Projective bichains

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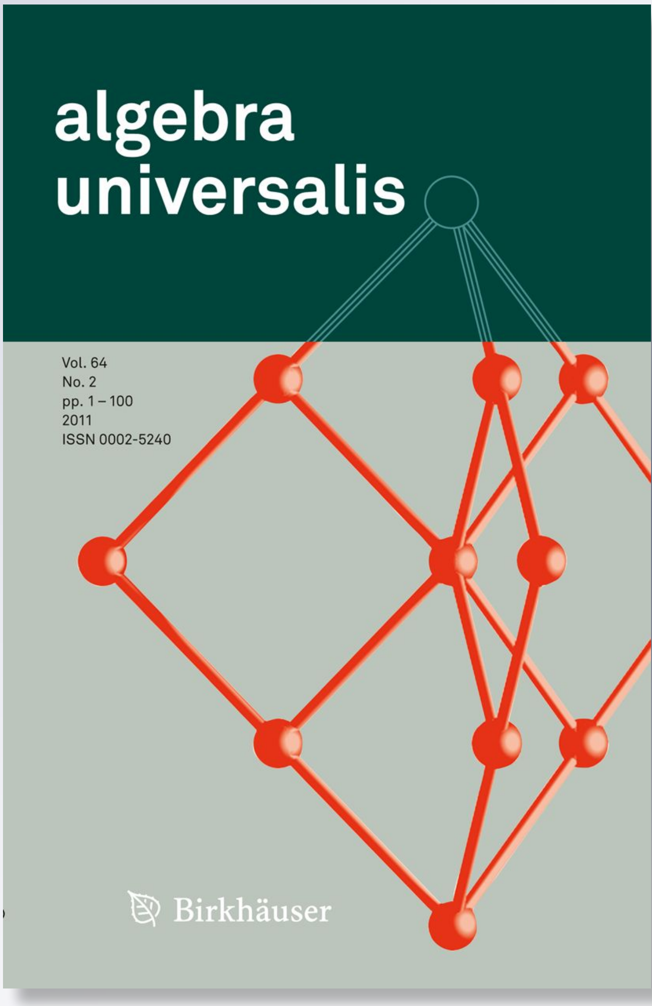

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Abstract. An algebra with two binary operations $\cdot$ and $+$ that are commutative, associative, and idempotent is called a bisemilattice. A bisemilattice that satisfies Birkhoff's equation $x \cdot (x + y) = x + (x \cdot y)$ is a Birkhoff system. Each bisemilattice determines, and is determined by, two semilattices, one for the operation $+$ and one for the operation $\cdot$. A bisemilattice for which each of these semilattices is a chain is called a bichain. In this note, we characterize the finite bichains that are weakly projective in the variety of Birkhoff systems as those that do not contain a certain three-element bichain. As subdirectly irreducible weak projectives are splitting, this provides some insight into the fine structure of the lattice of subvarieties of Birkhoff systems.

1. Introduction

A *bisemilattice* is an algebra with two binary operations $+$ and $\cdot$ that are commutative, associative, and idempotent. A *Birkhoff system* is a bisemilattice satisfying Birkhoff's equation $x \cdot (x + y) = x + (x \cdot y)$. There is a substantial literature on these structures, see for instance [2, 4, 9, 10, 11, 12, 13, 14].

A bisemilattice is given by two semilattice operations on the same underlying set, one for the operation $\cdot$ and one for the operation $+$. A bisemilattice for which each of these semilattices is a chain is called a *bichain*. It is not difficult to see that any bichain satisfies Birkhoff's equation and so is a Birkhoff system. A particular three-element bichain N is shown in Figure 1. In this diagram, the left side is the Hasse diagram for the meet semilattice given by $\cdot$, and the right side is the Hasse diagram for the join semilattice given by $+$. So N has three elements a, b, c, where $b \cdot c = b$, $b + c = c$, and a is absorbing.

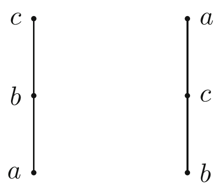

Figure 1. The 3-element bichain N.

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The purpose of this paper is to show that a finite bichain is weakly projective in the variety of Birkhoff systems if and only if it does not contain a subalgebra isomorphic to N. As every weakly projective subdirectly irreducible algebra is splitting, this sheds additional light on the lattice of subvarieties of Birkhoff systems.

2. Preliminaries

We begin with some basics about weakly projective algebras [1, p. 36].

Definition 2.1. An algebra P in a variety $\mathcal{V}$ is *weakly projective* in $\mathcal{V}$ if for any algebras $A, B \in \mathcal{V}$, any homomorphism $f\colon P \to B$, and any homomorphism $g\colon A \to B$ onto B, there is a homomorphism $h\colon P \to A$ with $g \circ h = f$.

This definition coincides with the usual categorical definition of projective if epimorphisms in the variety $\mathcal{V}$ are surjective. We do not know whether epimorphisms in the variety of Birkhoff systems are surjective. The following well-known result will be our operational form of weak projectivity.

Proposition 2.2. *An algebra P is weakly projective in $\mathcal{V}$ if and only if for any $A \in \mathcal{V}$ and any homomorphism $f\colon A \to P$ onto P, there is a subalgebra B of A so that the restriction of f to B is an isomorphism of B onto P.*

We recall that an algebra R is a *retract* of an algebra A if there are homomorphisms $f\colon A \to R$ and $g\colon R \to A$ with $f \circ g$ being the identity map on R. The following is well known [1, p. 36].

Proposition 2.3. *If P is weakly projective in $\mathcal{V}$ and R is a retract of P, then R is weakly projective in $\mathcal{V}$.*

The subdirectly irreducible weakly projective algebras have an additional feature making them of particular interest when considering subvarieties of a variety $\mathcal{V}$ [5, p. 345].

Proposition 2.4. *Suppose P is an algebra that is subdirectly irreducible and weakly projective in the variety $\mathcal{V}$. Then the class $\mathcal{V}_P$ of all algebras A in $\mathcal{V}$ that do not contain a subalgebra isomorphic to P is the largest subvariety of $\mathcal{V}$ not containing P. Thus, P is splitting in $\mathcal{V}$.*

We now turn our attention to Birkhoff systems. For notational convenience, we often write $x \cdot y$ simply as xy, and when working with the meet or join of a finite family of elements x_i for $(i \in I)$, we will write $\prod_I x_i$ and $\sum_I x_i$.

Definition 2.5. Given a bisemilattice $(A, \cdot, +)$, we let $\leq_L$ be the partial ordering on A determined by the meet operation $\cdot$, and $\leq_R$ the partial ordering on A determined by the join operation $+$. So,

$$x \leq_L y \Leftrightarrow xy = x \text{ and } x \leq_R y \Leftrightarrow x + y = y.$$

We call $\leq_L$ the left order, and $\leq_R$ the right order as we draw pairs of Hasse diagrams to depict bisemilattices with $\leq_L$ on the left, and $\leq_R$ on the right.

To describe a bisemilattice, it is enough to describe its partial orderings $\leq_L$ and $\leq_R$. For a bichain, these will be total orderings. We consider bichains on the underlying set $\{c_1, \ldots, c_n\}$ where $c_1 <_L \cdots <_L c_n$ and $c_{\sigma 1} <_R \cdots <_R c_{\sigma n}$ for some permutation σ of $\{1, \ldots, n\}$. See Figure 2. Up to isomorphism, every finite bichain occurs this way, and it follows that up to isomorphism, there are $n!$ bichains with n elements.

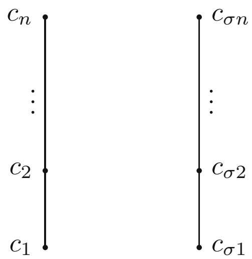

FIGURE 2. A typical finite bichain.

A small observation will be important. Clearly, c_i occurs in the i^{th} spot from the bottom on the left. Note that c_i appearing in the p^{th} spot from the bottom on the right means $i = \sigma p$. So c_i appears at the $(\sigma^{-1} i)^{th}$ spot from the bottom on the right.

Proposition 2.6. *Every subset of a bichain is a subalgebra; therefore, every quotient of a bichain is a retract.*

Proof. As $\leq_L$ is a chain under meet, xy is either x or y and as $\leq_R$ is a chain under join, $x + y$ is either x or y. So any subset is a subalgebra. If C is a bichain and $f\colon C \to A$ is a surjective homomorphism, choosing one element from each equivalence class of the kernel of f gives a subalgebra S of C mapping isomorphically onto A, so A is a retract of C. $\square$

We next collect a few basic facts about Birkhoff systems.

Lemma 2.7. *In any Birkhoff system*

(1) *If* $qa_i = q$ *for* $i \leq n$, *then* $q(a_1 + \cdots + a_n) = q$.
(2) *If* $q + a_i = q$ *for* $i \leq n$, *then* $q + a_1 \cdots a_n = q$.
(3) *If* $x_1 \leq_L \cdots \leq_L x_n$, *then* $x_n(x_1 + \cdots + x_{n-1}) = x_1 + \cdots + x_{n-1}$.
(4) *If* $x_n \leq_R \cdots \leq_R x_1$, *then* $x_n + (x_1 \cdots x_{n-1}) = x_1 \cdots x_{n-1}$.
(5) *If* $x_1 \leq_L \cdots \leq_L x_n$, *then* $(x_1 + \cdots + x_k)(x_{k+1} + \cdots + x_n) = x_1 + \cdots + x_k$.
(6) *If* $x_n \leq_R \cdots \leq_R x_1$, *then* $(x_1 \cdots x_k) + (x_{k+1} \cdots x_n) = x_1 \cdots x_k$.
(7) *If* $x_1 \leq_L \cdots \leq_L x_n$, *then* $(x_1 + \cdots + x_k)(x_1 + \cdots + x_n) = x_1 + \cdots + x_k$.
(8) *If* $x_n \leq_R \cdots \leq_R x_1$, *then* $(x_1 \cdots x_k) + (x_1 \cdots x_n) = x_1 \cdots x_k$.
(9) $xy(x + y) = xy$.
(10) $x + y + xy = x + y$.

