1$. As $f(V) \cap g(V) \subseteq S_{j_1}(V) \cap S_{i_1}(V)$ our result follows trivially unless $j_1 = i_1$ as $\mathcal{F}_1$ satisfies the OSC. Assuming $j_1 = i_1$ we have

$$f(V) \cap g(V) = S_{j_1}(S_{j_2 \cdots j_{s+1}} \circ S_r(V) \cap S_{i_2 \cdots i_{n+1}} \circ S_l(V)).$$

Our task reduces to showing $S_{j_2 \cdots j_{s+1}} \circ S_r(V) \cap S_{i_2 \cdots i_{n+1}} \circ S_l(V) = \emptyset$. Repeat the above argument to eliminate $j_2, \ldots, j_{s+1}$. If $s+1 = n+1$ our task reduces to showing $S_r(V) \cap S_l(V) = \emptyset$, and this follows as $\mathcal{F}_2$ satisfies the OSC and $r \neq l$ as f and g are distinct. If $s+1 < n+1$, our task reduces to showing $S_r(V) \cap S_{i_{s+2} \cdots i_n} \circ S_{i_{n+1}} \circ S_l(V) = \emptyset$, and this follows from Case 2.

Corollary 4. *Let $f_1, \ldots, f_r, g_1, \ldots, g_s \in \mathcal{F}_2 \cup \mathcal{F}_3$ with $r \leqslant s$ and*

$$f_1 \circ \cdots \circ f_r(V) \cap g_1 \circ \cdots \circ g_s(V) \neq \emptyset.$$

Then $f_i = g_i$ for $i = 1, \dots, r$, and $g_1 \circ \cdots \circ g_s(V) \subseteq f_1 \circ \cdots \circ f_r(V)$.

Proof. This is a simple consequence of the OSC.

Lemma 5. *For any $I \in \Sigma^*$ and $j \in \Sigma_2$, there exist $f_1, \dots, f_l$ in $\mathcal{F}_2 \cup \mathcal{F}_3$ with $S_I \circ S_j = f_1 \circ \cdots \circ f_l$.*

Proof. As the last term of the sequence Ij belongs to Σ_2 we can break Ij into segments, each consisting of an initial (possibly empty) sequence of terms from Σ_1 followed by a single element of Σ_2. Suppose $J = i_1 \cdots i_p k$ is such a sequence where $i_1, \dots, i_p \in \Sigma_1$ and $k \in \Sigma_2$. We prove by induction on p that S_J can be written as a composite of members of $\mathcal{F}_2 \cup \mathcal{F}_3$. If $p = 0$ this is trivial as S_k is a member of $\mathcal{F}_2$. Suppose $p > 0$. If $k \in D_{i_p}$ then by definition $S_J \in \mathcal{F}_3$. Otherwise $k \in C_{i_p}$ so there is some $j \in B_{i_p}$ with $k = n_{i_p j}$. Using $I_{i_p j} \in \Sigma_2^*$ provided by Assumption $\mathscr{H}$ (3) we have $S_{i_p} \circ S_k = S_{i_p} \circ S_{n_{i_p j}} = S_j \circ S_{I_{i_p j}}$. It follows that $S_J = S_{i_1 \cdots i_{p-1} j} \circ S_{I_{i_p j}}$. The inductive hypothesis shows that $S_{i_1 \cdots i_{p-1} j}$ is a composite of members of $\mathcal{F}_2 \cup \mathcal{F}_3$ and $S_{I_{i_p j}}$ is by definition a composite of members of $\mathcal{F}_2$.

For the IFS $\{S_i\}_{i=1}^m$ let $\rho_* = \min\{\rho_i : 1 \leqslant i \leqslant m\}$. Then for $b > 0$ set

(1) $\mathcal{I}_b = \{I \in \Sigma^* : \rho_I \leqslant b < \rho_I \rho_*^{-1}\}$,

(2) $\mathcal{A}_b = \{S_I : I \in \mathcal{I}_b\}$.

For a full account of the following, see [8, 9].

Definition 6. *An IFS $\{S_i\}_{i=1}^m$ is said to satisfy the weak separation condition (WSC) if there exist $x_0 \in \mathbb{R}^d$ and $\gamma > 0$ such that for every $0 < b < 1$, every $J \in \Sigma^*$ and every $x \in \mathbb{R}^d$*

$$\sharp\{S_I \in \mathcal{A}_b : S_I(S_J(x_0)) \in B_b(x)\} \leqslant \gamma,$$

where $B_b(x)$ is the ball centered at x with radius b.

