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Modal Logics of Stone Spaces

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Abstract Interpreting modal diamond as the closure of a topological space, we axiomatize the modal logic of each metrizable Stone space and of each extremally disconnected Stone space. As a corollary, we obtain that **S4.1** is the modal logic of the Pelczynski compactification of the natural numbers and **S4.2** is the modal logic of the Gleason cover of the Cantor space. As another corollary, we obtain an axiomatization of the intermediate logic of each metrizable Stone space and of each extremally disconnected Stone space. In particular, we obtain that the intuitionistic logic is the logic of the Pelczynski compactification of the natural numbers and the logic of weak excluded middle is the logic of the Gleason cover of the Cantor space.

Keywords Modal logic · Boolean algebra · Stone space · Metrizable space · Extremally disconnected space

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1 Introduction

Topological semantics of modal logic was first developed by McKinsey and Tarski [13], who interpreted modal diamond as the closure operator and consequently modal box as the interior operator of a topological space. The main result of [13] states that under the above interpretation Lewis' modal system **S4** is the modal logic

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of each dense-in-itself metrizable space. Our purpose here is to study modal logics of compact Hausdorff spaces having a basis of clopen sets, also known as Stone spaces.

Up to homeomorphism, the Cantor space **C** is the unique dense-in-itself metrizable Stone space. The result of McKinsey and Tarski implies the modal logic of the Cantor space is **S4**. The results of [5] yield a description of the modal logic of each countable Stone space, and in [3] we described the modal logic of the Stone–Čech compactification of the natural numbers (but under an additional set-theoretic assumption). To the best of our knowledge, these are the only results describing modal logics of Stone spaces. In this paper, to these results we add descriptions of the modal logic of each metrizable Stone space and of each extremally disconnected Stone space. Our result for extremally disconnected Stone spaces uses our earlier result on the modal logic of the Stone–Čech compactification, so makes use of a set-theoretic assumption beyond ZFC. As corollaries, we obtain that the modal logic of the Pelczynski compactification of the natural numbers is **S4.1** and the modal logic of the Gleason cover of the Cantor space is **S4.2**.

Using Stone duality [17] it is natural to extend our terminology and speak of the modal logic of a Boolean algebra, meaning the modal logic of its Stone space. Our results will then have two versions: one for Stone spaces, and one for their corresponding Boolean algebras. Under Stone duality, countable Boolean algebras correspond to metrizable Stone spaces, atomless Boolean algebras correspond to dense-in-itself Stone spaces, and complete Boolean algebras correspond to extremally disconnected Stone spaces. Our results then classify the modal logics of countable Boolean algebras and of complete Boolean algebras. The original result of McKinsey and Tarski, applied to the setting of Stone spaces, amounts to describing the modal logic of the unique (up to isomorphism) countable atomless Boolean algebra.

Each topological space not only gives a modal logic, but also an intermediate logic via its Heyting algebra of open sets. As a further corollary of our results we obtain an axiomatization of the intermediate logic of each metrizable Stone space and of each extremally disconnected Stone space. In particular, we obtain that the intuitionistic logic is the logic of the Pelczynski compactification of the natural numbers and the logic of weak excluded middle is the logic of the Gleason cover of the Cantor space.

This paper is organized in the following way. The second section gives necessary background, including our primary technical tools. In the third section we describe the modal logics of metrizable Stone spaces, and in the fourth we describe the modal logics of extremally disconnected Stone spaces. The fifth section applies these results to intermediate logics of these spaces. The final section makes some small observations on the problem of describing the modal logic of an arbitrary Stone space.

2 Preliminaries

We recall that **S4** is the least set of formulas of basic propositional modal language containing the axiom schemas:

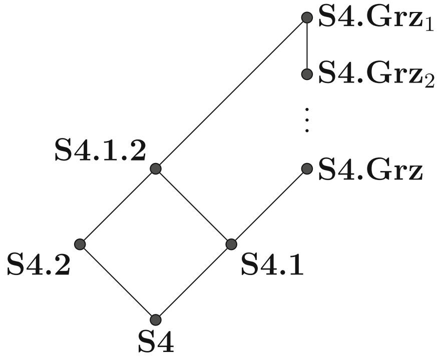
(1) $\Box(p \to q) \to (\Box p \to \Box q)$;
(2) $\Box p \to p$;
(3) $\Box p \to \Box\Box p$;

and closed under Modus Ponens ($\varphi, \varphi \to \psi/\psi$) and Generalization ($\varphi/\Box\varphi$).

We also recall that **S4.1** is obtained from **S4** by postulating $\Box\Diamond p \to \Diamond\Box p$ as a new axiom schema, that **S4.2** is obtained from **S4** by postulating $\Diamond\Box p \to \Box\Diamond p$ as a new axiom schema, and that **S4.1.2** is obtained from **S4** by postulating $\Box\Diamond p \leftrightarrow \Diamond\Box p$ as a new axiom schema; that is, **S4.1.2** is the join of **S4.1** and **S4.2** in the lattice of normal extensions of **S4**. We will also be interested in the modal logic **S4.Grz**, which is obtained from **S4** by postulating $\Box(\Box(p \to \Box p) \to p) \to p$ as a new axiom schema, and the modal logics $\mathbf{S4.Grz}_n$, for $n \geq 1$, which are obtained from **S4.Grz** by postulating bd_n as new axiom schemas, where bd_n are defined recursively as follows:

(1) $bd_1 = \Diamond\Box p_1 \to p_1$,
(2) $bd_{n+1} = \Diamond(\Box p_{n+1} \wedge \neg bd_n) \to p_{n+1}$.

The inclusion relation between the normal extensions of **S4** we are interested in is depicted in the figure below.



Definition 2.1 For a topological space X, let $M(X)$ be the modal algebra associated with X, namely the powerset algebra $\mathcal{P}(X)$ with $\Box$ and $\Diamond$ interpreted as interior and closure of X, respectively. For a modal formula φ, write $X \models \varphi$ if the identity $\varphi = 1$ holds in $M(X)$. Then $L(X) = \{\varphi : X \models \varphi\}$ is a modal logic over **S4**, and we call it the modal logic of X. The modal logic of a class $\mathfrak{K}$ of topological spaces is the intersection of the modal logics $L(X)$, where $X \in \mathfrak{K}$.

For more on topological semantics of modal logic we refer the reader to [1, 19]. We recall that X is *extremally disconnected* if the closure of each open subset of X is open in X. We also recall that $x \in X$ is an *isolated point* if $\{x\}$ is open in X, that X is *scattered* if each nonempty subspace of X has an isolated point, and that X is *weakly scattered* if the set of isolated points of X is dense in X. We view each ordinal as a topological space in the interval topology. The next proposition is well-known. To keep the paper self-contained, and also to introduce the necessary background, we outline a short proof of it later in the section using Proposition 2.5 below.

Proposition 2.2

(a) **S4** *is the modal logic of the class of all topological spaces.*
(b) **S4.1** *is the modal logic of the class of weakly scattered spaces.*
(c) **S4.2** *is the modal logic of the class of extremally disconnected spaces.*
(d) **S4.1.2** *is the modal logic of the class of weakly scattered extremally disconnected spaces.*

(e) **S4.Grz** *is the modal logic of the class of scattered spaces; in fact,* **S4**.**Grz** $= L(\alpha)$ *for each ordinal* α *such that* $\omega^\omega \leq \alpha$.
(f) **S4**.**Grz**$_n$ $= L(\alpha)$ *for each ordinal* α *such that* $\omega^n + 1 \leq \alpha \leq \omega^{n+1}$.

The standard way to prove these types of results is through the relational semantics of **S4**. We recall that $\mathfrak{F} = \langle W, R \rangle$ is an **S4**-*frame* if W is a nonempty set and R is a reflexive and transitive relation on W; in other words, an **S4**-frame is a quasi-ordered set. For $w \in W$ we let $R(w) = \{v \in W : wRv\}$ and $R^{-1}(w) = \{v \in W : vRw\}$.

Definition 2.3 For an **S4**-frame $\mathfrak{F} = \langle W, R \rangle$, we let $M(\mathfrak{F})$ be the modal algebra associated with $\mathfrak{F}$, namely the powerset $\mathcal{P}(W)$ where $\Box$ and $\Diamond$ are defined for $S \subseteq W$ by

$$\Box S = \{w \in W : R(w) \subseteq S\} \text{ and } \Diamond S = \{w \in W : R(w) \cap S \neq \emptyset\}.$$

For a modal formula φ, we write $\mathfrak{F} \models \varphi$ if the identity $\varphi = 1$ holds in $M(\mathfrak{F})$. Then $L(\mathfrak{F}) = \{\varphi : \mathfrak{F} \models \varphi\}$ is a modal logic over **S4** that we call the modal logic of $\mathfrak{F}$. The modal logic of a class $\mathfrak{K}$ of **S4**-frames is the intersection of the modal logics $L(\mathfrak{F})$, where $\mathfrak{F} \in \mathfrak{K}$.

For more on relational semantics of modal logic we refer the reader to [6, 7]. Given an **S4**-frame $\mathfrak{F} = \langle W, R \rangle$, we say w is a *root* of $\mathfrak{F}$ if $W = R(w)$. Also, for any $w \in W$ we let $C(w) = \{v \in W : wRv \text{ and } vRw\}$ and call $C(w)$ the *cluster generated by* w. A subset C of W is called a *cluster* if $C = C(w)$ for some $w \in W$. A cluster C is *simple* if it consists of a single point, *proper* if it consists of more than one point, and a *maximal cluster*, or *leaf*, if $w \in C$, $v \in W$, and wRv imply $v \in C$. If $W = C(w)$ for some $w \in W$, then we call $\mathfrak{F}$ a *cluster*. We also call a subset E of W a *chain* if for each $w, v \in E$ we have wRv or vRw.