Proof. We prove only the odd numbered statements. The even numbered ones are dual to them.

(1): We do this first for $n = 2$. By Birkhoff's equation,

$$\begin{aligned} q(a_1 + a_2) &= qa_1(a_1 + a_2) = q(a_1 + a_1a_2) = qa_1a_2(a_1 + a_1a_2) \\ &= q(a_1a_2 + a_1a_2a_1) = qa_1a_2 = q. \end{aligned}$$

For $n > 2$, we have $q(a_1 + \cdots + a_{n-1} + a_n) = q((a_1 + \cdots + a_{n-1}) + a_n)$ and the result follows from the inductive hypothesis and the specific case when $n = 2$.

(3): Induct on n. The base case $n = 2$ is trivial. For $n > 2$, the inductive hypothesis gives $x_n(x_2 + \cdots + x_{n-1}) = x_2 + \cdots + x_{n-1}$ and part (1) gives $x_1(x_2 + \cdots + x_{n-1}) = x_1$. Therefore,

$$\begin{aligned} x_n(x_1 + \cdots + x_{n-1}) &= x_n(x_1(x_2 + \cdots + x_{n-1}) + (x_2 + \cdots + x_{n-1})) \\ &= x_n(x_2 + \cdots + x_{n-1})(x_1 + \cdots + x_{n-1}) \\ &= (x_2 + \cdots + x_{n-1})(x_1 + \cdots + x_{n-1}) \\ &= x_1(x_2 + \cdots + x_{n-1}) + (x_2 + \cdots + x_{n-1}) = x_1 + \cdots + x_{n-1}. \end{aligned}$$

(5): For each $i \leq k$, we have $x_ix_j = x_i$ for each $j = k+1, \ldots, n$. It then follows from part (1) that $x_i(x_{k+1} + \cdots + x_n) = x_i$ for each $i \leq k$. Therefore, we have $x_1 \leq_L \cdots \leq_L x_k \leq_L (x_{k+1} + \cdots + x_n)$. It then follows from part (3) that $(x_1 + \cdots + x_k)(x_{k+1} + \cdots + x_n) = x_1 + \cdots + x_k$.

(7): This is a trivial consequence of part (5) and Birkhoff's identity.

(9): This follows from part (1) as $(xy)x = xy$ and $(xy)y = xy$. □

Note, when we dualize $x_1 \leq_L \cdots \leq_L x_n$ to $x_n \leq_R \cdots \leq_R x_1$ in the above lemma, the order reverses because $x_1 \leq_L x_2$ means $x_1x_2 = x_1$. So interchanging meet and join, the condition translates to $x_1 + x_2 = x_1$, and this means $x_2 \leq_R x_1$.

3. A technical lemma

In the previous section, we described how each bichain is isomorphic to one with underlying set $\{c_1, \ldots, c_n\}$ where $c_1 <_L \cdots <_L c_n$ and $c_{\sigma 1} <_R \cdots <_R c_{\sigma n}$ for some permutation σ of $\{1, \ldots, n\}$. In this section, we consider bichains given by a permutation σ with additional properties.

Definition 3.1. A permutation σ of $\{1, \ldots, n\}$ will be called *bipartite* if it maps $\{1, \ldots, (\sigma^{-1}n) - 1\}$ to itself.

The diagram in Figure 3 should be treated as a visual aid, and nothing more. We find it convenient to introduce $k = (\sigma^{-1}n) - 1$. Being bipartite means that the portion below the dashed line, called the *lower part*, is mapped to itself, and consequently the portion above the dashed line, called the *upper part*, is also mapped to itself. Extend this terminology and call a bichain C bipartite if its associated permutation is bipartite. We note that the bichains described in the introduction that do not contain a subalgebra isomorphic to the three-element bichain N are all bipartite, and remark that there are bichains given by bipartite permutations that do contain N.

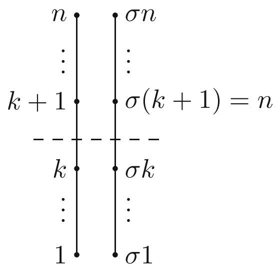

FIGURE 3. A bipartite bichain.

The following is the key technical result we shall need.

Lemma 3.2. *Suppose σ is a bipartite permutation of $\{1, \ldots, n\}$ and $x_1, \ldots, x_n$ are elements of a Birkhoff system A with $x_1, \ldots, x_n$ forming a chain under meet in A with x_1 at the bottom and x_n at the top. Define elements x_i^1, x_i^2 for $i \leq n$ as follows:*

$$x_{\sigma i}^1 = \sum \{x_{\sigma j} : 1 \leq j \leq i\},$$

$$x_i^2 = \prod \{x_j^1 : i \leq j \leq n\}.$$

Then $x_i^2 + x_n^2 = x_n^2$ for each $i \leq k = (\sigma^{-1} n) - 1$.

Before the proof, we consider a few pictures to illustrate the elements involved. To begin, things look as in Figure 4 with the left side a chain under meet.

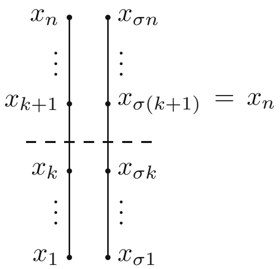

FIGURE 4

To construct the x_i^1, we make sums with progressively more summands so they form a chain under join as shown in Figure 5. Then to construct the x_i^2, we take products with progressively more factors so they form a chain under meet. This is shown in Figure 6.

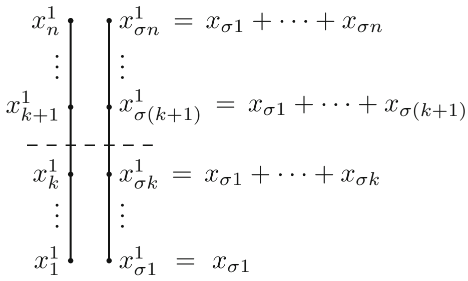

FIGURE 5

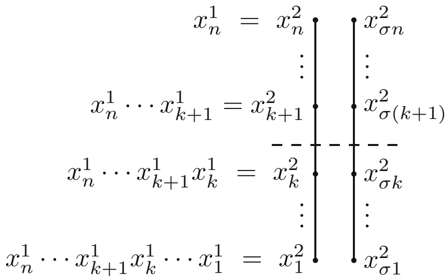

FIGURE 6

Proof of the Lemma. The proof is by induction on the number of elements in the upper part of the permutation, that is, on $n - (\sigma^{-1}n) + 1$.

The base case says there is one element in the upper part, or in other words, that our partition has $\sigma n = n$. Then $x_{\sigma 1}^1, \ldots, x_{\sigma n}^1$ form a chain under join with $x_{\sigma n}^1$ at the top, and x_i^2 is a meet of elements of this chain, so it follows from Lemma 2.7 (2) that $x_i^2 + x_n^2 = x_n^2$ as required.

For the inductive step, we assume the statement is proved for all bipartite permutations having some fixed number F of elements in their upper part. We must prove the statement for any bipartite permutation γ having $F+1$ elements in its upper part. Suppose γ is a bipartite permutation on $\{1, \ldots, n+1\}$. We note that n or $n+1$ is not directly part of our induction, and the choice of $n+1$ here is for notational convenience in our work. Let $k = (\gamma^{-1}(n+1)) - 1$ be the spot where the dividing line occurs for this γ, and let $q = \gamma(n+1)$. As we are not in the base case, $q \neq n+1$, and as γ is bipartite, $k < q < n+1$. Figure 7 describes γ. It is not intended to indicate any covering relationships involving q.

To use the inductive hypothesis, we create a bipartite permutation σ from γ where σ has one fewer element in its upper part. Informally, build σ by removing q from the picture for γ and then renumber things to not have the gap where q should be. Precisely, we define σ to be the partition of $\{1, \ldots, n\}$ with $\sigma i = \gamma i$ if $\gamma i < q$ and $\sigma i = (\gamma i) - 1$ if $q < \gamma i$. Then one sees σ is bipartite,

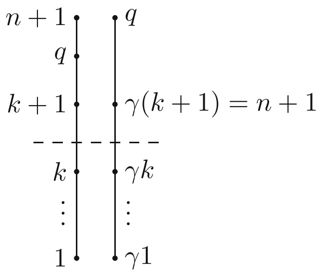

FIGURE 7

$k = (\sigma^{-1}n) - 1$, and σ has one fewer element than γ in its upper part. Figure 8 depicts σ.

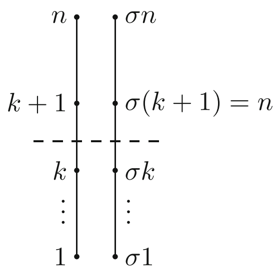

FIGURE 8

Let $x_1, \ldots, x_{n+1}$ be a chain under meet, in the stated order, in the Birkhoff system A. To make use of the inductive hypothesis as it applies to σ, define elements $y_1, \ldots, y_n$ of A by setting $y_1, \ldots, y_n$ to be $x_1, \ldots, x_{q-1}, x_{q+1}, \ldots, x_{n+1}$ in that order. We use x_i^1, x_i^2 for the elements built according to the formulas in the Lemma using the permutation γ and y_i^1, y_i^2 for the elements built from these formulas using the permutation σ. So $x_{\gamma i}^1 = x_{\gamma 1} + \cdots + x_{\gamma i}$ and $x_i^2 = x_{n+1}^1 \cdots x_i^1$, while $y_{\sigma i}^1 = y_{\sigma 1} + \cdots + y_{\sigma i}$ and $y_i^2 = y_n^1 \cdots y_i^1$.

Claim 1.

(1) $y_i = x_i$ *for* $1 \leq i < q$.
(2) $y_i = x_{i+1}$ *for* $q \leq i \leq n$.
(3) $y_{\sigma i} = x_{\gamma i}$ *for* $1 \leq i \leq n$.
(4) $y_{\sigma i}^1 = x_{\gamma i}^1$ *for* $1 \leq i \leq n$.
(5) $y_i^1 = x_i^1$ *for* $i < q$.
(6) $y_i^1 = x_{i+1}^1$ *for* $q \leq i \leq n$.
(7) $y_i^2 = x_{i+1}^2$ *for* $q \leq i \leq n$.
(8) $y_i^2 x_q^1 = x_i^2$ *for* $1 \leq i \leq q$.