Theorem 7. *The IFS $\{S_i\}_{i=1}^m$ satisfying Assumption $\mathscr{H}$ satisfies the WSC.*

Proof. Fix $x_0 \in V$ and $l \in \Sigma_2$. For any $x \in \mathbb{R}^d$, $J \in \Sigma^*$ and $0 < b < 1$, if $S_I \in \mathcal{A}_b$, and $S_I(S_J(x_0)) \in B_b(x)$, since $S_J(x_0) \in V$ and $\rho_I \leqslant b$, so $S_I(V) \subset B_{b(1+|V|)}(x)$. Hence $S_I(S_l(S_J(x_0))) \in S_I(V) \subset B_{b(1+|V|)}(x)$, where $|V|$ is the diameter of the bounded set V. Hence

$$\{S_I \in \mathcal{A}_b : S_I(S_J(x_0)) \in B_b(x)\} \subseteq \{S_I \in \mathcal{A}_b : S_I(S_l(S_J(x_0))) \in B_{b(1+|V|)}(x)\},$$

set

$$\{S_I \in \mathcal{A}_b : S_I(S_l(S_J(x_0))) \in B_{b(1+|V|)}(x)\} = \{S_{I_1},\ S_{I_2}, \dots, S_{I_q}\}.$$

We need only to show that there is a constant γ, not dependent on x, J or b, with $q \leqslant \gamma$. This establishes the WSC. We remark that the set above is finite as $I \in \mathcal{I}_b$ implies $b\rho_* < \rho_I$ which gives a bound on the length of I.

For each $i = 1, \dots, q$ Lemma 5 gives a family $f_1^i, \dots, f_{n_i}^i$ in $\mathcal{F}_2 \cup \mathcal{F}_3$ with $S_{I_i} \circ S_l = f_1^i \circ \cdots \circ f_{n_i}^i$. For convenience, set $\Psi_i = f_1^i \circ \cdots \circ f_{n_i}^i$, and note that this map has contraction ratio $\rho_{I_i}\rho_l$. As $S_{I_i} \in \mathcal{A}_b$ we have $\rho_{I_i} \leqslant b < \rho_{I_i}\rho_*^{-1}$, and it follows that the contraction ratio of Ψ_i is less than b. As $\Psi_i(S_J(x_0)) = S_{I_i}(S_l \circ S_J(x_0)) \in B_{b(1+|V|)}(x)$, it follows that $\Psi_i(V) \cap B_{b(1+|V|)}(x) \neq \emptyset$, since $S_J(x_0) \in V$. Hence $\Psi_i(V) \subseteq B_{b(1+2|V|)}(x)$.

Suppose $i, j \leqslant q$ and $I \in \Sigma^*$ is a sequence with $\Psi_i \circ S_I = \Psi_j$. Then $\rho_{I_i}\rho_l\rho_I = \rho_{I_j}\rho_l$, so $\rho_I = \rho_{I_j}/\rho_{I_i}$. But $\rho_{I_i} \leqslant b < \rho_{I_j}\rho_*^{-1}$, so $\rho_* < \rho_I$. There are a finite number of sequences $I \in \Sigma^*$ with this property (this number is at most $c = \frac{\ln \rho_*}{\ln \max_j\{\rho_j\}}$). Corollary 4 shows that for each i, j

with $\Psi_i(V) \cap \Psi_j(V) \neq \emptyset$ there is a sequence $I \in \Sigma^*$ with either $\Psi_i \circ S_I = \Psi_j$ or $\Psi_j \circ S_I = \Psi_i$. So for any $i = 1, \ldots, q$ we have $\sharp\{j : \Psi_i(V) \cap \Psi_j(V) \neq \emptyset\} \leqslant 2c$. It follows that there is a set $S \subseteq \{1, \ldots, q\}$ of cardinality $s \geqslant q/2c$ so that $\Psi_i(V) \cap \Psi_j(V) = \emptyset$ for any distinct $i, j \in S$. We assume for simplicity that this set is $\{1, \ldots, s\}$. Then as the contraction ratio of each Ψ_i is greater than $b\rho_*^2$ we have