Let $\mathfrak{F}$ be a rooted **S4**-frame. We call $\mathfrak{F}$ a *quasi-tree* if $R^{-1}(w)$ is a chain for each $w \in W$, and a *tree* if $\mathfrak{F}$ is a quasi-tree and each cluster of $\mathfrak{F}$ is simple. Let $\mathfrak{F} = \langle W, R \rangle$ be a tree and $w \in W$. We say that w is of *depth n* if there is an n-element chain $E \subseteq W$ with the root w, and every other chain with the root w does not contain more than n elements. We say that the *depth of* $\mathfrak{F}$ is n if the root of $\mathfrak{F}$ has depth n.

Definition 2.4 Let $\mathfrak{F} = \langle W, R \rangle$ be a quasi-tree and $\mathfrak{G} = \langle V, S \rangle$ be a cluster with $W \cap V = \emptyset$. We let $\mathfrak{F} \oplus \mathfrak{G}$ be the frame $\langle W \cup V, T \rangle$, where for $w, v \in W \cup V$ we have:

$$wTv \text{ if, and only if, } w, v \in W \text{ and } wRv \text{ or } w, v \in V \text{ or } w \in W \text{ and } v \in V.$$

We call $\mathfrak{F} \oplus \mathfrak{G}$ the *sum* of $\mathfrak{F}$ and $\mathfrak{G}$.

For an **S4**-frame $\mathfrak{F} = \langle W, R \rangle$, there is an equivalence relation $\sim$ on W given by $w \sim v$ if $C(w) = C(v)$, and a partial ordering $\leq$ on $W/{\sim}$ given by $w/{\sim} \leq v/{\sim}$ if wRv. Let $\mathfrak{F}/{\sim}$ be the **S4**-frame $(W/{\sim}, \leq)$. The above notions have natural interpretation in terms of the poset $\mathfrak{F}/{\sim}$. For instance, a cluster is an equivalence class of $\sim$; an **S4**-frame $\mathfrak{F}$ being a quasi-tree means $\mathfrak{F}/{\sim}$ is a tree, and the construction $\mathfrak{F} \oplus \mathfrak{G}$ can be viewed by noting $(\mathfrak{F} \oplus \mathfrak{G})/{\sim}$ is the ordinal sum of $\mathfrak{F}/{\sim}$ and $\mathfrak{G}/{\sim}$.

It is well known (see, e.g., [7, Sections 5.3, 5.5, and 8.6]) that **S4** is the modal logic of the class of all finite **S4**-frames, that **S4.1** is the modal logic of the class of finite **S4**-frames whose maximal clusters are simple, that **S4.2** is the modal logic of the class of finite **S4**-frames where each pair of elements with a lower bound has an upper bound, that **S4.Grz** is the modal logic of the class of all finite partially ordered **S4**-frames, and that **S4**.$\mathbf{Grz}_n$ is the modal logic of the class of all finite partially ordered **S4**-frames of depth at most n. Consequently, **S4.Grz** is the intersection of the logics **S4**.$\mathbf{Grz}_n$. We will use the following well-known sharpening of these results.

Proposition 2.5

- (i) **S4** *is the modal logic of the class of all finite quasi-trees.*
- (ii) **S4.1** *is the modal logic of the class of all finite quasi-trees whose maximal clusters are simple.*
- (iii) **S4.2** *is the modal logic of the class of sums $\mathfrak{F} \oplus \mathfrak{G}$, where $\mathfrak{F}$ is a finite quasi-tree and $\mathfrak{G}$ is a finite cluster.*
- (iv) **S4.1.2** *is the modal logic of the class of sums $\mathfrak{F} \oplus \mathfrak{G}$, where $\mathfrak{F}$ is a finite quasi-tree and $\mathfrak{G}$ is a simple cluster.*
- (v) **S4.Grz** *is the modal logic of the class of all finite trees.*
- (vi) **S4**.$\mathbf{Grz}_n$ *is the modal logic of the class of all finite trees of depth $\leq n$.*

Proof (Sketch) It is well-known (see, e.g., [7, Sections 5.3, 5.5, and 8.6]) that each of the logics in the proposition is determined by its finite rooted frames. The desired quasi-trees are now produced as in [2, Section 3]. □

Let $\mathfrak{F} = \langle W, R\rangle$ be an **S4**-frame. We recall that a subset U of W is an *upset* if $w \in U$ and wRv imply $v \in U$, and that the collection τ_R of upsets of W forms a topology on W, called the *Alexandroff* topology. Therefore, each **S4**-frame can be viewed as a topological space. In fact, it is a special topological space in which the intersection of open sets is again open.

We recall that a map f between two topological spaces X and Y is *continuous* if V open in Y implies $f^{-1}(V)$ is open in X, that f is *open* if U open in X implies $f(U)$ is open in Y, and that f is *interior* if f is continuous and open. We remark that the interior maps between **S4**-frames are precisely the "bounded morphisms" (also known as "p-morphisms") from the model theory of modal logic (see, e.g., [6, 7]). The following basic result [20, Proposition 2.9.2] is key in later proofs.

Proposition 2.6 *If $f : X \to Y$ is an onto interior map and $Y \not\models \varphi$, then $X \not\models \varphi$.*

In proving that a modal logic L is the modal logic of a topological space X, soundness is usually not difficult. The above proposition is key to proving completeness. For instance, to show **S4** is complete with respect to X, we must show that if **S4** $\nvdash \varphi$, then $X \not\models \varphi$. For this, it is enough to show that for each finite quasi-tree Y, there is an interior map from X onto Y, as any such φ will be falsified on some such finite quasi-tree. Similarly, to show **S4.1** is complete with respect to X it is enough to show there is an interior map from X onto each finite quasi-tree whose maximal clusters are simple, and so forth.

Proof of Proposition 2.2 Parts (a)–(f) of Proposition 2.2 will be proved using the above remarks and (i)–(vi) of Proposition 2.5. For (a) it is obvious that **S4** is sound with respect to all topological spaces. Conversely, if **S4** $\nvdash \varphi$, by (i) there is a finite quasi-tree $\mathfrak{F}$ such that $\mathfrak{F} \not\models \varphi$. Therefore, φ is refuted on a finite topological space, and so **S4** is complete with respect to all topological spaces. For (b) it is easy to see that each weakly scattered space is irresolvable (does not have two disjoint dense subsets), and so by [4, Proposition 2.1], **S4.1** is sound with respect to all weakly scattered spaces. Conversely, if $\mathbf{S4.1} \nvdash \varphi$, then by ii) there is a finite quasi-tree $\mathfrak{F}$ with simple maximal clusters such that $\mathfrak{F} \not\models \varphi$. Since simple maximal clusters are isolated points in the Alexandroff topology on $\mathfrak{F}$, it is easy to verify that each tree with simple maximal clusters viewed as a topological space is weakly scattered. Therefore, φ is refuted on a finite weakly scattered space, and so **S4.1** is complete with respect to all weakly scattered spaces. For (c) it follows from [10, Theorem 1.3.3] that **S4.2** is sound with respect to all extremally disconnected spaces. Conversely, if $\mathbf{S4.2} \nvdash \varphi$, by (iii) there is a finite quasi-tree $\mathfrak{F}$ and a finite cluster $\mathfrak{G}$ such that $\mathfrak{F} \oplus \mathfrak{G} \not\models \varphi$. Now it is easy to verify that $\mathfrak{F} \oplus \mathfrak{G}$ viewed as a topological space is extremally disconnected. Therefore, φ is refuted on a finite extremally disconnected space, and so **S4.2** is complete with respect to all extremally disconnected spaces. For (d) we again see **S4.1.2** is sound with respect to all weakly scattered extremally disconnected spaces. If $\mathbf{S4.1.2} \nvdash \varphi$, then by iv) there is a finite quasi-tree $\mathfrak{F}$ and a simple cluster $\mathfrak{G}$ such that $\mathfrak{F} \oplus \mathfrak{G} \not\models \varphi$. Viewed as a topological space, $\mathfrak{F} \oplus \mathfrak{G}$ is weakly scattered and extremally disconnected, providing completeness.

To establish (e) and (f), since each scattered space X is hereditarily irresolvable (that is, each subspace of X is irresolvable) and **S4.Grz** is sound with respect to hereditarily irresolvable spaces (see, e.g., [5, Section 6]), **S4.Grz** is also sound with respect to scattered spaces. As each ordinal is scattered, **S4.Grz** is sound with respect to any ordinal α. Moreover, $\mathbf{S4.Grz}_n$ is sound with respect to any ordinal $\alpha \leq \omega^{n+1}$. Conversely, if $\mathbf{S4.Grz}_n \nvdash \varphi$, then by vi) there is a finite tree $\mathfrak{F}$ of depth $\leq n$ such that $\mathfrak{F} \not\models \varphi$. Without loss of generality we may assume that the depth of $\mathfrak{F}$ is n. By [5, Lemma 3.4], there is an interior onto map $f : \omega^n + 1 \to \mathfrak{F}$. Therefore, $\omega^n + 1 \not\models \varphi$. It follows that $\alpha \not\models \varphi$ for each ordinal α with $\omega^n + 1 \leq \alpha \leq \omega^{n+1}$, and so $\mathbf{S4.Grz}_n$ is complete with respect to α. If $\mathbf{S4.Grz} \nvdash \varphi$, then by (v) there exists a finite tree $\mathfrak{F}$ with $\mathfrak{F} \not\models \varphi$. By the above, there exist $n \geq 1$ and an interior onto map $f : \omega^n + 1 \to \mathfrak{F}$. Therefore, $\omega^n + 1 \not\models \varphi$. But then $\omega^\omega \not\models \varphi$, and so $\gamma \not\models \varphi$ for each $\gamma \geq \omega^\omega$. So, **S4.Grz** is complete with respect to any $\gamma \geq \omega^\omega$. This establishes (e) and (f), completing the proof. □

3 Modal Logics of Metrizable Stone Spaces

In this section we characterize the modal logics of metrizable Stone spaces. We begin with a basic result used throughout the paper.