Proof of Claim. The first and second statements are by the definition of the y_i. For the third, if $\sigma i < q$, then by the definitions of σ and y_i, we have $y_{\sigma i} = y_{\gamma i} = x_{\gamma i}$, and if $q \leq \sigma i$, then $y_{\sigma i} = y_{(\gamma i)-1} = x_{\gamma i}$. The fourth statement is clear from the third and the definitions of x_i^1, y_i^1. The fifth and sixth statements follow from the fourth as $\gamma i = \sigma i$ when $\sigma i < q$, and $\gamma i = \sigma i+1$ when $\sigma i \geq q$. The seventh and eighth statements follow from the fifth and sixth using the definitions of y_i^2 and x_i^2. □

Returning to the proof of the lemma. Our objective is to show for any $i \leq k$ that

$$x_i^2 + x_{n+1}^2 = x_{n+1}^2. \tag{3.1}$$

As $k < q$, by Claim 1, parts (7) and (8), it is enough to show that

$$y_i^2 x_q^1 + y_n^2 = y_n^2. \tag{3.2}$$

As $y_1, \ldots, y_n$ are a chain in A under meet, the inductive hypothesis applied to σ gives for any $i \leq k$ (the lines for σ and γ both occur between the k and $k+1$ spots) that

$$y_i^2 + y_n^2 = y_n^2. \tag{3.3}$$

Therefore, it is enough to prove that for $i \leq k$, we have

$$y_i^2 x_q^1 = y_i^2. \tag{3.4}$$

Claim 2. *For $i \leq k$*

(1) $x_q^1 = x_q + y_{\sigma n}^1$.
(2) $y_i^2 y_{\sigma n}^1 = y_i^2$.
(3) $y_i^1 x_q = y_i^1$.
(4) $y_i^2 x_q = y_i^2$.
(5) $y_i^2 = y_i^2 x_q y_{\sigma n}^1$.

Proof of claim. (1): Since $q = \gamma(n+1)$, we then have

$$x_q^1 = x_{\gamma(n+1)}^1 = x_1 + \cdots + x_{n+1} = x_q + y_1 + \cdots + y_n = x_q + y_{\sigma n}^1.$$

(2): As σ is bipartite with dividing line between k and $k+1$, we have $\sigma n > k \geq i$. As $y_i^2 = y_n^1 \cdots y_i^1$, we have $y_{\sigma n}^1$ is one of the terms whose meet is taken to form y_i^2, and it follows that $y_i^2 y_{\sigma n}^1 = y_i^2$.

(3): As $i \leq k$ and σ is bipartite, we have $i = \sigma j$ for some $j \leq k$. Then $y_i^1 = y_{\sigma j}^1 = y_{\sigma 1} + \cdots + y_{\sigma j} = x_{\gamma 1} + \cdots + x_{\gamma j}$. As $x_{\gamma 1}, \ldots, x_{\gamma j}, x_q$ forms a chain under meet with x_q on the top, by Lemma 2.7 (3), we have $y_i^1 x_q = y_i^1$.

(4): As $y_i^2 = y_n^1 \cdots y_i^1$, we clearly have $y_i^2 = y_i^2 y_i^1$. Then by part (3), $y_i^2 = y_i^2 y_i^1 x_q = y_i^2 x_q$.

(5): This is a direct consequence of parts (2) and (4). □

Concluding the proof of the lemma. Using Claim 2 and Lemma 2.7, we have

$$y_i^2 x_q^1 = y_i^2 x_q y_{\sigma n}^1 (x_q + y_{\sigma n}^1) = y_i^2 x_q y_{\sigma n}^1 = y_i^2.$$

This establishes equation (3.4), and so concludes the proof of the lemma. □

4. Main Theorem

In this section, we prove that a finite bichain that does not contain a subalgebra isomorphic to N is weakly projective. To work inductively, we need the following stronger form of weak projectivity.

Definition 4.1. Let C be a finite bichain with left ordering $c_1 <_L \cdots <_L c_n$ and right ordering $c_{\sigma 1} <_R \cdots <_R c_{\sigma n}$ and let $x_1, \ldots, x_n$ be elements of a Birkhoff system A. Form elements $x_1^0, \ldots, x_n^0$ by setting $x_i^0 = x_i \cdots x_n$. Then define for each $p \geq 0$,

$$x_{\sigma i}^{2p+1} = x_{\sigma 1}^{2p} + \cdots + x_{\sigma i}^{2p},$$

$$x_i^{2p+2} = x_i^{2p+1} \cdots \; x_n^{2p+1}.$$

We say C is LR-*projective* if there is a p so that $x_i^p = x_i^{p+1}$ for each $i = 1, \ldots, n$ for any choice of elements $x_1, \ldots, x_n$ in any Birkhoff system A. The least such p is called the LR-*length* of C.

Note that to check whether C is LR-projective, it suffices to consider free generators $x_1, \ldots, x_n$ of a free Birkhoff system, and the elements x_i^p built from them. In this case, if they stabilize, then the least p for which $x_i^p = x_i^{p+1}$ for each $i = 1, \ldots, n$ will be the LR-length of C.

Proposition 4.2. *If a finite bichain C is* LR*-projective, then it is weakly projective in the variety of Birkhoff systems.*

Proof. Suppose A is a Birkhoff system and $f\colon A \to C$ is a surjective homomorphism. If $c_1, \ldots, c_n$ are the elements of C, choose $x_1, \ldots, x_n$ in A with $f(x_i) = c_i$. Form the elements x_i^p using the definition of LR-projectivity of C, and note $f(x_i^p) = c_i$ for each $i \leq n$ and $p \geq 0$. From the construction of the x_i^p, we have that $x_1^{2p}, \ldots, x_n^{2p}$ form a chain under meet, and $x_1^{2p+1}, \ldots, x_n^{2p+1}$ form a chain under join. As C is LR-projective, there is some p so that $x_i^{2p} = x_i^{2p+1}$ for $i = 1, \ldots, n$. So $x_1^{2p}, \ldots, x_n^{2p}$ form a subalgebra of A that is mapped isomorphically onto C. So C is weakly projective. $\square$

Before proceeding to our main result, we need a small lemma.

Lemma 4.3. *If C is a finite bichain that is* LR*-projective, then the bichain C^* formed by placing an element q at the top of the left side of C and the bottom of the right side of C is also* LR*-projective.*

Proof. Start with elements $x_1, \ldots, x_n, y$ in a Birkhoff system A. As our first step, we form $x_1^0 \leq_L \cdots \leq_L x_n^0 \leq_L y^0 = y$. As q occurs on the bottom of the right of C^*, we set $y^1 = y$, then set $x_{\sigma i}^1 = y + x_{\sigma 1}^0 + \cdots + x_{\sigma i}^0$. Then as q occurs on the top of the left of C^*, we get $y^2 = y$ and $x_i^2 = y x_n^1 \cdots x_i^1$. We note x_i^1 is of the form $y + x^0_{\sigma 1} + \cdots + x^0_{\sigma j}$ for some j. By Lemma 2.7 (3), we have

$$\begin{aligned} yx^1_i &= y(y + x^0_{\sigma 1} + \cdots + x^0_{\sigma j}) = y + y(x^0_{\sigma 1} + \cdots + x^0_{\sigma j}) \\ &= y + x^0_{\sigma 1} + \cdots + x^0_{\sigma j} = x^1_i. \end{aligned}$$

It follows that $x^2_i = x^1_i \cdots x^1_n$. Next, note $x^3_{\sigma i} = y + x^2_{\sigma 1} + \cdots + x^2_{\sigma i}$. But $x^2_{\sigma i} = x^1_{\sigma i} \cdots x^1_n$. As $y <_R x^1_{\sigma 1} <_R \cdots <_R x^1_{\sigma i}$, we have by Lemma 2.7 (4), that $y + x^2_{\sigma i} = x^2_{\sigma i}$, and therefore that $x^3_{\sigma i} = x^2_{\sigma 1} + \cdots + x^2_{\sigma i}$. Continuing in this way, we see for $p \geq 0$,

$$\begin{aligned} x^{2p+2}_i &= x^{2p+1}_i \cdots x^{2p+1}_n, \\ x^{2p+3}_{\sigma i} &= x^{2p+2}_{\sigma 1} + \cdots + x^{2p+2}_{\sigma i}. \end{aligned}$$

In other words, after the first two steps, y does not occur. So in effect, the terms x^{2p+2}_i, x^{2p+3}_i we produce using the LR-process for C^* on the input $x_1, \ldots, x_n, y$ are exactly the terms x^{2p}_i, x^{2p+1}_i we produce using the LR-process for C on the input $x^1_1, \ldots, x^1_n$. So they stabilize after some finite number of steps. As the other term we are using always equals y, it obviously stabilizes. So the LR-length of C^* stabilizes in at most 2 more steps than it takes the LR-process for C to stabilize. So the LR-length of C^* is at most 2 more than that of C. □

Theorem 4.4. *Any finite bichain that does not contain N is* LR*-projective.*

Proof. The proof is by induction on the number of elements n in the bichain. Suppose C is a finite bichain whose left side is $c_1 <_L \cdots <_L c_n$ and whose right side is $c_{\sigma 1} <_R \cdots <_R c_{\sigma n}$. If C does not contain N, then σ is a bipartite permutation of $\{1, \ldots, n\}$ whose dividing line lies between k and $k+1$ for some $0 \leq k < n$.

The base case: $n = 1$ is trivial.

Consider the inductive case $n > 1$. If the top left element c_n of C is equal to the bottom right element $c_{\sigma 1}$, then removing this element gives a subchain not containing N, hence by the inductive hypothesis, a subchain that is LR-projective. Then Lemma 4.3 shows that C is also LR-projective. So we may assume, without loss of generality, that the top left element of C is not equal to the bottom right element of C, or in other words, that $k \geq 1$. This is key, as we may then split C into two strictly smaller pieces, $\{c_1, \ldots, c_k\}$ and $\{c_{k+1}, \ldots, c_n\}$, and use that these are LR-projective to show C is LR-projective.