$$\mathcal{L}(B_{b(1+2|V|)}(x)) \geqslant \mathcal{L}\left(\bigcup_{i=1}^{s} \Psi_i(V)\right) = \sum_{i=1}^{s} \mathcal{L}(\Psi_i(V)) \geqslant \frac{q}{2c}(b\rho_*^2)^d \mathcal{L}(V),$$

where $\mathcal{L}(V)$ is the Lebesgue measure of V. Therefore

$$q \leqslant \frac{2c\mathcal{L}(B_{b(1+2|V|)}(x))}{(b\rho_*^2)^d \mathcal{L}(V)} = \frac{2c\mathcal{L}(B_{(1+2|V|)}(x))}{\rho_*^{2d} \mathcal{L}(V)}.$$

As $\mathcal{L}(B_{(1+2|V|)}(x))$ depends only on the diameter of V and the dimension of the space, we have $q \leqslant \gamma$ for some universal constant γ.

The above theorem extends Proposition 4.4 in [12], and our proof is also simpler.

We now present the main theorem of the paper.

Theorem 8. *Suppose that the IFS* $\{S_i\}_{i=1}^m$ *satisfies Assumption* $\mathscr{H}$. *Then the Hausdorff dimension* $s = \dim K$ *of the attractor* K *is computed as follows*:

(i) *If* $D_i = \emptyset$ *for all* $i \in \Sigma_1$, *then* s *is the unique solution to*

$$\sum_{j=k+1}^{m} \rho_j^s = 1;$$

(ii) *If* $D_i \neq \emptyset$ *for at least one* $i \in \Sigma_1$, *then* s *is the unique solution to*

$$\sum_{j=1}^{m} \rho_j^s - \sum_{i=1}^{k} \rho_i^s \sum_{j \in C_i} \rho_j^s = 1.$$

In either case (i) *or case* (ii), $0 < \mathcal{H}^s(K) < \infty$.

Proof. (i) As $\mathcal{F}_2$ satisfies the OSC it suffices to show $K = K_2$, or equivalently $K \subseteq K_2$, since the reversed inclusion is obvious. Let $a \in K_2$ be the fixed point of S_m. As K is the closure of $\bigcup_{I \in \Sigma^*} S_I(a)$, we need only show $S_I(a) \in K_2$ for any $I \in \Sigma^*$. For any such $I \in \Sigma^*$, Lemma 5 provides $f_1, \ldots, f_l \in \mathcal{F}_2 \cup \mathcal{F}_3$ with $S_I \circ S_m = f_1 \circ \cdots \circ f_l$. But $D_i = \emptyset$ for all $i \in \Sigma_1$, so $\mathcal{F}_3 = \emptyset$, hence $f_1, \ldots, f_l$ belong to $\mathcal{F}_2$. Therefore $S_I(a) = S_I \circ S_m(a) = f_1 \circ \cdots \circ f_l(a) \in K_2$.

(ii) We first show $\dim K = \max\{\dim K_1, \dim K_{2,3}\}$ where $K_{2,3}$ is the attractor of the IFS $\mathcal{F}_2 \cup \mathcal{F}_3$. Surely $K_1, K_{2,3} \subseteq K$, so we trivially have $\dim K_1, \dim K_{2,3} \leqslant \dim K$. For the other inequality note that for an infinite sequence $\mathbf{i} = i_1 i_2 \cdots \in \Sigma^\infty$ the intersection below is of a decreasing family of compact sets, hence is a singleton $x_{\mathbf{i}} = \bigcap_{j=1}^{\infty} S_{i_1} \circ \cdots \circ S_{i_j}(X)$.

By the definition of the attractor, $K = \{x_{\mathbf{i}} : \mathbf{i} \in \Sigma^\infty\}$. Break K into two pieces. The first piece, A, consists of those $x_{\mathbf{i}}$ where infinitely many terms of $\mathbf{i}$ belong to Σ_2, and the second piece, B, consists of those $x_{\mathbf{i}}$ where only finitely many terms of $\mathbf{i}$ belong to Σ_2. It is not important whether A and B are disjoint, provided their union is K.