Proposition 3.1 *If U, V are complementary clopen subsets of a topological space X, then the modal logic of X is the intersection of the modal logics of the subspaces U and V.*

Proof If U and V are complementary clopen subsets of X, then X is the disjoint union of U and V. The result now follows from [20, Proposition 2.9.3]. □

For a Boolean algebra B and $a \in B$, let a' denote the complement of a in B. Then the intervals $[0, a]$ and $[0, a']$ naturally form Boolean algebras and B is isomorphic to $[0, a] \times [0, a']$. The above result, restricted to the setting of Stone spaces, translates to the following.

Proposition 3.2 *For B a Boolean algebra and $a \in B$, the modal logic of B is the intersection of the modal logics of* $[0, a]$ *and* $[0, a']$.

The McKinsey–Tarski theorem shows **S4** is the modal logic of a metrizable dense-in-itself Stone space. Using this, and the above result, we can see that if X is a metrizable Stone space which is not weakly scattered, then the modal logic of X is **S4**. Indeed, such an X has a clopen subset that is metrizable and dense-in-itself, so by the above result, the logic of X is contained in **S4**, but **S4** is contained in the modal logic of every space.

By Stone duality, each Stone space is homeomorphic to the Stone space of a Boolean algebra. It is well known [12, pp. 103–104] that if B is a Boolean algebra with Stone space X, then B is countable if, and only if, X is metrizable; B is atomless if, and only if, X is dense-in-itself; and B is atomic if, and only if, X is weakly scattered. These equivalences will be used throughout the paper to state equivalent versions of the results. To begin, we have the following.

Proposition 3.3 *Let X be a metrizable Stone space and B be a countable Boolean algebra.*

(1) *If X is not weakly scattered, its modal logic is* **S4**.
(2) *If B is not atomic, its modal logic is* **S4**.

It remains to consider matters for metrizable Stone spaces that are weakly scattered, or equivalently, for countable Boolean algebras that are atomic. The result will depend on whether the Boolean algebra satisfies a stronger form of atomicity.

Definition 3.4 For a Boolean algebra B we say $a \in B$ is of finite height if $[0, a]$ is finite. Define recursively for each ordinal α an ideal I_α of B as follows.

(1) $I_0 = \{0\}$.
(2) $I_{\alpha+1} = \{a \in B : a/I_\alpha$ is of finite height in $B/I_\alpha\}$.
(3) $I_\beta = \bigcup\{I_\alpha : \alpha < \beta\}$.

Eventually $I_\alpha = I_{\alpha+1}$ for some α. Set $I = I_\alpha$. We say B is superatomic if B/I is trivial.

It is well known [12, pp. 271–275] that B is superatomic if, and only if, every homomorphic image of B is atomic, and that this is equivalent to the Stone space of B being scattered.

Proposition 3.5 *Let X be a metrizable Stone space and B be a countable Boolean algebra.*

(1) *If X is scattered, its modal logic is either* **S4.Grz** *or* $\mathbf{S4.Grz}_n$ *for some* $n \geq 1$.
(2) *If B is superatomic, its modal logic is either* **S4.Grz** *or* $\mathbf{S4.Grz}_n$ *for some* $n \geq 1$.

Proof It is enough to show the first statement as it is equivalent to the second. As X is a scattered metrizable Stone space, it is well known [16, Theorem 8.6.10] that X is homeomorphic to a countable compact ordinal α under the interval topology. Such α can be constructed from X using techniques from [9]. In this case, results of [5, Section 6] show the modal logic of X is either **S4.Grz** or $\mathbf{S4.Grz}_n$ for some $n \geq 1$, depending whether $\alpha \geq \omega^\omega$ or $\omega < \alpha \leq \omega^{n+1}$. □

It remains to consider the case where X is a weakly scattered, but not scattered, metrizable Stone space. As every Stone space is homeomorphic to the Stone space of a Boolean algebra, we assume X is the Stone space of a Boolean algebra B, and by the above remarks, that B is countable and atomic, but not superatomic. We will show **S4.1** is the modal logic of X and B. To do so, we show each finite rooted quasi-tree whose maximal clusters are simple is an interior image of X, and then make use of Propositions 2.5 and 2.6.

As B is countable and not superatomic, the quotient $A = B/I$ is nontrivial, where I is the ideal provided by Definition 3.4. Note that by construction, A is atomless, and as B is countable, A is countable. Therefore, the Stone space of A is homeomorphic to the Cantor space $\mathbf{C}$. Finally, we let $\kappa : B \to A$ be the canonical homomorphism.

Since B is countable, it is well known [12, p. 247] that there is a chain C of B that generates B and contains the bounds 0, 1. As B is atomic, each interval of C contains a cover. We let D be the image of C under κ and note that D is a chain of A that generates A and contains its bounds. Because A is atomless, D contains no covers. Although we do not use the fact, we remark that the chain D can be constructed directly from C by repeatedly collapsing covers of C in a process known as taking the condensation of C [15, Chapter 5].

Let $\mathcal{D}(C)$ be the set of all nonempty proper downsets of C; that is, all downsets $V \subseteq C$ containing 0 and not 1. Then $\mathcal{D}(C)$ is in bijective correspondence with the prime ideals of B. Moreover, $\mathcal{D}(C)$ itself forms a chain under set inclusion, and it is well known that under the interval topology (where closed intervals are a basis of closed sets) $\mathcal{D}(C)$ is homeomorphic to the Stone space X of B [12, pp. 241–246]. Similarly the collection $\mathcal{D}(D)$ of all nonempty proper downsets of D with its interval topology is homeomorphic to the Stone space Y of A, and hence to the Cantor space $\mathbf{C}$.

We identify X with $\mathcal{D}(C)$ and Y with $\mathcal{D}(D)$. For $U \subseteq C$ and $V \subseteq D$, we use κU for the image of U under κ and $\kappa^{-1}V$ for the preimage of V. By Stone duality, Y is homeomorphic to the subspace $\kappa^{-1}Y = \{\kappa^{-1}y : y \in Y\}$ of X. We note that an element $y \in Y$ is a proper downset of D, so the preimage $\kappa^{-1}y$ is a proper downset of C, hence an element of X. We develop further the relationship between X and $\kappa^{-1}Y$.

Lemma 3.6 *For $x \in X$, the following conditions are equivalent:*

(1) $x \in \kappa^{-1}Y$.
(2) $x = \kappa^{-1}\kappa x$.
(3) $x \supseteq \kappa^{-1}\kappa x$.

Proof Since it is always the case that $x \subseteq \kappa^{-1}\kappa x$, we have (2) is equivalent to (3). To see that (1) implies (2), let $x \in \kappa^{-1}Y$. Then $x = \kappa^{-1}y$ for some $y \in Y$. As $\kappa : C \to D$ is onto, $\kappa\kappa^{-1}y = y$, giving $x = \kappa^{-1}\kappa\kappa^{-1}y = \kappa^{-1}\kappa x$. Finally, to see that (2) implies (1), let $x = \kappa^{-1}\kappa x$. Then as x is proper, $1 \notin x = \kappa^{-1}\kappa x$, and this shows $1 \notin \kappa x$. Then as κ is onto, κx is a proper downset of D, hence $\kappa x \in Y$. □

For an element d in the chain D, we use $\downarrow d$ for the principal downset $\{e \in D : e \leq d\}$ generated by d and $\Downarrow d$ for the downset $\downarrow d - \{d\} = \{e \in D : e < d\}$. If $d \in D - \{0, 1\}$, then both $\downarrow d$ and $\Downarrow d$ belong to Y, and $\Downarrow d$ is covered by $\downarrow d$. The least element of Y is $\downarrow 0 = \{0\}$ and the largest is $\Downarrow 1 = D - \{1\}$. By [12, pp. 241–246], the sets of the form $[\downarrow 0, \downarrow d)$ and $(\Downarrow d, \Downarrow 1]$, where $d \in D - \{0, 1\}$, are a subbasis for the topology on Y consisting of clopen sets. Similar comments hold for an element $c \in C$ and for the topology on X.

For $d \in D - \{0\}$ we let d^- be the element $\kappa^{-1}(\Downarrow d)$ of X, and for $d \in D - \{1\}$ we let d^+ be the element $\kappa^{-1}(\downarrow d)$ of X. So as κ^{-1} is a homeomorphism from Y to the subspace $\kappa^{-1}Y$ of X, the intersections with $\kappa^{-1}Y$ of sets of the form $[0^+, d^+)$ and $(d^-, 1^-]$ form a subbasis for the subspace topology on $\kappa^{-1}Y$.