To show that C is LR-projective, suppose A is a Birkhoff system and $x_1, \ldots, x_n$ are elements of A. We form the elements x^p_i for $i \leq n$ and $p > 0$ by the process outlined in the definition of LR-projectivity of C. We must show that this process stabilizes. We first collect a few technical facts.

Claim 3. *If $i \leq k$, then $x^3_{\sigma i} = x^2_{\sigma 1} + \cdots + x^2_{\sigma i}$, and if $i \geq k+1$, then we have $x^3_{\sigma i} = x^2_{\sigma(k+1)} + \cdots + x^2_{\sigma i}$.*

Proof of claim. The first statement is simply from the definition of $x^3_{\sigma i}$ when $i \leq k$. Suppose $i \geq k+1$. Lemma 3.2 shows that if $j \leq k$, then $x^2_{\sigma j} + x^2_n = x^2_n$.

By definition of k, $\sigma(k+1) = n$, so $x^3_{\sigma i} = x^2_{\sigma 1} + \cdots + x^2_{\sigma k} + x^2_{\sigma(k+1)} + \cdots + x^2_{\sigma i}$, and the first batch of terms is absorbed into $x^2_{\sigma(k+1)}$. □

Claim 4. *If $i \geq k+1$, then $x^4_i = x^3_i \cdots x^3_n$, and if $i \leq k$, then $x^4_i = x^3_i \cdots x^3_k$.*

Proof of claim. The first statement is simply from the definition of x^4_i when $i \geq k+1$. For the second half, suppose $i \leq k$. Then by definition, $x^4_i = x^3_i \cdots x^3_k x^3_{k+1} \cdots x^3_n$. The term x^3_i is the sum of elements from $\{x^2_1, \ldots, x^2_k\}$. As we have just shown in Claim 3, for $j \geq k+1$, we have that x^3_j is a sum of elements from $\{x^2_{k+1}, \ldots, x^2_n\}$. As $x^2_1 <_L \cdots <_L x^2_n$, Lemma 2.7 (5) shows that $x^3_i x^3_j = x^3_i$ when $j \geq k+1$. So in forming x^4_i, the first batch of terms is absorbed into the term x^3_i, giving $x^4_i = x^3_i \cdots x^3_k$. □

Claim 5. *For any $p \geq 1$,*

$$x^{2p+1}_{\sigma i} = \begin{cases} x^{2p}_{\sigma 1} + \cdots + x^{2p}_{\sigma i} & \text{if } i \leq k, \\ x^{2p}_{\sigma(k+1)} + \cdots + x^{2p}_{\sigma i} & \text{if } i \geq k+1; \end{cases}$$

$$x^{2p+2}_{i} = \begin{cases} x^{2p+1}_{i} \cdots x^{2p+1}_{k} & \text{if } i \leq k, \\ x^{2p+1}_{i} \cdots x^{2p+1}_{n} & \text{if } i \geq k+1. \end{cases}$$

Proof of claim. For $p = 1$, this is the content of Claims 3 and 4. Suppose then that $p > 1$. Set $y_1 = x^{2p-3}_1, \ldots, y_n = x^{2p-3}_n$. We may apply the LR-construction for C to the elements $y_1, \ldots, y_n$ of A. We note $y^0_i = y_i \cdots y_n$ which is equal to x^{2p-2}_i, and extending this, $y^q_i = x^{2p-2+q}_i$ for each $i \leq n$ and $q \geq 0$. As Claims 1 and 2 apply also to the elements produced from the initial sequence $y_1, \ldots, y_n$, we then obtain the above for $p > 1$. □

Continuing with the proof of the theorem. We let B be the subchain $\{c_1, \ldots, c_k\}$ consisting of the bottom portion of C, and let T be the subchain $\{c_{k+1}, \ldots, c_n\}$ consisting of the top portion of C. As $k \geq 1$, both are proper subchains and neither contains N as a subalgebra. So by the inductive hypothesis, both are LR-projective. Let $u^p_1, \ldots, u^p_k$ be the result of the p^{th} step of the LR-process for B applied to the starting input $x^3_1, \ldots, x^3_k$ and let $v^p_{k+1}, \ldots, v^p_n$ be the result of the p^{th} step of the LR-process for T applied to the starting input $x^3_{k+1}, \ldots, x^3_n$.

Claim 6. *$u^p_i = x^{p+4}_i$ for each $i \leq k$, and $v^p_i = x^{p+4}_i$ for each $i \geq k+1$.*

Proof of claim. By definition, $u^0_i = x^3_i \cdots x^3_k$, and by Claim 5, this is x^4_i. By definition, $u^1_{\sigma i} = u^0_{\sigma 1} + \cdots + u^0_{\sigma i}$ which is then equal to $x^4_{\sigma 1} + \cdots + x^4_{\sigma i}$, and hence to $x^5_{\sigma i}$. By definition, $u^2_i = u^1_i \cdots u^1_k$, which equals $x^5_i \cdots x^5_k$, and by the above result, is equal to x^6_i. Continuing in this fashion, we obtain $u^p_i = x^{p+4}_i$ for all $i \leq k$ and $p \geq 0$. The argument to show $v^p_i = x^{p+4}_i$ for all $i \geq k+1$ and $p \geq 0$ is similar. □

Concluding the proof of the theorem. As B and T are LR-projective, there is some p for which $u_i^p = u_i^{p+1}$ for all $i \leq k$, and $v_i^p = v_i^{p+1}$ for all $i \geq k+1$. It follows that $x_i^p = x_i^{p+1}$ for all $i \leq n$, showing C is LR-projective and the LR-length of C is at most 4 more than the maximum of the LR-lengths of B and T. □

5. The converse

In this section, we show that any finite bichain containing a subalgebra isomorphic to N is not weakly projective. For this we shall require a construction.

Suppose we have a bichain with $c_1 <_L \cdots <_L c_n$ and $c_{\sigma 1} <_R \cdots <_R c_{\sigma n}$. We wish to eventually construct a new chain from C by inserting a new element. We can insert this new element into any spot in the left side of C, and any spot in the right side of C. So we divide the left side of C into two parts, a lower segment, and an upper segment, and the same with the right side. This simply amounts to choosing $0 \leq k \leq n$ to divide the left, and $0 \leq m \leq n$ to divide the right. We point out that none of these parts need have the same size as any of the others, and some of them can be empty. The situation is shown in Figure 9.

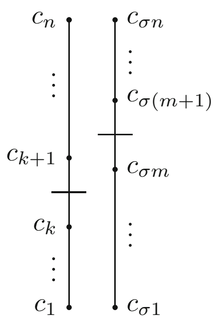

FIGURE 9

Definition 5.1. Using the division spots k, m, we now construct a new algebra (not a bichain) on the set $A = \{c_1, \ldots, c_n\} \cup \{a_1, \ldots, a_n\}$. Define the relation $<_L$ on A as follows.

$c_i <_L c_j$ iff $i < j$,
$c_i <_L a_j$ iff $i \leq k$,
$a_i <_L c_j$ iff $i \leq j$ and $k < j$,

$$a_i <_L a_j \text{ iff one of the following holds: } \begin{cases} i \leq k \text{ and } j > k, \\ i < j \text{ and } i, j > k, \\ \sigma^{-1} j < \sigma^{-1} i \text{ and } i, j \leq k. \end{cases}$$

And we define the relation $<_R$ on A as follows.

$c_{\sigma i} <_R c_{\sigma j}$ iff $i < j$,
$c_{\sigma i} <_R a_{\sigma j}$ iff $i \leq j$ and $i \leq m$,
$a_{\sigma i} <_R c_{\sigma j}$ iff $j > m$,

$$a_{\sigma i} <_R a_{\sigma j} \text{ iff one of the following holds: } \begin{cases} i \leq m \text{ and } j > m, \\ i < j \text{ and } i, j \leq m, \\ \sigma j < \sigma i \text{ and } i, j > m. \end{cases}$$

Note: The left ordering for $a_1, \ldots, a_k$ is given by $a_i <_L a_j$ if $\sigma^{-1}j < \sigma^{-1}i$. This means that if $c_i <_R c_j$, then $a_j <_L a_i$ and conversely. So the order of $a_1, \ldots, a_k$ on the left is obtained by reversing the order $c_1, \ldots, c_k$ appearing on the right. Similarly, the order of $a_{\sigma(m+1)}, \ldots, a_{\sigma n}$ on the right is obtained by looking at the indices of the elements $c_{\sigma(m+1)}, \ldots, c_{\sigma n}$, that is, the order in which they appear on the left, then reversing this ordering to get the ordering of $a_{\sigma(m+1)}, \ldots, a_{\sigma n}$ on the right. This is shown in Figure 10.

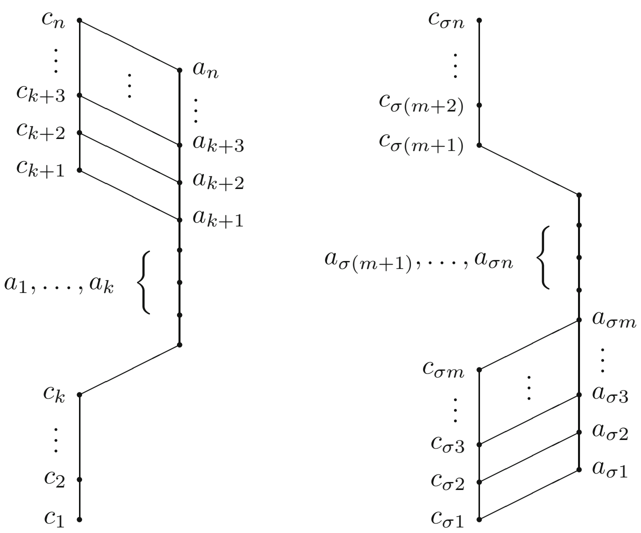

FIGURE 10. The algebra A constructed from C using spots k, m.