If $\mathbf{i}$ has infinitely many terms in Σ_2, then it can be split into infinitely many disjoint pieces of the form $i_j \cdots i_r$ where $i_r \in \Sigma_2$. It follows from Lemma 5 that each $S_{i_j} \circ \cdots \circ S_{i_r}$ is equal to a composite of members of $\mathcal{F}_2 \cup \mathcal{F}_3$. From this, one obtains that A is contained in $K_{2,3}$. If $\mathbf{i}$ has only finitely many terms in Σ_2, then $\mathbf{i} = I\mathbf{j}$ where I is a finite sequence and $\mathbf{j}$ is an infinite sequence whose terms all belong to Σ_1. In this case $x_{\mathbf{i}} = S_I(x_{\mathbf{j}})$, and we note that $x_{\mathbf{j}}$ belongs to K_1. It follows that $B \subseteq \bigcup\{S_I(K_1) : I \in \Sigma^*\}$, and as Σ^* is countable, B is contained in the union of countably many sets all having the same Hausdorff dimension as K_1. So $\dim K \leqslant \max\{\dim K_1, \dim K_{2,3}\}$.

We consider now $\dim K_{2,3}$. Set

$$q(t) = \sum_{j=k+1}^{m} \rho_j^t + \sum_{r=0}^{\infty} \sum_{I\in\Sigma_1^r} \rho_I^t \sum_{i=1}^{k} \sum_{j\in D_i} \rho_i^t \rho_j^t,$$

and denote the partial sums of $q(t)$ by

$$q_n(t) = \sum_{j=k+1}^{m} \rho_j^t + \sum_{r=0}^{n} \sum_{I\in\Sigma_1^r} \rho_I^t \sum_{i=1}^{k} \sum_{j\in D_i} \rho_i^t \rho_j^t.$$

In Lemma 3 we have shown the IFS $\mathcal{F}_2 \cup \mathcal{F}_3$ satisfies the OSC, it is easy to verify by using the Arzela-Ascoli theorem that it is relatively compact, therefore by Proposition 2 $\dim K_{2,3} = \inf\{t \in \mathbb{R} : q(t) \leqslant 1\}$.

As $q_n(t)$ involves only a finite sum, it is a continuous decreasing function. It follows that there is a real number s_n with $q_n(s_n) = 1$. But s_n is the Hausdorff dimension of a subsystem of $\mathcal{F}_1 \cup \mathcal{F}_2$, so $s_n \leqslant d$ where d is the dimension of the Euclidean space $\mathbb{R}^d$. As the s_n are bounded above, we may define $s = \sup\{s_n : n \geqslant 0\}$.

As q_n is decreasing, $q_n(s) \leqslant q_n(s_n) = 1$, and as $q(s)$ is the limit of the partial sums $q_n(s)$, $q(s) \leqslant 1$. Noting that $\sum_{I\in\Sigma_1^r} \rho_I^s = (\sum_{i=1}^k \rho_i^s)^r$, it follows from $q(s) \leqslant 1$ that $\sum_{i=1}^k \rho_i^s < 1$. As the IFS $\mathcal{F}_1$ satisfies the OSC, $\dim K_1$ is the unique solution to $\sum_{i=1}^k \rho_i^t = 1$. So $\dim K_1 < s$.

Let I be the open interval $(\dim K_1, \infty)$. Then $s \in I$ and for all $t \in I$

$$q(t) = \sum_{j=k+1}^{m} \rho_j^t + \Big(1 - \sum_{i=1}^{k} \rho_i^t\Big)^{-1} \sum_{i=1}^{k} \sum_{j\in D_i} \rho_i^t \rho_j^t.$$

This shows $q(t)$ is continuous on I. As $s_n \in I$ for large n and $q(s_n) > q_n(s_n) = 1$, then as s is the limit of the s_n, it follows that $q(s) \geqslant 1$. Therefore $q(s) = 1$ and we conclude $\dim K_{2,3} = s$. As we have shown that $\dim K_1 < s$, we have $\dim K = s$.