Lemma 3.7 *For $x \in X$, exactly one of the following holds.*

(1) $x \in \kappa^{-1}Y$.
(2) $x > 1^-$.
(3) $x < 0^+$.
(4) $x \in (d^-, d^+)$ *for some* $d \in D - \{0, 1\}$.

Proof Clearly these cases are exclusive. We show they are exhaustive. Let $x \notin \kappa^{-1}Y$. By Lemma 3.6, $\kappa^{-1}\kappa x \supset x$, so there is $c \in \kappa^{-1}\kappa x - x$. If $\kappa c = 1$, we have $x > 1^-$; if $\kappa c = 0$, we have $x < 0^+$; and if $\kappa c = d$ for some $d \in D - \{0, 1\}$, we have $x \in (d^-, d^+)$. □

Lemma 3.8 *Suppose $b, c \in C - \{0, 1\}$.*

(a) *If $\kappa c = 0$, then $[\downarrow 0, \downarrow c)$ is disjoint from $\kappa^{-1}Y$.*
(b) *If $\kappa b = 1$, then $(\Downarrow b, \Downarrow 1]$ is disjoint from $\kappa^{-1}Y$.*
(c) *If $\kappa c \leq \kappa b$, then $(\Downarrow b, \downarrow c)$ is disjoint from $\kappa^{-1}Y$.*

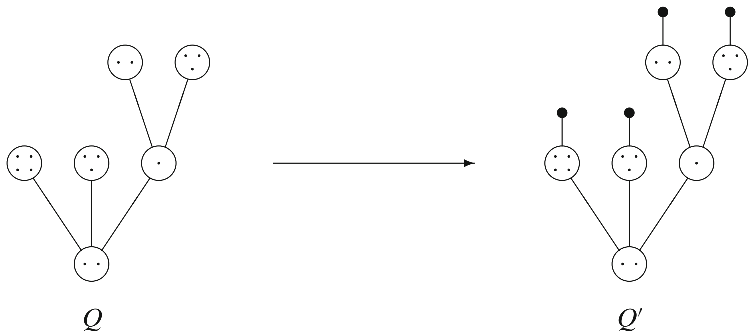
Proof (a) If $\kappa c = 0$, then $c \in 0^+ = \kappa^{-1}(\downarrow 0)$, and the result follows as 0^+ is the smallest member of $\kappa^{-1}Y$. (b) If $\kappa b = 1$, then $b \notin 1^- = \kappa^{-1}(\Downarrow 1)$, so $1^- < \downarrow b$. This shows $1^- \leq \Downarrow b$, and as 1^- is the largest element of $\kappa^{-1}Y$, the result follows. (c) If $c \leq b$, then $(\Downarrow b, \downarrow c)$ is empty, so we may assume $b < c$ and as we also assume $\kappa c \leq \kappa b$, that $\kappa b = \kappa c$. Note $(\kappa b)^- = \{a : \kappa a < \kappa b\}$, so $b \notin (\kappa b)^-$, showing $(\kappa b)^- < \downarrow b$, hence $(\kappa b)^- \leq \Downarrow b$. Also, $(\kappa c)^+ = \{a : \kappa a \leq \kappa c\}$, so $\downarrow c \leq (\kappa c)^+$. Thus $(\Downarrow b, \downarrow c) \subseteq ((\kappa b)^-, (\kappa c)^+)$. But as $\kappa b = \kappa c$ we have $(\kappa b)^-$ is covered by $(\kappa c)^+$ in $\kappa^{-1}Y$. □

Lemma 3.9 *For each $d \in D - \{0, 1\}$, the interval $[d^-, d^+]$ of X contains a cover which we denote $\mu_{d^-} < \mu_{d^+}$.*

Proof If d^- is covered by d^+, there is nothing to show. Otherwise there is some $x \in X$ with $d^- < x < d^+$. This means there exist $b \in x$ and $c \notin x$ with $\kappa b = \kappa c = d$. Then $d^- < \downarrow b < \downarrow c \leq d^+$. As each interval of C contains a cover, the interval $[b, c]$ contains a cover $p < q$. We set $\mu_{d^-} = \downarrow p$ and $\mu_{d^+} = \downarrow q$. □

Definition 3.10 For a finite rooted quasi-tree Q, let Q' be the quasi-tree obtained from Q by adding a single simple leaf off of each leaf of Q. We note that a leaf of Q may be a single node or a cluster, but the added leaf of Q' above each leaf of Q is a single node.

A typical example of building Q' from Q is given in the figure below.

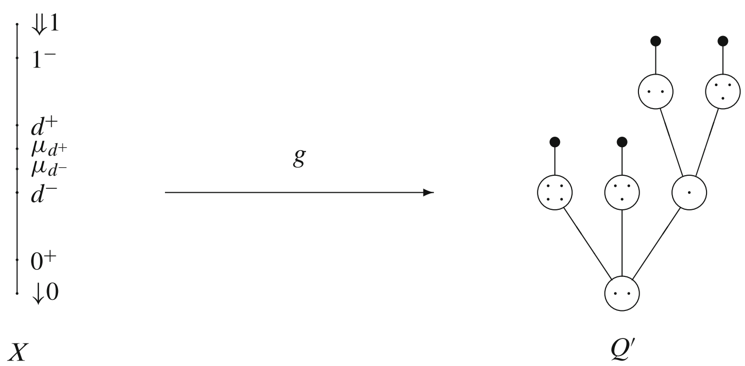


We equip Q with the Alexandroff topology, that is, the topology where the open sets are the upsets. As noted above, the Stone space of $A = B/I$ is homeomorphic to the Cantor space $\mathbf{C}$, so by [1, Lemma 4.5] there is an interior map from the Stone space of A onto Q, hence there is an onto interior map $f : \kappa^{-1}Y \to Q$, where $\kappa^{-1}Y$ has the subspace topology.

Definition 3.11 Define $g : X \to Q'$ as follows. On $\kappa^{-1}Y$ let g agree with f; on $(1^-, \Downarrow 1]$ let $g(x)$ be some leaf of Q' above $f(1^-)$; on $[\downarrow 0, 0^+)$ let $g(x)$ be a leaf of Q' above $f(0^+)$; and for each $d \in D - \{0, 1\}$ let $g(x)$ be a leaf of Q' above $f(d^-)$ on $(d^-, \mu_{d^-}]$, and let $g(x)$ be a leaf of Q' above $f(d^+)$ on $[\mu_{d^+}, d^+)$.

Remark 3.12 In some cases μ_{d^-} will equal d^- or μ_{d^+} will equal d^+ or both. So the intervals $(d^-, \mu_{d^-}]$ and $[\mu_{d^+}, d^+)$ can be empty. But this poses no difficulty. There is also ambiguity in the definition as to which leaf of Q' will be chosen as the constant value of g on the various intervals, at least in the case when any of $f(1^-)$, $f(0^+)$, $f(d^-)$, $f(d^+)$ has more than one leaf of Q above it. We could remove this ambiguity by requiring g to take some leftmost leaf of Q' above the given element of Q, but this is not necessary. Any map g with the above properties will suit our purpose, and there is at least one.

A typical situation is given in the figure below.



We equip Q' with the Alexandroff topology and show that $g : X \to Q'$ is an onto interior map.

Lemma 3.13 *g is continuous.*

Proof Note that $(1^-, \Downarrow 1]$ and $[\downarrow 0, 0^+)$ are open, and for $d \in D - \{0, 1\}$, as μ_{d^-} is covered by μ_{d^+}, we have $(d^-, \mu_{d^-}] = (d^-, \mu_{d^+})$ and $[\mu_{d^+}, d^+) = (\mu_{d^-}, d^+)$, so these intervals are open as well (although possibly empty). As g is constant on these intervals, it is continuous on them. It remains to show that g is continuous at each point of $\kappa^{-1}Y$.

Suppose $x \in \kappa^{-1}Y$ and $x \neq 0^+, 1^-$. As $f : \kappa^{-1}Y \to Q$ is continuous, there is a neighborhood of x in $\kappa^{-1}Y$ mapped by f into the upset of Q generated by $f(x) = g(x)$. From the description of the subspace topology on $\kappa^{-1}Y$ given before Lemma 3.7, we may assume this neighborhood is of the form $\kappa^{-1}Y \cap (c^-, e^+)$ for some $c < e$ in $D - \{0, 1\}$. By Lemma 3.7, (c^-, c^+) and (e^-, e^+) are disjoint from $\kappa^{-1}Y$, so $c^+ \leq x \leq e^-$. Then as $\mu_{c^-} < c^+$ and $e^- < \mu_{e^+}$, we have (μ_{c^-}, μ_{e^+}) is a neighborhood of x in X. The points in this interval belonging to $\kappa^{-1}Y$ are mapped by f, and hence by g, above $f(x) = g(x)$. A point z in this interval not in $\kappa^{-1}Y$ lies in (d^-, d^+) for some $d \in D - \{0, 1\}$ with $c < d < e$, so is mapped by g above either $f(d^-)$ or $f(d^+)$. As $d^-, d^+ \in \kappa^{-1}Y \cap (\mu_{c^-}, \mu_{e^+})$, they are mapped by f above $g(x)$, and it follows that z is mapped by g above $g(x)$. So this neighborhood is mapped by g into the upset of Q' generated by $g(x)$, showing g is continuous at x. The cases where $x = 0^+, 1^-$ are handled similarly using a neighborhood $[\downarrow 0, \mu_{c^+})$ if $x = 0^+$ and $(\mu_{c^-}, \Downarrow 1]$ if $x = 1^-$. □

Lemma 3.14 *g is open.*

Proof Suppose U is an open subset of X. We must show $g(U)$ is an upset of Q'. As $U \cap \kappa^{-1}Y$ is an open subset of $\kappa^{-1}Y$ and g agrees with the open map f on $\kappa^{-1}Y$, the image under g of $U \cap \kappa^{-1}Y$ is an upset of Q. Also, each point of $X - \kappa^{-1}Y$ is mapped by g to a simple leaf of Q'. So it remains to show that if $x \in U \cap \kappa^{-1}Y$ and l is a leaf of Q' above $f(x)$, then l is in the image of U.