It is clear that $<_L$ and $<_R$ are partial orderings giving lattice structures, so $<_L$ defines a meet operation and $<_R$ defines a join operation on A. We must show that the bisemilattice A satisfies Birkhoff's equation $x(x+y) = x+xy$. As we know this to be true in any bichain, and both $\{c_1, \ldots, c_n\}$ and $\{a_1, \ldots, a_n\}$ are sub-bichains of A, it is enough to do this for $\{x, y\} = \{c_i, a_j\}$ for some

$i, j \leq n$. We then must show that

$$c_i(c_i + a_j) = c_i + c_i a_j \tag{5.1}$$

$$a_j(a_j + c_i) = a_j + a_j c_i \tag{5.2}$$

It is convenient to introduce some terminology. By the "bottom left" we mean the set $\text{BL} = \{c_1, \ldots, c_k, a_1, \ldots, a_k\}$; by the "top left" we mean the set $\text{TL} = \{c_{k+1}, \ldots, c_n, a_{k+1}, \ldots, a_n\}$; by the "bottom right" we mean the set $\text{BR} = \{c_{\sigma 1}, \ldots, c_{\sigma m}, a_{\sigma 1}, \ldots, a_{\sigma m}\}$, and finally by the "top right" we mean the set $\text{TR} = \{c_{\sigma(m+1)}, \ldots, c_n, a_{\sigma(m+1)}, \ldots, a_n\}$.

Lemma 5.2. *Given c_i, a_j for some $i, j \leq n$,*

$$c_i a_j = \begin{cases} c_i & \text{if } c_i \in \text{BL}, \\ a_i a_j & \text{if } c_i \in \text{TL}; \end{cases}$$

$$c_i + a_j = \begin{cases} a_i + a_j & \text{if } c_i \in \text{BR}, \\ c_i & \text{if } c_i \in \text{TR}. \end{cases}$$

Lemma 5.3. *If $c_i \in \text{TR}$, then* (5.1) *holds.*

Proof. Recall (5.1) is $c_i(c_i + a_j) = c_i + c_i a_j$. We let $\text{LHS} = c_i(c_i + a_j)$ and $\text{RHS} = c_i + c_i a_j$. As $c_i \in \text{TR}$ we have $c_i + a_j = c_i$ so $\text{LHS} = c_i$. The only possibilities for $c_i a_j$ are c_i or a_k for some k (actually $k = i$ or $k = j$). Clearly if $c_i a_j = c_i$ then $\text{RHS} = c_i$. If $c_i a_j = a_k$, then as $c_i \in \text{TR}$ we have $\text{RHS} = c_i + a_k = c_i$. In any event, we have $\text{LHS} = \text{RHS}$. □

Lemma 5.4. *If $c_i \in \text{BL}$, then* (5.1) *holds.*

Proof. Again we set $\text{LHS} = c_i(c_i + a_j)$ and $\text{RHS} = c_i + c_i a_j$. As $c_i \in \text{BL}$, we have $c_i a_j = c_i$, so $\text{RHS} = c_i$. The only possibilities for $c_i + a_j$ are c_i or a_k for some k (actually $k = i$ or $k = j$). Clearly if $c_i + a_j = c_i$, then $\text{LHS} = c_i$. If $c_i + a_j = a_k$, then as $c \in \text{BL}$, we have $\text{LHS} = c_i a_k = c_i$. In any event, we have $\text{LHS} = \text{RHS}$. □

Lemma 5.5. *If $c_i \in \text{TL} \cap \text{BR}$, then* (5.1) *and* (5.2) *both hold.*

Proof. As $c_i \in \text{TL}$, we have $c_i a_k = a_i a_k$ for any k, and as $c_i \in \text{BR}$, we have $c_i + a_k = a_i + a_k$ for any k. Then for (5.1), noting that $a_i + a_j$ is either a_i or a_j, and that $a_i a_j$ is either a_i or a_j, we have

$$c_i(c_i + a_j) = c_i(a_i + a_j) = a_i(a_i + a_j) = a_i + a_i a_j = c_i + a_i a_j = c_i + c_i a_j.$$

Here we are using the fact that $\{a_1, \ldots, a_n\}$ is a sub-bichain of A, so satisfies Birkhoff's equation. The calculation for (5.2) is similar,

$$a_j(a_j + c_i) = a_j(a_j + a_i) = a_j + a_j a_i = a_j + a_j c_i.$$

So both (5.1) and (5.2) hold. □

Lemma 5.6. *If* $c_i \in \mathrm{BL} \cap \mathrm{TR}$, *then* (5.2) *holds.*

Proof. As $c_i \in \mathrm{BL}$, we have $c_i a_j = c_i$ and as $c_i \in \mathrm{TR}$, we have $c_i + a_j = c_i$. Therefore, $a_j(a_j + c_i) = a_j c_i = c_i = a_j + c_i = a_j + a_j c_i$. So (5.2) holds. □

Lemma 5.7. *If* $c_i \in \mathrm{TL} \cap \mathrm{TR}$, *then* (5.2) *holds.*

Proof. As $c_i \in \mathrm{TR}$, we have $a_j + c_i = c_i$ and as $c_i \in \mathrm{TL}$, we have $a_j c_i = a_j a_i$. So in this case, verifying (5.2) amounts to showing that $a_j a_i = a_j + a_j a_i$. If $j \leq i$, then as $c_i \in \mathrm{TL}$, it follows that $a_j a_i = a_j$, and then clearly $a_j a_i = a_j + a_j a_i$. So we are left to consider $i < j$. In this case, as $c_i \in \mathrm{TL}$, we have $c_j \in \mathrm{TL}$ and $a_j a_i = a_i$. So our job of showing that $a_j a_i = a_j + a_j a_i$ reduces to showing that $a_i = a_j + a_i$, or in other words, that $a_j <_R a_i$. As $c_i \in \mathrm{TR}$, if $c_j \in \mathrm{BR}$, then it is clear that $a_j <_R a_i$. It remains to consider the case where $i < j$ and $c_j \in \mathrm{TR}$. Then as both $c_i, c_j \in \mathrm{TR}$, having $c_i <_L c_j$ implies $a_j <_R a_i$ as mentioned above. To fill in the details, suppose $i = \sigma p$ and $j = \sigma q$. Then as $c_i, c_j \in \mathrm{TR}$, we have $p, q > m$ and $\sigma p < \sigma q$. By the definition of $<_R$, we have $a_{\sigma q} <_R a_{\sigma p}$, hence $a_j <_R a_i$ as required. □

The reader may have noticed a certain duality between Lemma 5.3 and Lemma 5.4. Indeed, this is the case, and a similar symmetry exists between the previous lemma and the following one. But it is a bit subtle, and it is perhaps easier and more convincing to provide the short argument directly.

Lemma 5.8. *If* $c_i \in \mathrm{BL} \cap \mathrm{BR}$, *then* (5.2) *holds.*

Proof. As $c_i \in \mathrm{BL}$, we have $a_j c_i = c_i$ and as $c_i \in \mathrm{BR}$, we have $a_j + c_i = a_j + a_i$. So in this case, verifying (5.2) amounts to showing that $a_j(a_j + a_i) = a_j + a_i$. If $c_i \leq_R c_j$, then as $c_i \in \mathrm{BR}$, it follows that $a_j + a_i = a_j$, and then clearly $a_j(a_j + a_i) = a_j$. So we are left to consider $c_j <_R c_i$. In this case, as $c_i \in \mathrm{BR}$, we have $c_j \in \mathrm{BR}$ and $a_j + a_i = a_i$. So our job of showing that $a_j(a_j + a_i) = a_j + a_j a_i$ reduces to showing that $a_j a_i = a_i$, or in other words, that $a_i <_L a_j$. As $c_i \in \mathrm{BL}$, if $c_j \in \mathrm{TL}$, then it is clear that $a_i <_L a_j$. It remains to consider the case where $c_j <_R c_i$ and $c_j \in \mathrm{BL}$. Then as both $c_i, c_j \in \mathrm{BL}$, having $c_j <_R c_i$ implies $a_i <_L a_j$ as mentioned above. To fill in the details, suppose $i = \sigma p$ and $j = \sigma q$, so c_i occurs in the p^{th} spot from the bottom on the right and c_j occurs in the q^{th} spot from the bottom. As $c_j <_R c_i$, we have $q < p$, hence $\sigma^{-1} j < \sigma^{-1} i$. As $c_i, c_j \in \mathrm{BL}$, we have $i, j \leq k$ and we have shown that $\sigma^{-1} j < \sigma^{-1} i$. By the definition of $<_L$, we have $a_j <_L a_i$ as required. □

Proposition 5.9. *The bisemilattice A created by applying Definition 5.1 to C at the spots k, m satisfies Birkhoff's equation, hence is a Birkhoff system.*

Proof. This amounts to collecting the results in the above lemmas. □

Theorem 5.10. *If a finite bichain D contains a subalgebra isomorphic to N, then it is not weakly projective in the variety of Birkhoff systems.*

Proof. If D contains only a single copy of N, the proof is a fairly simple application of the above construction. But D may contain multiple copies of N and it is delicate to set things up correctly and to choose the correct copy of N to use with the above construction.

To begin, we assume that among all weakly projective bichains containing a subalgebra isomorphic to N, that D is of minimal cardinality. We then argue for a contradiction. The next step is to narrow down those subalgebras isomorphic to N we will consider.

To this end, we introduce the notion of a sandwiched subalgebra. A subalgebra $\{a, b, c\}$ of D with $a <_L b <_L c$ and $b <_R c <_R a$ is said to be *sandwiched* if there is some $x \in D$ with $x <_L a$ and $a <_R x$. A subalgebra of D that is isomorphic to N that is not sandwiched is called *non-sandwiched.*

The situation for a sandwiched subalgebra is shown in Figure 11.

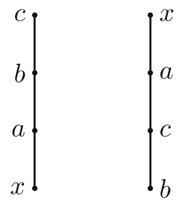

FIGURE 11. A sandwiched subalgebra

We need one more notion, that of the size of a subalgebra of D. The *size* of a subalgebra S of D is the sum of the number of elements of D between the least and largest elements of S under the $<_L$ order and the number of elements of elements of D between the least and largest elements of S under the $<_R$ order.