It remains to show s is the unique solution to the equation given in condition (ii). Let

$$h(t) = \sum_{j=1}^{m} \rho_j^t - \sum_{i=1}^{k} \sum_{j\in C_i} \rho_i^t \rho_j^t.$$

Then as

$$q(t) = \sum_{j=k+1}^{m} \rho_j^t + \Big(1 - \sum_{i=1}^{k} \rho_i^t\Big)^{-1} \sum_{i=1}^{k} \rho_i^t \Big(\sum_{j=k+1}^{m} \rho_j^t - \sum_{j\in C_i} \rho_j^t \Big),$$

we have

$$\Big(1 - \sum_{i=1}^{k} \rho_i^t\Big) q(t) = \sum_{j=k+1}^{m} \rho_j^t - \sum_{i=1}^{k} \rho_i^t \sum_{j\in C_i} \rho_j^t.$$

It follows that

$$\left(1-\sum_{i=1}^{k}\rho_i^t\right)q(t)=h(t)-1+\left(1-\sum_{i=1}^{k}\rho_i^t\right).$$

Hence $h(t)=1$ if and only if $q(t)=1$, and $s=\dim K$ is the unique solution of the equation in (ii).

For the further remark, Theorem 7 shows our IFS satisfies WSC. A result of [9] then gives $0<\mathcal{H}^s(K)<\infty$.

4 Examples

We give several examples to illustrate the use of the main theorem.

Example 9. Suppose $0<p,q<1$ and let $K\subset\mathbb{R}$ be the attractor of the IFS

$$S_1(x)=px+1,\quad S_2(x)=qx,\quad S_3(x)=px+1/q.$$

If $2p+q-pq<1$, then $0<\mathcal{H}^s(K)<\infty$, where $s=\dim K$ is the unique solution to $2p^s+q^s-p^sq^s=1$. Otherwise the attractor K is an interval.

Proof. Suppose $2p+q-pq<1$. It is enough to show this IFS satisfies Assumption $\mathscr{H}$ with $\Sigma_1=\{1\}$, $\Sigma_2=\{2,\ 3\}$ and $V=(0,b)$, where $b=1/q(1-p)$.

The condition $2p+q-pq<1$ implies $S_2(b)<S_1(b)<S_3(0)<S_3(b)=b$ and so $S_i(V)\subseteq V$ for $i=1,2,3$ and $S_2(V)\cap S_3(V)=\emptyset$, the first assumption. Also $S_2(0)<S_1(0)<S_2(b)<S_1(b)$ implies $B_1=\{2\}\neq\emptyset$, the second assumption. Note that $S_1(V)\cap S_2(V)=(1,pb+1)\cap(0,qb)=(1,qb)$. The observation $pqb+1=qb$ provides $S_1(V)\cap S_2(V)=S_1S_2(V)$, and a computation shows $S_1S_2=S_2S_3$. This provides the final assumption with $n_{12}=2$ and $I_{12}=3$.

Conversely, assume that $2p+q-pq\geqslant 1$, then we have $S_1(b)\geqslant S_3(0)$ and as $S_2(b)=qb>1=S_1(0)$. It follows that $S_2([0,b])\cup S_1([0,b])\cup S_3([0,b])=[0,b]$. Hence $[0,b]$ is the attractor.

Remark 10. Example 9 was presented by the third author at a Seminar in Fractals at the Chinese University of Hong Kong in 2001, and was later studied by Lau and Wang in [12, Proposition 4.4]. Note that the IFS in the example is related to the well known $(0,1,3)$ problem by letting $p=q=1/3$. In this case, the Hausdorff dimension of the attractor $s=\dim K$ satisfies $3\cdot 3^{-s}-3^{-2s}=1$, so $x\approx 0.87604$.

Example 11. Suppose $0<r_2,r_4\leqslant r_1\leqslant 1/4$, $r_1r_3=r_2r_4$ and let K be the attractor of the IFS

$$S_1(x)=r_1x+1,\quad S_2(x)=r_2x+1+3r_1,\quad S_3(x)=r_3x+3,\quad S_4(x)=r_4x.$$

Then $0<\mathcal{H}^s(K)<\infty$, where $s=\dim K$ is the unique solution to $r_1^s+r_2^s+r_3^s+r_4^s-r_1^sr_3^s=1$.

Proof. It is enough to show this IFS satisfies Assumption $\mathscr{H}$ with $\Sigma_1=\{1\}$, $\Sigma_2=\{2,3,4\}$, $B_1=\{2\},n_{12}=3$, $I_{12}=4$ and $V=(0,b)$, where $b=3/(1-r_3)$.