As the image of $U \cap \kappa^{-1}Y$ is an upset of Q, there is some $y \in U \cap \kappa^{-1}Y$ with $f(y)$ lying in a cluster leaf L of Q where the cluster L lies above $f(x)$ and has l as the unique leaf of Q' above it. As f is continuous and L is an open subset of Q, there is some neighborhood of y in $\kappa^{-1}Y$ mapped by f into L. Therefore, there is some open subset V of X containing y so that $V \cap \kappa^{-1}Y$ is mapped by f into L, and clearly V can be chosen to be contained in U.

Claim For any open set T of X that contains a point y of $\kappa^{-1}Y$ there is some $d \in D - \{0, 1\}$ with $[d^-, d^+]$ contained in T and either $d^- < \mu_{d^-}$ or $\mu_{d^+} < d^+$, or both.

Proof of Claim The point y will be contained in some basic open interval of X that is contained in T, and the discussion following Lemma 3.6 shows this interval will be of one of the forms (i) $[\downarrow 0, \downarrow c)$, (ii) $(\Downarrow b, \Downarrow 1]$, or (iii) $(\Downarrow b, \downarrow c)$ for some $b, c \in C - \{0, 1\}$. In the first case, Lemma 3.8(a) shows $\kappa c \neq 0$, so as κ is onto and D is a dense chain, there is $b \in C - \{0, 1\}$ with $\kappa b < \kappa c$ and $(\Downarrow b, \downarrow c) \subseteq T$. In the second case, Lemma 3.8(b) shows $\kappa b \neq 1$, then giving some $c \in C - \{0, 1\}$ with $\kappa b < \kappa c$ and $(\Downarrow b, \downarrow c) \subseteq T$. In the final case, Lemma 3.8(c) shows $\kappa b < \kappa c$ and $(\Downarrow b, \downarrow c) \subseteq T$.

As D is dense, κ is onto, and each interval of C contains a cover, we can find elements $b < b_1 < p < q < c_1 < c$ in C so that $\kappa b < \kappa b_1 < \kappa c_1 < \kappa c$ and $p < q$ is a cover. As κ collapses covers, $\kappa p = \kappa q$, and we set $d = \kappa p = \kappa q$. As $b \in d^-$, $p \notin d^-$, $q \in d^+$, and $c \notin d^+$, we have $\Downarrow b < d^- < \downarrow p < \downarrow q \leq d^+ < \downarrow c$. So $[d^-, d^+]$ has more than two elements and is contained in $(\Downarrow b, \downarrow c)$, hence contained in T. As μ_{d^-} is covered by μ_{d^+}, our claim follows. □

To conclude the proof of the lemma, the set V contains a point of $\kappa^{-1}Y$, so we may apply the above claim to find some $[d^-, d^+]$ contained in V with either $d^- < \mu_{d^-}$ or $\mu_{d^+} < d^+$. As d^-, d^+ belong to $\kappa^{-1}Y$ and f maps $V \cap \kappa^{-1}Y$ into the cluster L, both $f(d^-)$ and $f(d^+)$ belong to L. By Lemma 3.7, at least one of μ_{d^-}, μ_{d^+} does not belong to $\kappa^{-1}Y$, so will be mapped to the leaf l of Q' above the cluster L. □

Lemma 3.15 *g is onto.*

Proof As f is onto Q, the image of g contains the elements in the cluster root of Q'. As the image of g is an upset of Q', it must be all of Q'. □

Theorem 3.16 *Let X be a metrizable Stone space and B be a countable Boolean algebra.*

(1) *If X is weakly scattered but not scattered, its modal logic is* **S4.1**.
(2) *If B is atomic, but not superatomic, its modal logic is* **S4.1**.

Proof These statements are equivalent. To prove them we use the method described in the previous section. By Proposition 2.5 it is sufficient to show that each finite rooted quasi-tree having simple clusters as its leaves is an interior image of X. Let Q be such a finite rooted quasi-tree having simple clusters as its leaves. From the above results, there is an interior map g from X onto Q'. But there is an interior map h from Q' onto Q: one maps elements of Q' belonging to Q to themselves, and maps a leaf l of Q' to the unique element in the simple leaf of Q beneath it. Then $h \circ g$ is an interior map from X onto Q. □

Assembling the above results we now axiomatize the modal logic of each metrizable Stone space.

Theorem 3.17 *Let X be a metrizable Stone space. Then:*

(1) *If X is scattered, its modal logic is* **S4.Grz** *or* $\mathbf{S4.Grz}_n$ *for some $n \geq 1$.*
(2) *If X is weakly scattered but not scattered, then its modal logic is* **S4.1**.
(3) *If X is not weakly scattered, its modal logic is* **S4**.

In algebraic form this becomes the following.

Theorem 3.18 *Let B be a countable Boolean algebra. Then:*

(1) *If B is superatomic, its modal logic is* **S4.Grz** *or* $\mathbf{S4.Grz}_n$ *for some $n \geq 1$.*
(2) *If B is atomic but not superatomic, its modal logic is* **S4.1**.
(3) *If B is not atomic, its modal logic is* **S4**.

This applies directly to zero-dimensional metrizable compactifications of ω [14].

Corollary 3.19 *If X is a zero-dimensional metrizable compactification of ω, then the modal logic of X is either* **S4.Grz**, $\mathbf{S4.Grz}_n$ *for some $n \geq 1$, or* **S4.1**. *Further,* **S4.1** *is the modal logic of the Pelczynski compactification of ω.*

Proof Let X be a zero-dimensional metrizable compactification of ω. Then ω is dense in X. Therefore, X is a weakly scattered Stone space, and it follows from Theorem 3.17 that its modal logic is either **S4.Grz**, $\mathbf{S4.Grz}_n$ for some $n \geq 1$, or **S4.1**. If X is the Pelczynski compactification of ω, then the remainder of X is homeomorphic to the Cantor space. Therefore, X is weakly scattered but not scattered, and so by Theorem 3.17, the modal logic of X is **S4.1**. □

4 Modal Logics of Extremally Disconnected Stone Spaces

We recall that a topological space X is *extremally disconnected* if the closure of each open subset of X is open in X; that is, the closure of each open subset of X is clopen in X. Equivalently, X is extremally disconnected if, and only if, regular open subsets of X coincide with clopen subsets of X. It is well known [18] that if X is the Stone space of a Boolean algebra B, then X is extremally disconnected if, and only if, B is complete. Our purpose in this section is to axiomatize the modal logics of extremally disconnected Stone spaces, or equivalently, of complete Boolean algebras.

An important example of an extremally disconnected Stone space is the Stone-Čech compactification $\beta(\omega)$ of ω. This is the Stone space of the power set $\mathcal{P}(\omega)$ of ω. In [3] we showed that the modal logic of $\beta(\omega)$ is **S4.1.2**. However, our proof used the set-theoretic assumption that each infinite maximal almost disjoint family of subsets of ω has cardinality 2^{ω}, which is not provable in ZFC. We use this result of [3] to prove the results in this section, so the results in this section depend, indirectly, on this additional set-theoretic assumption. We remark that if this assumption can be removed from the result for $\beta(\omega)$, then it will be removed from the results here as well.

Let X be an extremally disconnected Stone space. If X is finite, then X is discrete, and so the modal logic of X is $\mathbf{S4.Grz}_1$. Therefore, without loss of generality we may assume that X is infinite.

Proposition 4.1 *Let X be an infinite extremally disconnected Stone space. If X is weakly scattered, then the modal logic of X is* **S4.1.2**.

Proof Since X is extremally disconnected, $X \models \Diamond\Box p \to \Box\Diamond p$, and as X is weakly scattered, $X \models \Box\Diamond p \to \Diamond\Box p$. Therefore, $X \models \Box\Diamond p \leftrightarrow \Diamond\Box p$, and so $X \models \mathbf{S4.1.2}$ (see Proposition 2.2). Because X is infinite, the Boolean algebra B of clopen subsets of X is an infinite complete and atomic Boolean algebra. Let A be a countable set of atoms of B and let $b = \bigvee A$. Then the interval $[0, b]$ is isomorphic to the power set $\mathcal{P}(\omega)$ of ω. By Stone duality, the Stone space of $\mathcal{P}(\omega)$ is a clopen subset of X. But the Stone space of $\mathcal{P}(\omega)$ is homeomorphic to $\beta(\omega)$. Therefore, $\beta(\omega)$ is homeomorphic to a clopen subset of X. By [3, Theorem 5.2], the modal logic of $\beta(\omega)$ is **S4.1.2**. Thus, by Proposition 3.1, the modal logic of X is contained in **S4.1.2**. This together with $X \models \mathbf{S4.1.2}$ gives us the modal logic of X is **S4.1.2**. □

The proof of the next lemma was suggested to us by D. Monk.