Consider all subalgebras of D that are isomorphic to N. Among these, consider those that are non-sandwiched, and among these choose one of maximal size. Suppose that this subalgebra is $\{a, b, c\}$ where $a <_L b <_L c$ and $b <_R c <_R a$. Let $C = D - \{a\}$, let d be the element of D that covers a under the $<_L$ order, and let e be the element covered by a under the $<_R$ order. Apply the construction of Definition 5.1 to the bichain C with the division immediately beneath d on the left, and immediately above e on the right, to create the Birkhoff system A shown in Figure 12. We note that this diagram shows $d <_L b$ and $c <_R e$, but it can be the case that $d = b$ or $c = e$ or both.

We alter notation a bit and use a_d for the new element inserted in constructing A from C and corresponding to the element d of C, and similarly for a_b, a_c, a_e. So the elements of A are $C \cup \{a_p : p \in C\}$.

Define $\varphi\colon A \to D$ by setting $\varphi(p) = p$ and $\varphi(a_p) = a$ for each $p \in C$. It is easily seen that φ is a surjective homomorphism. As we have assumed D is weakly projective, there must be a subalgebra T of A that is mapped isomorphically onto D. Surely this subalgebra must consist of C and one

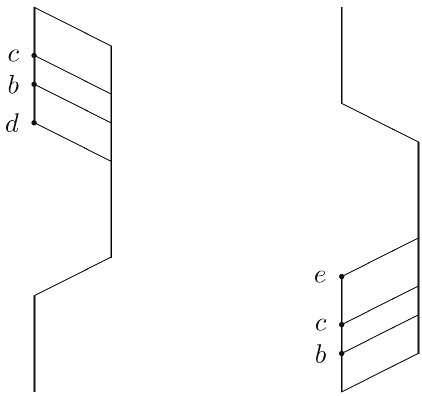

FIGURE 12

additional element a_p for some $p \in C$. We now consider various possibilities for this element a_p.

Claim 7. *For the element a_p with $T = C \cup \{a_p\}$,*

(1) *either $p = d$ or $p \in$ BL;*
(2) *either $p = e$ or $p \in$ TR.*

Proof of claim. (1): Otherwise we would have $d <_L p$ and this would imply that a_p is incomparable to d under $<_L$, contrary to T being a bichain.

(2): This is similar to part (1), otherwise a_p is incomparable to e under $<_R$. □

Claim 8. *For the element a_p with $T = C \cup \{a_p\}$,*

(1) *we cannot have both $p \in$ BL and $p \in$ TR;*
(2) *we cannot have both $p = d$ and $p = e$.*

Proof of claim. (1): If this were the case, then $\{a, b, c\}$ would be sandwiched by the element p.

(2): If this were the case, then $d = e$, and we would have a covered by d on the left, and a covering $d = e$ on the right. So by collapsing just a, d we would have a congruence θ on D. We cannot have $d = c$ as considering the left, this would give $b = c$, and we cannot have $d = b$ as this would imply $e = b$, and then considering the right, this would give $b = c$. So D/θ has a subalgebra isomorphic to N, namely $a/\theta = d/\theta = e/\theta, b/\theta, c/\theta$. As D is of minimal cardinality among those that contain a copy of N and are weakly projective, we have D/θ is not weakly projective. But D/θ is a quotient of the bichain D, so by Proposition 2.6 D/θ is a retract of D. But a retract of a weak projective is weakly projective, a contradiction. □

The following eliminates the remaining possibilities for a_p.

Claim 9. *For the element a_p with $T = C \cup \{a_p\}$,*

(1) *we cannot have $p = d$ and $p \in$ TR;*
(2) *we cannot have $p = e$ and $e \in$ BL.*

Proof of claim. We prove the first statement, the second follows by symmetry. The situation is shown in Figure 13.

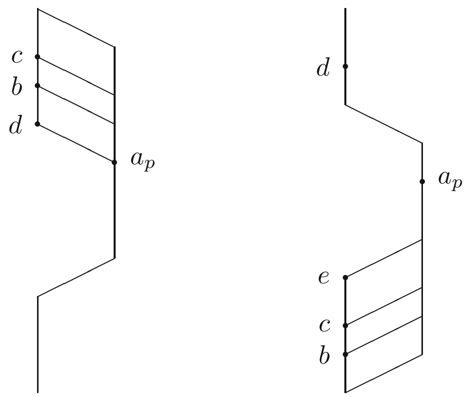

FIGURE 13

First note that in this case, $d \neq b$ as $b \in$ BR and $d \in$ TR, so $\{d, b, c\}$ is isomorphic to N. Then as d covers a in the $<_L$ order and $a <_R d$, it follows that $\{d, b, c\}$ is non-sandwiched because $\{a, b, c\}$ is non-sandwiched. Compare the size of $\{d, b, c\}$ to that of $\{a, b, c\}$. When looking at the number of elements on the left of D between d and c, we see that this is one less than the number of elements on the left of D between a and c because d covers a on the left. As $\{a, b, c\}$ is of maximal size, it follows that the number of elements on the right of D between b and d can be at most one more than the number of elements on the right of D between b and a. This means that d must cover a on the right.

Then as d covers a on both the left and right of D, collapsing a, d gives a congruence θ on D. We have seen that $d \neq b$ and clearly $d \neq c$. So this quotient D/θ has a copy of N, namely $a/\theta = d/\theta, b/\theta, c/\theta$. Then as D/θ is of smaller cardinality than D, it follows that D/θ is not weakly projective. But it is a retract of the weakly projective D, a contradiction. □

Concluding the proof of the theorem. We have exhausted all possibilities for the element a_p used to form a subalgebra of A mapped isomorphically by φ onto D. It follows that D is not weakly projective, concluding the proof of the theorem. □

Corollary 5.11. *For a finite bichain C, the following are equivalent.*

(1) *C is weakly projective.*
(2) *C is* LR*-projective.*
(3) *C does not contain a subalgebra isomorphic to N.*

Proof. (1) $\Rightarrow$ (3): This is is Theorem 5.10.
(3) $\Rightarrow$ (2): This is Theorem 4.4.
(2) $\Rightarrow$ (1): This is Proposition 4.2. □

Remark 5.12. Our attention here has been confined to determining which finite bichains are weakly projective. Of course, there are other weak projectives in the variety of Birkhoff systems, free ones for instance, and for two or more generators, these are not bichains. There are also other finite weak projectives as the free Birkhoff system on two generators is finite. There is a natural generalization of the notion of LR-projectivity to apply to any finite Birkhoff system. In this setting LR-projectivity would still imply weak projectivity. We do not know whether the converse of this would hold in this more general setting as well.

6. Weakly projective and splitting bichains

In this section we give recursive methods to construct and count the finite weakly projective bichains, and the finite subdirectly irreducible weakly projective bichains.

Proposition 6.1. *Let $C = \{c_1, \ldots, c_n\}$ be a finite bichain with left order $c_1 <_L \cdots <_L c_n$ and right order $c_{\sigma 1} <_R \cdots <_R c_{\sigma n}$. Then for $k = (\sigma^{-1}n) - 1$, we have C is weakly projective if and only if both*

(1) *C is bipartite;*
(2) *the sub-bichains $\{c_1, \ldots, c_k\}$ and $\{c_{k+1}, \ldots, c_{n-1}\}$ are weakly projective.*

Proof. $\Rightarrow$: C failing to be bipartite implies there is some $i > k$ with c_i occurring below the $(k+1)^{st}$ spot on the right, and therefore there is some $j \le k$ with c_j occurring above the $(k+1)^{st}$ spot on the right. Then $\{c_j, c_i, c_n\}$ is a subalgebra of C isomorphic to N. So the failure of condition (1) implies C is not weakly projective, and surely the failure of condition (2) implies C has a subalgebra isomorphic to N, so is not weakly projective.

$\Leftarrow$: This is equivalent to showing that if C is not weakly projective and satisfies the first condition, then it fails the second. So assume C is bipartite and has a subalgebra c_p, c_q, c_r with $c_p <_L c_q <_L c_r$ and $c_q <_R c_r <_R c_p$. Note C being bipartite means that if $i \le k < j$, then $c_i <_R c_j$. So if $p \le k$, we must also have $q, r \le k$, showing that $\{c_1, \ldots, c_k\}$ has a subalgebra isomorphic to N. On the other hand, if $k < p$, then as $c_p <_L c_q <_L c_r$, we must have $k < p < q < r$. We note that we cannot have $r = n$ since c_r occurs at the $(k+1)^{st}$ spot on the right and $c_q <_R c_r$. So $k < p, q, r < n$, showing that $\{c_{k+1}, \ldots, c_{n-1}\}$ has a subalgebra isomorphic to N. □

Remark 6.2. The above result gives a recursive method to construct the n-element weakly projective bichains having the property that the element that occurs at the top of the $<_L$ order occurs in the $(k+1)^{st}$ spot in the $<_R$ order. Take any weakly projective k-element bichain D, and any weakly projective $(n-k-1)$-element bichain E. Place E on top of D, then add a new element to the top of the $<_L$ order of the bichain produced, and between the elements of D and E of the $<_R$ order of the bichain produced. Allowing k to range from 0 to $n-1$ produces all n-element weakly projective bichains.

Proposition 6.3. *Let P_n be the number of isomorphism classes of weakly projective n-element bichains. Then clearly $P_0 = 1$, and for any $n \geq 1$ we have*

$$P_n = \sum_{k=0}^{n-1} P_k P_{n-k-1}.$$

Then P_n is the n^{th} Catalan number.

Proof. The recursive method given in the above remark shows that the number of nonisomorphic n-element weakly projective bichains where the largest element in the $<_L$ order occurs in the $(k+1)^{st}$ spot of the $<_R$ order is given by the product of the number of nonisomorphic k-element weakly projective bichains and the number of $(n-k-1)$-element weakly projective bichains, so by $P_k P_{n-k-1}$. Allowing k to range from 0 to $n-1$ we obtain the above formula. This formula is equivalent to having $P_{n+1} = \sum_{k=0}^{n} P_k P_{n-k}$, the familiar recursive formula for the $(n+1)^{st}$ Catalan number. □

We next turn our attention to subdirectly irreducible weakly projective bichains.