Noting that $3<b\leqslant 4$ and that b is the fixed point of S_3 it follows that $f_i(V)\subseteq V$ for each $i=0,1,2,3$ and $S_4(b)<S_2(0)<S_2(b)<S_3(0)$ implying that $S_2(V),S_3(V),S_4(V)$ are pairwise disjoint, providing the first assumption. As $S_1(V)=(1,1+r_1b)$ and $S_2(V)=(1+3r_1,r_2b+1+3r_1)$ we have $S_1(V)\cap S_2(V)=(1+3r_1,1+r_1b)$ and $S_1(V)\cap S_2(V)=S_1S_3(V)$. Finally, a computation shows $S_1S_3=S_2S_4$, hence the last two assumptions are satisfied.

Example 12 (A modified Sierpinski triangle)**.** Assume that u, v are two independent vectors in $\mathbb{R}^2$. For any l, $n \in \mathbb{N}$, let $(1-r)(1-\frac{1}{2^n}) \geqslant \frac{1}{2}$ and let K be the attractor of the IFS:

$$S_1(x) = \frac{1}{2^l}x + \frac{1}{2}\left(1-\frac{1}{2^l}\right)v, \quad S_2(x) = \frac{1}{2^n}x + (1-r)\left(1-\frac{1}{2^n}\right)u,$$
$$S_3(x) = \frac{1}{2}x, \quad S_4(x) = \frac{1}{2}x + \frac{1}{2}v, \quad S_5(x) = rx + (1-r)u.$$

Then $0 < \mathcal{H}^s(K) < \infty$, where $s = \dim K$ is the unique solution to the equation $(\frac{1}{2})^{ls} + (\frac{1}{2})^{ns} + 2(\frac{1}{2})^s + r^s - 2(\frac{1}{2})^{(l+1)s} - (\frac{1}{2})^{ns}r^s = 1$ (see Figure 1).

Proof. Let $\Sigma_1 = \{1,2\}$, $\Sigma_2 = \{3,4,5\}$, $B_1 = \{3,4\}$, $B_2 = \{5\}$ and $V = \{pv+qu : p > 0, q > 0, p+q < 1\}$. Then u, v and the origin 0 are fixed points of S_5, S_4 and S_3, respectively. Furthermore, V is an invariant open set for our IFS and both $\{S_1, S_2\}$ and $\{S_3, S_4, S_5\}$ satisfy the OSC with respect to the open set V. By the definition of S_i, it is not difficult to check that $S_1(V) \cap S_3(V) = S_1S_3(V)$ and $S_1S_3 = S_3S_4^l$, i.e., $n_{1\,3} = 3$ and $I_{1\,3} = 44\cdots4$ with length l; $S_1(V) \cap S_4(V) = S_1S_4(V)$ and $S_1S_4 = S_4S_3^l$, i.e., $n_{1\,4} = 4$ and $I_{1\,4} = 33\cdots3$ with length l; $S_2(V) \cap S_5(V) = S_2S_5(V)$ and $S_2S_5 = S_5S_3^n$, i.e., $n_{2\,5} = 5$ and $I_{2\,5} = 33\cdots3$ with length n. Thus Assumption $\mathscr{H}$ is satisfied. Note that $C_1 = \{3,4\}$ and $C_2 = \{5\}$. So $s = \dim K$ is the unique solution to the equation

$$\left(\frac{1}{2^l}\right)^s + \left(\frac{1}{2^n}\right)^s + 2\left(\frac{1}{2}\right)^s + r^s - \left(\frac{1}{2^l}\right)^s\left[\left(\frac{1}{2}\right)^s + \left(\frac{1}{2}\right)^s\right] - \left(\frac{1}{2^n}\right)^s r^s = 1.$$

This proves the statement.

Example 13. Assume that $r^l + 2r \leqslant 1$ and $0 < r_i < 1/3$ for $i = 3, 4$, where l is a positive integer. Let K be the attractor of the IFS on the plane:

$$S_0(x) = r^l x + \frac{1}{2}(1-r^l, 1-r^l), \quad S_1(x) = \frac{1}{2}x, \quad S_2(x) = rx + (1-r, 1-r),$$
$$S_3(x) = r_3x + (0, 1-r_3), \quad S_4(x) = r_4x + (1-r_4, 0).$$

Then $0 < \mathcal{H}^s(K) < \infty$, where $s = \dim K$ is the unique solution to the equation $r^{ls} + (\frac{1}{2})^s + r^s + r_3^s + r_4^s - (\frac{1}{2})^s r^{ls} = 1$ (see Figure 2).