Lemma 4.2 *Let B be an infinite complete Boolean algebra. Then $\mathcal{P}(\omega)$ is a homomorphic image of B.*

Proof Since B is an infinite Boolean algebra, B has a countable pairwise orthogonal set $P = \{a_n : n \in \omega\}$. Let S be the subalgebra of B generated by P. Clearly S is isomorphic to the Boolean algebra $\mathrm{FC}(\omega)$ of finite and cofinite subsets of ω. Therefore, there is an embedding $i : S \to \mathcal{P}(\omega)$. As S is a subalgebra of B and $\mathcal{P}(\omega)$ is complete, hence injective, we can extend i to $j : B \to \mathcal{P}(\omega)$. We show that j is onto. Let $A \subseteq \omega$ and let A^* be the corresponding set of elements of P. Then $A^* = \{a_n : n \in A\}$. Let $b = \bigvee A^*$. We show that $j(b) = A$. If $n \in A$, then $a_n \leq b$, so $j(a_n) \leq j(b)$, and so $n \in j(b)$. If $n \notin A$, then $a_n \notin A^*$, and so $a_n \wedge a = 0$ for each $a \in A^*$. Since B is complete, the infinite distributive law holds in B, by which $a_n \wedge \bigvee A^* = 0$. Therefore, $a_n \wedge b = 0$, so $j(a_n) \wedge j(b) = 0$, and so $n \notin j(b)$. Thus, $j(b) = A$, and so j is onto. Consequently, $\mathcal{P}(\omega)$ is a homomorphic image of B. □

Proposition 4.3 *Let X be an infinite extremally disconnected Stone space. If X is dense-in-itself, then the modal logic of X is* **S4.2***.*

Proof Since X is extremally disconnected, $X \models \Diamond\Box p \to \Box\Diamond p$. Therefore, $X \models$ **S4.2**. Let **S4.2** $\not\vdash \varphi$. By Proposition 2.5, there is a finite quasi-tree Q and a cluster $C = \{c_1, \ldots, c_k\}$ such that $Q \oplus C \not\models \varphi$. We define an interior map f from X onto $Q \oplus C$.

Let B be the Boolean algebra of clopen subsets of X. Then B is an infinite complete Boolean algebra. By Lemma 4.2, $\mathcal{P}(\omega)$ is a homomorphic image of B. By Stone duality, $\beta(\omega)$ is homeomorphic to a closed subset Y of X. Then Y is the disjoint union of Y_1 and Y_2, where Y_1 is homeomorphic to ω, and Y_2 is homeomorphic to the remainder $\omega^* = \beta(\omega) - \omega$. Because ω^* is closed, it is compact. Therefore, Y_2 is closed in X. Since X is dense-in-itself, so is the open set $X - Y_2$. Therefore, $X - Y_2$ is a dense-in-itself locally compact Hausdorff space. Thus, by [11, p. 332], we can split $X - Y_2$ into k nonempty disjoint pieces $C_1, \ldots C_k$ so that each open set that intersects $X - Y_2$ nontrivially intersects each of these sets nontrivially.

Since ω^* is homeomorphic to Y_2, by [3, Section 4] there is an onto interior map $g : Y_2 \to Q$. We define $f : X \to Q \oplus C$ so that f agrees with g on Y_2 and f takes value c_i on C_i for $i \leq k$. Clearly f is a well-defined onto map. To see that f is continuous, note that $g^{-1}(C) = X - Y_2$, which is open in X. Moreover, for each $q \in Q$, we have $f^{-1}(\uparrow q) = g^{-1}(\uparrow q) \cup (X - Y_2)$, which is again open in X. Therefore, f is continuous. To see that f is open, let U be a nonempty open subset of X. If $U \subseteq X - Y_2$, then as each C_i is dense in $X - Y_2$, U has a nonempty intersection with each C_i, and so $f(U)$ contains all of the cluster C, which is an upset of $Q \oplus C$. Otherwise, $U \cap Y_2 \neq \emptyset$, and as ω is dense in $\beta(\omega)$, the construction of Y, Y_1, and Y_2 shows $U \cap Y_1 \neq \emptyset$. Therefore, $f(U) = g(U \cap Y_2) \cup C$. Thus, $f(U)$ is an upset of $Q \oplus C$.

Consequently, $f : X \to Q \oplus C$ is an onto interior map, and so $X \not\models \varphi$. Thus, **S4.2** is the modal logic of X. □

Corollary 4.4 *The modal logic of the Gleason cover of the Cantor space is* **S4.2***.*

Proof Since the Cantor space **C** is dense-in-itself, so is the Gleason cover $\widehat{\mathbf{C}}$ of **C**. Therefore, by Proposition 4.3, the modal logic of $\widehat{\mathbf{C}}$ is **S4.2**. □

Putting the obtained results together, we arrive at the characterization of the modal logic of each extremally disconnected Stone space.

Theorem 4.5 *Let X be an extremally disconnected Stone space. Then:*

(1) *If X is finite, its modal logic is* $\mathbf{S4.Grz}_1$.
(2) *If X is infinite and weakly scattered, its modal logic is* **S4.1.2**.
(3) *If X is infinite and not weakly scattered, its modal logic is* **S4.2**.

Proof We already saw that (1) and (2) hold. To see that (3) holds, let X be an extremally disconnected Stone space and let Y be the closure of the set of isolated points of X. Then Y is a proper clopen subset of X. If $Y = \emptyset$, then X is dense-in-itself, and so by Proposition 4.3, the modal logic of X is **S4.2**. Suppose that $Y \neq \emptyset$ and let $Z = X - Y$. Then Y is a weakly scattered extremally disconnected Stone space, Z is a dense-in-itself extremally disconnected Stone space, and X is the disjoint union of Y and Z. By Proposition 4.1, the modal logic of Y is **S4.1.2**; by Proposition 4.3, the modal logic of Z is **S4.2**; so by Proposition 3.1 the modal logic of X is the intersection of the modal logic of Y and the modal logic of Z. Now as $\mathbf{S4.2} \subseteq \mathbf{S4.1.2}$, it follows that the modal logic of X is **S4.2**. □

Remark 4.6 Representing X as the disjoint union of Y and Z algebraically corresponds to representing a complete Boolean algebra B as the product $A \times C$, where A is complete and atomic and C is complete and atomless.

Theorem 4.7 *Let B be a complete Boolean algebra. Then:*

(1) *If B is finite, its modal logic is* $\mathbf{S4.Grz}_1$.
(2) *If B is infinite and atomic, its modal logic is* **S4.1.2**.
(3) *If B is infinite and not atomic, its modal logic is* **S4.2**.

We conclude this section by mentioning that it is still an open problem whether the characterization of Theorem 4.5 of the modal logic of each extremally disconnected Stone space can be done within ZFC.

5 Intermediate Case

We recall that logics containing the intuitionistic propositional logic **Int** and contained in the classical propositional logic **Cl** are called *intermediate logics*.

As we saw in Section 2, the modal logic of a topological space X is determined by the modal algebra $M(X)$ associated with X. If instead we work with the Heyting algebra $H(X)$ of open subsets of X, then we obtain the intermediate logic of X. Our purpose in this section is to describe the intermediate logics of metrizable Stone spaces and of extremally disconnected Stone spaces.

There are well-known methods (see, e.g., [7, Sections 3.9 and 9.6]) to translate an intermediate logic L into a normal extension L^* of **S4**, and to translate a normal extension S of **S4** into an intermediate logic S^*. We have $L^{**} = L$, but there can be $S_1 \neq S_2$ with $S_1^* = S_2^*$. We say S is a *modal companion* of L if $S^* = L$. For any topological space X, the modal logic of X is such a modal companion of the intermediate logic of X. These results will allow us to translate our results on modal logics of Stone spaces to the intuitionistic setting.

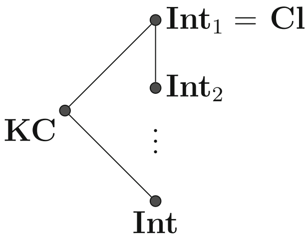
Definition 5.1 Let $\mathbf{KC} = \mathbf{Int} + (\neg p \vee \neg\neg p)$ be the logic of the weak excluded middle and $\mathbf{Int}_n$ denote $\mathbf{Int} + ibd_n$, where:

$$ibd_1 = p_1 \vee \neg p_1.$$

$$ibd_{n+1} = p_{n+1} \vee (p_{n+1} \to bd_n).$$

Here ibd_n is an intuitionistic version of the formula bd_n used in Section 2 to define $\mathbf{S4.Grz}_n$.

The following diagram shows containments between the intermediate logics described above.



It is well known (see, e.g., [7, Section 9.6]) that **S4**, **S4.1**, and **S4.Grz** are modal companions of **Int**; that **S4.2** and **S4.1.2** are modal companions of **KC**; and that $\mathbf{S4.Grz}_n$ is a modal companion of $\mathbf{Int}_n$ for each $n \geq 1$. Putting this together with our earlier results, we obtain:

Theorem 5.2

(1) *If X is a metrizable Stone space, then the intermediate logic of X is either* **Int** *or* $\mathbf{Int}_n$ *for some $n \geq 1$. In particular,* **Int** *is the intermediate logic of the Pelczynski compactification of ω.*

(2) *If X is an extremally disconnected Stone space, then the intermediate logic of X is* **KC** *or* **Cl***. In particular,* **KC** *is the intermediate logic of the Gleason cover of the Cantor space.*

Proof (1) Let X be a metrizable Stone space. First suppose that X is not scattered. By Theorem 3.17, the modal logic of X is either **S4** or **S4.1**. Since both of these are modal companions of **Int**, it follows that the intermediate logic of X is **Int**. In particular, since the Pelczynski compactification of ω is a metrizable non-scattered space, it follows that its intermediate logic is **Int**. Next suppose that X is scattered. By Proposition 3.5, the modal logic of X is either **S4.Grz** or $\mathbf{S4.Grz}_n$ for some $n \geq 1$, depending whether $\alpha \geq \omega^\omega$ or $\omega^n < \alpha < \omega^{n+1}$. Since **S4.Grz** is a modal companion of **Int** and $\mathbf{S4.Grz}_n$ is a modal companion of $\mathbf{Int}_n$, the intermediate logic of X is either **Int** or $\mathbf{Int}_n$ for some $n \geq 1$.