Lemma 6.4. *If $C = \{c_1, \ldots, c_n\}$ is a subdirectly irreducible weakly projective bichain with $c_1 <_L \cdots <_L c_n$, then either c_n occurs at the top of the $<_R$ order, c_n occurs at the bottom of the $<_R$ order, or c_n is second from the bottom of the $<_R$ order.*

Proof. Suppose c_n occurs at the $(k+1)^{st}$ spot of the $<_R$ order for some $0 \leq k < n$. As the permutation σ for C is bipartite, there is a congruence θ collapsing $\{c_1, \ldots, c_k\}$ and not collapsing any other elements, and a congruence ϕ collapsing $\{c_{k+1}, \ldots, c_n\}$ and not collapsing any other elements. Clearly θ and ϕ intersect in the diagonal. So for C to be subdirectly irreducible, we must have one of these congruences trivial, and this implies either $k = 0$, $k = 1$, or $k = n-1$. □

Lemma 6.5. *Suppose $n \geq 3$ and $C = \{c_1, \ldots, c_n\}$ is a finite bichain with $c_1 <_L \cdots <_L c_n$ where c_n is at the top of its $<_R$ order. Then C is subdirectly irreducible and weakly projective if and only if $C - \{c_n\}$ is a subdirectly irreducible weakly projective bichain with c_{n-1} not at the top of its $<_R$ order.*

Proof. $\Rightarrow$: Every subalgebra of C is weakly projective. Further, as c_n sits at the end of both orders of C, for any congruence θ on $C - \{c_n\}$, we have $\theta \cup \{(c_n, c_n)\}$ is a congruence on C. As C is subdirectly irreducible, it follows that $C - \{c_n\}$ is subdirectly irreducible. Finally, if c_{n-1} were the top of the $<_R$ order of $C - \{c_n\}$, then we would have a congruence θ collapsing only c_{n-1}, c_n, and a congruence ϕ collapsing only $c_1, \ldots, c_{n-2}$. As $n \geq 3$, both are non-trivial congruences, and clearly they intersect to the diagonal, contradicting that C is subdirectly irreducible.

$\Leftarrow$: Suppose $C - \{c_n\}$ is subdirectly irreducible, weakly projective, and there is some $p < n-1$ with $c_{n-1} <_R c_p$. Then $C - \{c_n\}$ has no subalgebra isomorphic to N, and as c_n is on the top of both orders for C, it follows that C has no subalgebra isomorphic to N, so is weakly projective. As $C - \{c_n\}$ is subdirectly irreducible, it has a critical pair. Take any congruence θ on C collapsing c_n with an element c_i with $i < n$. Then θ must collapse the interval from c_i to c_n in the $<_L$ order of C, so must collapse c_{n-1} and c_n. But then as c_p lies between c_{n-1} and c_n in the $<_R$ order, θ must collapse c_{n-1} and c_p, and as both belong to $C - \{c_n\}$, θ must collapse the critical pair of $C - \{c_n\}$. So C is subdirectly irreducible. $\square$

The proof of the following is similar to that of the previous result.

Lemma 6.6. *Suppose $n \geq 3$ and $C = \{c_1, \ldots, c_n\}$ is a finite bichain with $c_1 <_L \cdots <_L c_n$ where c_n is at the bottom of its $<_R$ order. Then C is subdirectly irreducible and weakly projective if and only if $C - \{c_n\}$ is subdirectly irreducible, weakly projective, and c_{n-1} not at the bottom of its $<_R$ order.*

We require one more case. Here we note that any weakly projective bichain whose left order is $c_1 <_L \cdots <_L c_n$ and having c_n as the second from bottom under the $<_R$ order must have c_1 at the bottom of the right order because it is bipartite.

Lemma 6.7. *Suppose $n \geq 4$ and $C = \{c_1, \ldots, c_n\}$ is a finite bichain with $c_1 <_L \cdots <_L c_n$ where $c_1 <_R c_n$ is the bottom of its $<_R$ order. Then C is subdirectly irreducible and weakly projective if and only if $C - \{c_1, c_n\}$ is subdirectly irreducible, weakly projective, and c_{n-1} is not the bottom of its $<_R$ order.*

Proof. $\Rightarrow$: As before, $C - \{c_1, c_n\}$ is weakly projective and subdirectly irreducible. If c_{n-1} was at the bottom of the $<_R$ order of $C - \{c_1, c_n\}$, then we would have congruences θ collapsing c_{n-1}, c_n and ϕ collapsing $c_2, \ldots, c_{n-1}$ with both non-trivial and intersecting to the diagonal, contrary to C being subdirectly irreducible.

$\Leftarrow$: As $C - \{c_1, c_n\}$ has no subalgebra isomorphic to N and $c_1 <_R c_n$ on the bottom of the $<_R$ order of C, there can be no subalgebra of C isomorphic to N, so C is weakly projective. As c_{n-1} does not cover c_n in the $<_R$ order of C, it follows that any non-trivial congruence on C must collapse two elements of $C - \{c_1, c_n\}$, hence must collapse a critical pair of $C - \{c_1, c_n\}$. So C is subdirectly irreducible. □

We next use the above lemmas to count the number of subdirectly irreducible weakly projective bichains up to isomorphism. We note that the argument could easily be used to provide a recursive method for constructing such bichains.

Proposition 6.8. *Let S_n be the number of isomorphism classes of subdirectly irreducible weakly projective n-element bichains. Then $S_0 = 1$, $S_1 = 1$, and for any $n \geq 2$,*

$$S_n = S_{n-1} + S_{n-2}.$$

Therefore, S_n is the $(n+1)^{st}$ Fibonacci number.

Proof. Define A_n to be the number, up to isomorphism, of subdirectly irreducible weakly projective n-element bichains where the top element of the $<_L$ order is at the top of the $<_R$ order; let B_n be similar, but having the top of the $<_L$ order be at the bottom of the $<_R$ order; and C_n be similar, but having the top of the $<_L$ order be second from the bottom of the $<_R$ order. We note that for small values of n, some bichains will be counted in more than one of these categories, but this does not occur for bichains with at least 3 elements. The above lemmas then give

(1) $A_n = B_{n-1} + C_{n-1}$ for each $n \geq 4$.
(2) $B_n = A_{n-1} + C_{n-1}$ for each $n \geq 4$.
(3) $C_n = A_{n-2} + C_{n-2}$ for each $n \geq 5$.

One can directly calculate $A_3 = B_3 = C_3 = C_4 = 1$. By comparing the second and third formulas, it follows from a simple induction that

$$A_n = B_n \quad \text{for all} \quad n \geq 3. \tag{6.1}$$

Therefore,

$$A_n = A_{n-1} + C_{n-1} \quad \text{for all} \quad n \geq 4. \tag{6.2}$$

Comparing this with the third formula and noting $C_4 = A_3$, we have

$$C_n = A_{n-1} \quad \text{for all} \quad n \geq 4. \tag{6.3}$$

Combining the previous two equations, we have

$$A_n = A_{n-1} + A_{n-2} \quad \text{for all} \quad n \geq 5. \tag{6.4}$$

Combining (6.4) with (6.1) and (6.3), we have

$$B_n = B_{n-1} + B_{n-2} \quad \text{for all} \quad n \geq 5. \tag{6.5}$$

$$C_n = C_{n-1} + C_{n-2} \quad \text{for all} \quad n \geq 6. \tag{6.6}$$

For $n \geq 3$, every subdirectly irreducible weakly projective n-element bichain is counted in exactly one of A_n, B_n, C_n. So $S_n = A_n + B_n + C_n$ for each $n \geq 3$, and applying (6.4), (6.5) and (6.6) to this, we have

$$S_n = S_{n-1} + S_{n-2} \quad \text{for all} \quad n \geq 6. \tag{6.7}$$

Applying the above formulas with $A_3 = B_3 = C_3 = C_4 = 1$ gives $A_4 = B_4 = C_5 = 2$ and $A_5 = B_5 = 3$. So $S_3 = 3$, $S_4 = 5$, $S_5 = 8$, and we can directly calculate that $S_0 = 1, S_1 = 1$ and $S_2 = 2$. It follows that (6.7) holds also for $n = 2, 3, 4, 5$. $\square$

7. A connection to fuzzy logic

In this section, we describe progress on a problem in fuzzy logic that first motivated our study of bichains. The basic ingredient is an algebra denoted M that arises in the study of type-2 fuzzy sets [15]. The underlying set of M is the collection of all functions f from the unit interval to itself. Define the operations $\cdot, +, *$ via a type of convolution as follows:

$$\begin{aligned}(f \cdot g)(x) &= \sup\{f(y) \wedge g(z) : y \wedge z = x\},\\ (f + g)(x) &= \sup\{f(y) \wedge g(z) : y \vee z = x\},\\ f^*(x) &= \sup\{f(y) : 1 - y = x\},\end{aligned}$$

and define constants $\overline{0}, \overline{1}$ to be the characteristic functions of $\{0\}$ and $\{1\}$.

The algebra M serves as a truth values algebra for type-2 fuzzy sets. So finding a decision procedure to determine whether an equation holds in M, and finding an equational basis for M, is of interest. To simplify matters, we consider the reduct of M having just $\cdot$ and $+$ as basic operations.

It turns out [7] this reduct is a Birkhoff system, and the variety generated by this reduct is generated by the four-element bichain we call B that is shown in Figure 14. This solves the matter of finding an algorithm to determine when an equation holds in this reduct — one simply tests whether the equation holds in the finite algebra B. But the problem of finding an equational basis for the variety generated by the $\cdot, +$ reduct of M remains open. We note it is also shown in [7] that the variety generated by the full algebra M is generated by a finite algebra, although one that is somewhat more complicated.