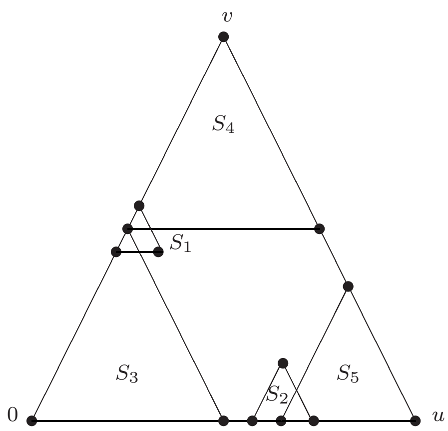

Figure 1

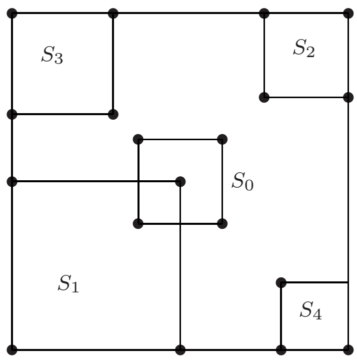

Figure 2

Proof. Note $A=(0,0)$, $B=(1,1)$, $C=(0,1)$, $D=(1,0)$ are the fixed points of S_1, S_2, S_3, S_4 respectively. It is enough to verify that this IFS satisfies Assumption $\mathscr{H}$ with $\Sigma_1=\{0\}$, $\Sigma_2=\{1,\ 2,\ 3,\ 4\}$, $B_0=\{1\}$, $n_{01}=1$, $I_{01}=2\cdots 2$ with length l and V the interior of $\Box ABCD$. These calculations are left to the reader.

In [6] Ngai and Wang introduced a property which is called the finite type condition and developed a method to compute the Hausdorff dimension of the attractor of an overlapping IFS satisfying the finite type condition. Ngai and Wang showed that their methods include many of the instances where the dimension of the attractor of an IFS can be calculated. We show that our methods do not lie within the scope of Ngai and Wang's finite type condition.

We review the definition of the finite type condition as it applies to a family of similarities $S_0,\dots,S_m$ on $\mathbb{R}$. Let $r_0,\dots,r_m$ be the contraction ratios of these maps and $r=\min r_i$. For $\mathbf{i}=i_1,\dots,i_k$ a sequence of integers from $\{0,\dots,m\}$ let $r_{\mathbf{i}}=r_{i_1}\cdots r_{i_k}$ and $S_{\mathbf{i}}=S_{i_1}\cdots S_{i_k}$. Let Γ_k be the collection of all triples $s=(r_{\mathbf{i}},S_{\mathbf{i}}(0),k)$ where i is a sequence with $r_{\mathbf{i}}\leqslant r^k$ and $r^k<r_{\mathbf{j}}$ for all proper subsequences j of i. For such a triple s define the map $S_{\mathbf{s}}$ by setting $S_{\mathbf{s}}(x)=r_{\mathbf{i}}x+S_{\mathbf{i}}(0)$. For V an open set we put

$$V(\mathbf{s})=\{\mathbf{t}\in\Gamma_k : S_{\mathbf{s}}(V)\cap S_{\mathbf{t}}(V)\neq\emptyset\}.$$

The crucial point is that for $s\in\Gamma_k$ and $t\in\Gamma_n$ we define $V(s)\equiv V(t)$ if there is a map $\tau(x)=r^{n-k}x+b$ with $S_{\mathbf{t}}=\tau\circ S_{\mathbf{s}}$ and $\{S_{\mathbf{t}'} : t'\in V(t)\}=\{\tau\circ S_{\mathbf{s}'} : s'\in V(s')\}$. The IFS is said to be of a finite type if there is a bounded open set V with $S_i(V)\subseteq V$ for each $i=0,\dots,m$ such that the equivalence relation $\equiv$ for V has only finitely many equivalence classes.

If the IFS satisfies the finite type condition, it is easy to show that there are rationals $t_1,\dots,t_m$ such that $r_j=r_0^{t_j}$, $j=1,\dots,m$. Therefore, the above examples do not satisfy the finite type condition in general. On the other hand, an IFS that satisfies the finite type condition does not necessarily satisfy Assumption $\mathscr{H}$ as illustrated by the following IFS defined on $\mathbb{R}$: $S_1(x)=x/2+1/7$, $S_2(x)=x/4$ and $S_3(x)=x/8+7/8$.

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