(2) Let X be an extremally disconnected Stone space. If X is finite, then by Theorem 4.5 the modal logic of X is $\mathbf{S4.Grz}_1$, which is a modal companion of $\mathbf{Int}_1 = \mathbf{Cl}$. Therefore, the intermediate logic of X is **Cl**. On the other hand, if X is infinite, then Theorem 4.5 implies that the modal logic of X is **S4.2** or **S4.1.2**. Since both of these are modal companions of **KC**, it follows that the intermediate logic of X is **KC**. In particular, **KC** is the intermediate logic of the Gleason cover of the Cantor space. □

Putting Theorem 5.2 with [3] yields:

Int	=	The intermediate logic of the Pelczynski compactification of ω
	=	The intermediate logic of any ordinal $\alpha \geq \omega^\omega$
	=	The intermediate logic of the remainder ω^* of $\beta(\omega)$.
KC	=	The intermediate logic of the Gleason cover of the Cantor space
	=	The intermediate logic of $\beta(\omega)$.

Again, our proof of Theorem 5.2(2) and the above result for **KC** rely on Theorem 4.5, so we use an assumption of set theory past ZFC. We do not know if these results can be proved in ZFC.

6 The General Problem

In this section we make some remarks on the general problem of describing the modal logics of Stone spaces. First, we collect, and slightly extend, some results established above.

Theorem 6.1 *Each modal logic that is an intersection of some collection of the logics* **S4***,* **S4.1***,* **S4.2***,* **S4.1.2***,* **S4.Grz***, and* $\mathbf{S4.Grz}_n$ *where* $n \geq 1$*, is the modal logic of some Stone space.*

Proof By the original result of McKinsey and Tarski [13], **S4** is the modal logic of the Cantor space. Theorem 3.16(2) shows **S4.1** is the modal logic of the Stone space of the Boolean algebra generated by the chain $\mathbb{Q} \times 2$, with lexicographic order, as each interval in this chain has a cover, and the quotient obtained by collapsing covers is $\mathbb{Q}$. Theorem 4.7 shows **S4.2** is the modal logic of the Stone space of any complete atomless Boolean algebra, and **S4.1.2** is the modal logic of the Stone space of any complete atomic Boolean algebra. Proposition 3.5 shows **S4.Grz** is the modal logic of $\omega^\omega + 1$ with the interval topology, and $\mathbf{S4.Grz}_n$ is the modal logic of $\omega^n + 1$ with the interval topology. Noting that the disjoint union of finitely many Stone spaces is a Stone space, it follows from Proposition 3.1 that any finite intersection of these logics can be obtained as the modal logic of some Stone space. As **S4.Grz** is the intersection of the $\mathbf{S4.Grz}_n$, any intersection of these logics can be obtained as a finite intersection. □

Question 6.2 Is there a Stone space whose modal logic is not an intersection of some collection of the logics **S4**, **S4.1**, **S4.2**, **S4.1.2**, **S4.Grz**, and $\mathbf{S4.Grz}_n$ where $n \geq 1$.

Understanding better when a finite quasi-tree is an interior image of a Stone space X would perhaps shed light on this question. For all the cases considered above, when X had an interior image of a certain kind, it would have all members of one of the classes in Proposition 2.2 as interior images. For instance, in all cases above, if X had a tree of depth 2 with at least two branches as an interior image, then every tree of depth 2 was an interior image of X. This allowed us to show the modal logic of X was contained in $\mathbf{S4.Grz}_2$.

We collect below a number of fairly simple observations about when some particularly easy quasi-trees are interior images of Stone spaces. Still, we do not know if there is a Stone space X having a tree of depth 2 with two branches as an interior image, but not a tree of depth 2 with three branches as an interior image. Such an example, if one exists, would perhaps be a place to look to settle Question 6.2. Our results here show there is no such X that is metrizable or extremally disconnected.

To begin our observations, we require the following definition.

Definition 6.3 An ideal of a Boolean algebra B is a dense ideal if it is non-principal and has join 1.

Recall that ideals of a Boolean algebra correspond to open subsets of its Stone space. Since a subset D of a Stone space X is dense if, and only if, X is the only clopen set containing D, it follows that dense ideals correspond to dense open subsets. We recall also that each ideal I of a Boolean algebra B gives a congruence, and the associated quotient is written B/I.

Lemma 6.4 *Suppose B is a Boolean algebra with Stone space X, I is an ideal of B, and there is an interior map from the Stone space of B/I onto an* **S4***-frame $\mathfrak{F}$.*

(1) *If I is dense, there is an interior map from X onto $\mathfrak{F}^\top = \mathfrak{F} \oplus \{t\}$.*
(2) *If $\mathfrak{F}$ has a simple maximum cluster, there is an interior map from X onto $\mathfrak{F}$.*

Proof (1) Suppose the Stone space of B is X and U is the dense open subset of X corresponding to the ideal I. Then the Stone space of B/I is the closed subspace $Y = X - U$ of X. Let $f : Y \to \mathfrak{F}$ be the given interior map. Define $g : X \to \mathfrak{F}^\top$ by letting g agree with f on Y and having g send everything in U to the top element t of $\mathfrak{F}^\top$. Surely g is onto. To see it is continuous, it is clearly continuous at each point of the open set U. For any point $x \in Y$ we have $g(x) = f(x)$. As f is continuous at x, there is an open neighborhood of x in Y that maps into the upset of $\mathfrak{F}$ generated by $f(x)$. This neighborhood is of the form $V \cap Y$ for some open neighborhood V of x in X. Then g maps V into the upset of $\mathfrak{F}^\top$ generated by $f(x)$. So g is continuous. To see g is open, take any nonempty open set V of X. As U is dense, we have V intersects U nontrivially. Then the image $g(V)$ is equal to $f(V \cap Y) \cup \{t\}$. As f is interior, $f(V \cap Y)$ is an upset of $\mathfrak{F}$, so $g(V)$ is an upset of $\mathfrak{F}^\top$.

The proof of (2) is similar, mapping all elements of U to the existing top element of $\mathfrak{F}$. □

In the terminology of Definition 3.4, for any superatomic Boolean algebra B, there is an ordinal α with $I_\alpha = I_{\alpha+1}$. We call the *degree* of a superatomic Boolean algebra the least α for which this happens. So B has degree 0 if it is trivial, degree 1 if it is finite but nontrivial, degree 2 if it is infinite and B modulo its elements of finite height is finite, and so forth.

Proposition 6.5 *If B is a nontrivial superatomic Boolean algebra, then there is an interior map from the Stone space of B onto the n-element chain if, and only if, the degree of B is at least n.*

Proof We show by induction that if B is superatomic with degree at least n, then there is an interior map from the Stone space of B onto the n-element chain. For the base case, there is an interior map from the Stone space of any nontrivial Boolean algebra onto the 1-element chain. Suppose B has degree at least $n+1$. As B is superatomic, hence atomic, the ideal I of elements of finite height in B is a dense ideal. As B/I has degree at least n, there is an interior map from the Stone space of B/I onto the n-element chain. Then by Lemma 6.4 there is an interior map from the Stone space of B onto the $(n+1)$-element chain.

For the converse, we show by induction that if B has degree at most n, there can be no interior map from the Stone space of B onto a finite chain with more than n elements. The base case follows as the Stone space of a finite Boolean algebra is discrete. Suppose B has degree at most n, that X is the Stone space of B, and f is an interior map from X onto a chain C with more than n elements. Let t be the top of C, let s be the element beneath it, and let $D = C - \{t\}$. So $C = D \cup \{t\}$ and s is the top element of D. Let U be the set of isolated points of X, let $Y = X - U$, and note that f maps all elements in U to t. Define $g : Y \to D$ by setting $g(y) = s$ if $f(y) = t$, and letting g agree with f otherwise. Then Y is the Stone space of a Boolean algebra of degree at most $n-1$ and g is an interior map from Y onto a chain with more than $n-1$ elements, contrary to our inductive hypothesis. □

Before settling the matter of having an n-element chain as an interior image of the Stone space of a Boolean algebra, we establish a lemma whose proof was suggested to us by I. Juhasz. The key ingredient in this proof is the notion of an *irreducible* map $f : X \to Y$ between topological spaces. This is a continuous map that is onto, but has no restriction to a proper closed subspace that is onto. A simple Zorn's lemma argument shows that any continuous onto map $f : X \to Y$ between compact Hausdorff spaces has a restriction to a closed subset of X that is irreducible. Further, if $f : X \to Y$ is irreducible and X, Y are compact Hausdorff spaces, then the image of any nonempty open set of X contains a nonempty open set of Y. For further details see [8, p. 55].