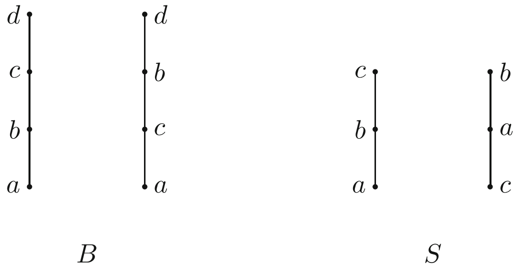

FIGURE 14

Consider the 3-element bichain S shown in Figure 14. We conjecture that the variety generated by the $\cdot, +$ reduct of M is the splitting variety of S in the variety $BiCh$ generated by the collection of all bichains. If this is the case, then an equational basis for our variety will consist of an equational basis for *BiCh* (finding such also remains open) together with the splitting equation for S. We outline our results on this topic below.

Proposition 7.1. *The algebra S is subdirectly irreducible and weakly projective in the variety BiCh and its splitting equation in BiCh is given by the generalized distributive law*

$$x(y+z)(xy+xz) \approx x(y+z) + (xy+xz). \tag{7.1}$$

Proof. Clearly S is subdirectly irreducible, and from the results above is weakly projective in the variety of Birkhoff systems, hence also in *BiCh*. Let F be the free Birkhoff system on the generators x, y, z and $\varphi\colon F \to S$ the homomorphism mapping x, y, z to a, b, c, respectively. By repeatedly applying Birkhoff's equation, one can check that

$$\{(z+yz+xyz)(z+xyz), z+yz+xyz, z\}$$

is a subalgebra of F mapped isomorphically onto S. Since a, b is the critical pair of S, it follows from general considerations that the elements of this subalgebra mapped to a, b give the splitting equation for S in the variety of Birkhoff systems:

$$(z+yz+xyz)(z+xyz) \approx z+yz+xyz \tag{7.2}$$

Using the software packages Prover9 and Mace4 [8], we can find an example to show that equation (7.2) is not equivalent to (7.1) in the variety of Birkhoff systems. However, consider the equations

$$x(x+y)(xz+y) \approx x(x+y)(xz+y+z). \tag{7.3}$$

$$x(xy+xz) \approx xy+xz. \tag{7.4}$$

Considering cases, one checks these equations are valid in every bichain, so are valid in the variety *BiCh*. Prover9 shows that in the presence of the identities for Birkhoff systems, equations (7.2), (7.3), and (7.4) together imply (7.1), and equations (7.1), (7.3), and (7.4) together imply (7.2). So in the variety *BiCh*, we have (7.2) and (7.1) are equivalent, showing that (7.1) is the splitting equation for S in the variety *BiCh*. □

Remark 7.2. Consider the splitting variety $BiCh_S$ of S in the variety *BiCh*, and the splitting variety $Birk_S$ of S in the variety *Birk* of Birkhoff systems. As noted in the above proof, the splitting equation for S in *BiCh* is not equivalent in the variety of Birkhoff systems to the splitting equation of S in *Birk*. Therefore, $BiCh_S \neq Birk_S$. Clearly $BiCh_S$ is strictly contained in $Birk_S$.

We note that the bichain B does not have a subalgebra isomorphic to S, so B belongs to $BiCh_S$. It is our conjecture that B generates the variety $BiCh_S$. If so, this would say that the splitting equation (7.1) together with equations defining the variety *BiCh* is an equational basis for the variety generated by

B. We have not been able to prove this conjecture, but the following result does lend it some credence.

Proposition 7.3. *For a bichain C the following are equivalent.*

(1) *C belongs to the variety $\mathcal{V}(B)$ generated by B.*
(2) *C does not have a subalgebra isomorphic to S.*
(3) *C satisfies the equation $x(y+z)(xy+xz) \approx x(y+z)+(xy+xz)$.*

Proof. (1) $\Rightarrow$ (3): This of course is simply a matter of checking that equation (7.1) holds in B, but the situation is more interesting than this. Note there is a congruence on B that collapses the two middle elements $\{b, c\}$, and the resulting quotient is a distributive lattice. Take any equation $s \approx t$ that holds in all distributive lattices. If this equation is to fail in B for some choice of elements, it must be that s, t evaluate to b and c. As $\{b, c\}$ is a subalgebra of B isomorphic to the two-element semilattice, it then follows that $st \approx s+t$ holds in B. Equation (7.1) is an instance of this, taking $s \approx t$ to be the meet distributive law.

(3) $\Rightarrow$ (2): Take $x = b, y = a, z = c$ to see S does not satisfy equation (7.1).

(2) $\Rightarrow$ (1): It is enough to show this in the case that C is finite since every finitely generated subalgebra of a bichain is finite. We show by induction on $n = |C|$ that if S is not isomorphic to a subalgebra of C, then $C \in \mathcal{V}(B)$.

For $n = 1, 2$, each n-element bichain is isomorphic to a subalgebra of B. Consider $n = 3$. Up to isomorphism, there are six 3-element bichains. Three of these are isomorphic to subalgebras of B. One has its meet operation equal to its join operation (roughly, is a semilattice) so belongs to $\mathcal{V}(B)$ as B has a 2-element semilattice as a subalgebra. One is the algebra S, so the claim is vacuous. The remaining bichain has $1 <_L 2 <_L 3$ and $2 <_R 3 <_R 1$. It is easily seen this is a subalgebra of a product of the two 2-element bichains, so belongs to $\mathcal{V}(B)$. Before considering $n \geq 4$, we establish the following.

Claim 10. *For a finite bichain C, let $C \cup \{\infty\}$ be the bichain formed from C by adding a new element to the bottom of the $<_L$ order of C and to the top of the $<_R$ order; let $C \cup \{b\}$ be formed by adding a new element to the bottom of both orders of C, and let $C \cup \{t\}$ be formed by adding a new element to the top of both orders of C. Then if $C \in \mathcal{V}(B)$, so are $C \cup \{\infty\}$, $C \cup \{b\}$, and $C \cup \{t\}$.*

Proof of claim. We first note $B \cup \{\infty\}$, $B \cup \{b\}$, and $B \cup \{t\}$ belong to $\mathcal{V}(B)$. Indeed, $B \cup \{\infty\}$ is a quotient of the product of B and the two-element semilattice; $B \cup \{b\}$ is a subalgebra of the product of B and the two-element distributive lattice, and $B \cup \{t\}$ is a subalgebra of the product of B and the two-element semilattice.

Assume $C \in \mathcal{V}(B)$. Then there is a set I, a subalgebra $S \leq B^I$, and a surjective homomorphism $\varphi\colon S \to C$. Consider the constant function $\overline{\infty}$ in $(B \cup \{\infty\})^I$. Then $S \cup \{\overline{\infty}\}$ is a subalgebra of this power, and φ extends to a homomorphism from $S \cup \{\overline{\infty}\}$ onto $C \cup \{\infty\}$. Similar arguments apply to the situations for $C \cup \{b\}$ and $C \cup \{t\}$. □

Concluding the proof of the proposition. Assume $C = \{c_1, \ldots, c_n\}$ where $c_1 <_L \cdots <_L c_n$ and $c_{\sigma 1} <_R \cdots <_R c_{\sigma n}$. If the bottom element of the $<_R$ order is c_1, then C is isomorphic to $C' \cup \{b\}$ where C' is the sub-bichain $\{c_2, \ldots, c_n\}$. By the inductive hypothesis and the above claim, $C \in \mathcal{V}(B)$. A similar argument handles the cases where either c_1 or c_n is at the top of the $<_R$ order. Set

$$U = \{k : c_k <_R c_1\},$$
$$V = \{k : c_1 <_R c_k\}.$$

As c_1 is not the bottom or the top of the $<_R$ order, U and V are non-empty. Also, as S is not a subalgebra of C, if $u \in U$ and $v \in V$, then $u < v$. And as c_n is not the top element of the $<_R$ order, V must have at least two elements. So there is some $2 \le k \le n-2$ with $U = \{2, \ldots, k\}$ and $V = \{k+1, \ldots, n\}$.

There are congruences θ, ϕ on C with θ collapsing $\{c_1, \ldots, c_k\}$ and nothing else, and ϕ collapsing $\{c_{k+1}, \ldots, c_n\}$ and nothing else. Note C/θ is isomorphic to the sub-bichain $\{c_1, c_{k+1}, \ldots, c_n\}$ of C, and C/ϕ is isomorphic to the sub-bichain $\{c_1, \ldots, c_k, c_{k+1}\}$ of C. It follows from the inductive hypothesis that C/θ and C/ϕ belong to $\mathcal{V}(B)$. As θ and ϕ intersect to the diagonal, C is a subalgebra of their product, so belongs to $\mathcal{V}(B)$. □

Remark 7.4. If *BiCh* were a congruence distributive variety, then the above result would imply that $\mathcal{V}(B)$ is the splitting variety of S in *BiCh*. But *BiCh* is not a congruence distributive variety. It is meet semi-distributive since each algebra in *Birk* has a semilattice reduct, but this is not sufficient to help in this instance.

8. Concluding remarks and open questions

There are many further questions one can ask about Birkhoff systems and bichains. Even questions directly related to the matter of projectives and splittings seem to present difficult open questions. For instance, one might well seek a characterization of all weak projectives or splitting algebras in the variety of Birkhoff systems. We present below several more modest questions, related directly to the work at hand.

Question 8.1. Are epimorphisms surjective in the variety of Birkhoff systems?

Question 8.2. Is the variety $\mathcal{V}(B)$ the splitting variety of S in *BiCh*?

Question 8.3. Give an equational basis for the variety *BiCh*.

Question 8.4. Give an equational basis for $\mathcal{V}(B)$.

We briefly mention the interesting role played by the software packages Prover9 and Mace4 in the preparation of this work. With the exception of proving the equivalence of equations (7.1) and (7.2) in the variety *BiCh* in Proposition 7.1, all proofs in this paper are constructed and verified by humans.

However, these software packages were useful in conducting exploratory work on medium sized specific instances, pointing out which of our conjectures were likely to be true, and which were false. This was the case in developing conjectures that lead to the technical lemma in Section 3. Here many instances were tested on bichains with 8 to 10 elements and could take several hours to run. The proofs constructed by Prover9 in these instances were not helpful in finding the main arguments in Section 3, but did reveal small facts that proved useful. We conclude by mentioning that the problems considered here, particularly those in Section 3, could be of interest to those hoping to further improve the capacities of Prover9.

9. Acknowledgements

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