Lemma 6.6 *If B is an atomless Boolean algebra, then B has a dense ideal I with B/I atomless.*

Proof We prove a stronger topological statement, that if X is a dense-in-itself compact Hausdorff space, then X has a dense open subset Y with $X - Y$ dense-in-itself. By [16, Theorem 8.5.4], for such X there is a continuous map $f : X \to \mathbf{I}$ onto the unit interval $\mathbf{I} = [0, 1]$. Let U be an open set so that the restriction of f to $X - U$ is an irreducible map onto $\mathbf{I}$, and let $g : (X - U) \to \mathbf{I}$ be this irreducible restriction. For the Cantor set $\mathbf{C}$ let $V = g^{-1}(\mathbf{I} - \mathbf{C})$. Then V is open in $X - U$, and as g is irreducible and $\mathbf{I} - \mathbf{C}$ is dense in $\mathbf{I}$, we have V is dense in $X - U$ since a nonempty open subset of $X - U$ disjoint from V would have its image contain a nonempty open subset of $\mathbf{I}$ that is contained in $\mathbf{C}$. So $U \cup V$ is a dense open subset of X. As $g : (X - (U \cup V)) \to \mathbf{C}$ is onto, there is an open subset W of $X - (U \cup V)$ so that the restriction h of g to $X - (U \cup V \cup W)$ is an irreducible map onto $\mathbf{C}$. Set $Y = U \cup V \cup W$ and note that Y is dense open in X. Further, as $h : (X - Y) \to \mathbf{C}$ is irreducible and $\mathbf{C}$ is dense-in-itself, it follows that $X - Y$ is dense-in-itself. Indeed, if $\{x\}$ is an isolated point of $X - Y$, then the irreducibility of h would imply that $\{h(x)\}$ contains a nonempty open set of $\mathbf{C}$, an impossibility. □

Proposition 6.7 *If B is a nontrivial Boolean algebra that is not superatomic, then there is an interior map from the Stone space of B onto the n-element chain for each $n \geq 1$.*

Proof Suppose B is not superatomic. By [12, p. 271], there is a quotient Q of B that is not atomic. Take an element $q \in Q$ with no atoms beneath it. As the interval $[0, q]$ is isomorphic to a quotient of Q, there is a quotient Q' of B that is atomless. By the second part of Lemma 6.4 it is enough to show that for each $n \geq 1$, the n-element chain is an interior image of the Stone space of Q'. We prove by induction on n that the n-element chain is an interior image of the Stone space of any atomless Boolean algebra. For $n = 1$ this is trivial. Assume the statement is true for n, and let A be an atomless Boolean algebra. By Lemma 6.5, A has a dense ideal I with A/I atomless. By the inductive hypothesis, there is an interior map from the Stone space of A/I onto the n-element chain, so by the first part of Lemma 6.4 there is an interior map from the Stone space of A onto the $n + 1$-element chain. □

Combining Propositions 6.5 and 6.7 then gives the following.

Corollary 6.8 *Let B be a nontrivial Boolean algebra with Stone space X and C be the n-element chain.*

(1) *If B is not superatomic, then there is an interior map from X onto C.*
(2) *If B is superatomic, then there is an interior map from X onto C if, and only if, the degree of B is at least n.*

In particular, a Boolean algebra has an interior map from its Stone space onto the 2-element chain if, and only if, it is infinite.

Having settled matters for chains, we consider a few other configurations.

Definition 6.9 Let V be the three-element tree having one root and two maximal nodes.

Proposition 6.10 *There is an interior map from the Stone space of B onto V if, and only if, B is not complete.*

Proof If B is not complete, there is a regular open subset R of the Stone space X that is not clopen. Then $R' = \text{int}(X - R)$ is also regular open, $R \cup R'$ is dense open, and $R \cup R' \neq X$. Define $f : X \to V$ by sending everything in R to one of the two maximal nodes, everything in R' to the other maximal node, and the rest to the root. Clearly f is continuous and onto. To see f is open, we need only show that if U is an open set containing a point of X not belonging to $R \cup R'$, then U intersects both R and R' nontrivially. But this follows as regularity shows each of R, R' is the interior of the complement of the other. Conversely, suppose B is complete and $f : X \to V$ is an interior map. Let p, q be the two maximal nodes of V. Then $\{p\}$ and $\{q\}$ are regular open in V, so $f^{-1}(p)$ and $f^{-1}(q)$ are regular open in X, and as B is complete, this means they are clopen in X. But $\{p, q\}$ is dense in V, so $f^{-1}(p) \cup f^{-1}(q)$ is dense in X, and as this is clopen, it must be all of X. Therefore, there can be nothing sent by f to the root of V, so f is not onto. □

We consider the four-element diamond $\Diamond = V \oplus \{t\}$.

Proposition 6.11 *If B is a Boolean algebra with Stone space X, then there is an interior map from X onto the three-element chain* 3 *if, and only if, there is an interior map from X onto the four-element diamond $\Diamond$.*

Proof If there is an interior map from X onto $\Diamond$, then, as there is an interior map from $\Diamond$ onto 3, there is an interior map from X onto 3. Conversely, suppose there is an onto interior map $f : X \to 3$. For u the top element of 3 we have $\{u\}$ is a dense open subset of 3, so $U = f^{-1}(u)$ is a dense open subset of X. Let I be the dense ideal of B that corresponds to U. Then the Stone space of B/I is homeomorphic to $X - U$. As f restricts to an interior map from $X - U$ onto the two-element chain, by Corollary 6.8, B/I is infinite. As $\mathcal{P}(\omega)$ has an incomplete quotient, namely the quotient by the ideal of finite subsets of ω, and Lemma 4.2 shows every infinite complete Boolean algebra has a quotient isomorphic to $\mathcal{P}(\omega)$, it follows that every infinite Boolean algebra has an incomplete quotient. So there is a quotient of B/I that is incomplete, hence an ideal $J \supseteq I$ corresponding to this quotient, with B/J incomplete. By Proposition 6.10, there is an interior map from the Stone space of B/J onto V, and as J is dense, Lemma 6.4 shows there is an interior map from X onto $\Diamond$. □

Remark 6.12 Increasing the branching size from two to three seems to impart substantial difficulties. Consider the simplest case of the four element tree W with one root and three maximal nodes. We have no useful characterization, past a fairly simple direct translation in terms of regular open sets or normal ideals, of when there will be an interior map from the Stone space of a Boolean algebra onto W. This seems a basic problem in determining the possible modal logic of the Stone space of an arbitrary Boolean algebra.

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References

1. Aiello, M., van Benthem, J., Bezhanishvili, G.: Reasoning about space: the modal way. J. Log. Comput. **13**(6), 889–920 (2003)
2. Bezhanishvili, G., Gehrke, M.: Completeness of S4 with respect to the real line: revisited. Ann. Pure Appl. Logic **131**(1–3), 287–301 (2005)
3. Bezhanishvili, G., Harding, J.: The modal logic of $\beta(\mathbb{N})$. Arch. Math. Log. **48**(3–4), 231–242 (2009)
4. Bezhanishvili, G., Mines, R., Morandi, P.J.: Scattered, Hausdorff-reducible, and hereditarily irresolvable spaces. Topol. Appl. **132**(3), 291–306 (2003)

5. Bezhanishvili, G., Morandi, P.J.: Scattered and hereditarily irresolvable spaces in modal logic. Arch. Math. Log. **49**(3), 343–365 (2010)
6. Blackburn, P., de Rijke, M., Venema, Y.: Modal Logic. Cambridge University Press, Cambridge (2001)
7. Chagrov, A., Zakharyaschev, M.: Modal Logic. Oxford Logic Guides, vol. 35. The Clarendon Press Oxford University Press, New York (1997)
8. Comfort, W.W., Negrepontis, S.: The Theory of Ultrafilters. Die Grundlehren der Mathematischen Wissenschaften, Band 211. Springer, New York (1974)
9. de Groot, J.: Topological classification of all closed countable and continuous classification of all countable pointsets. Indag. Math. **7**, 42–53 (1945)
10. Gabelaia, D.: Modal definability in topology. Master's thesis, ILLC, University of Amsterdam (2001)
11. Hewitt, E.: A problem of set-theoretic topology. Duke Math. J. **10**, 309–333 (1943)
12. Koppelberg, S.: Handbook of Boolean Algebras, vol. 1. North-Holland, Amsterdam (1989)
13. McKinsey, J.C.C., Tarski, A.: The algebra of topology. Ann. Math. **45**, 141–191 (1944)
14. Pełczyński, A.: A remark on spaces 2^X for zero-dimensional X. Bull. Acad. Pol. Sci. Sér. Sci. Math. Astron. Phys. **13**, 85–89 (1965)
15. Rosenstein, J.G.: Linear Orderings. Pure and Applied Mathematics, vol. 98. Academic, New York (1982)
16. Semadeni, Z.: Banach Spaces of Continuous Functions, vol. I. PWN—Polish Scientific Publishers, Warsaw (1971)
17. Stone, M.H.: The theory of representations for Boolean algebras. Trans. Am. Math. Soc. **40**(1), 37–111 (1936)
18. Stone, M.H.: Algebraic characterizations of special Boolean rings. Fundam. Math. **29**, 223–302 (1937)
19. van Benthem, J., Bezhanishvili, G.: Modal logics of space. In: Handbook of Spatial Logics, pp. 217–298. Springer, Dordrecht (2007)
20. van Benthem, J., Bezhanishvili, G., Gehrke, M.: Euclidean hierarchy in modal logic. Stud. Log. **75**(3), 327–344 (2003)