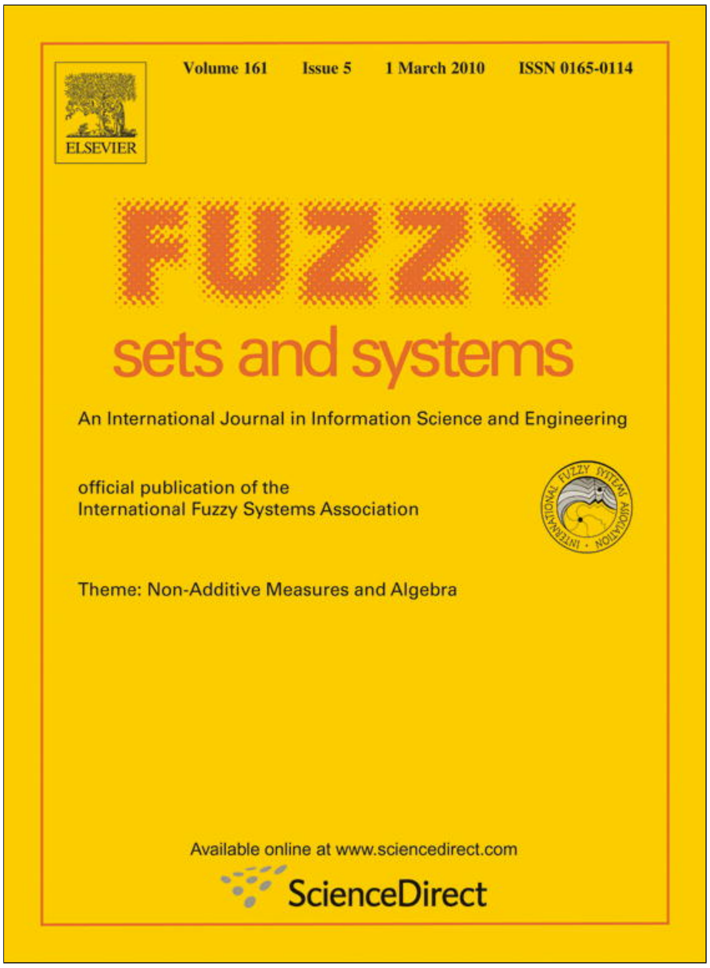
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The variety generated by the truth value algebra of type-2 fuzzy sets

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Abstract

This paper addresses some questions about the variety generated by the algebra of truth values of type-2 fuzzy sets. Its principal result is that this variety is generated by a finite algebra, and in particular is locally finite. This provides an algorithm for determining when an equation holds in this variety. It also sheds light on the question of determining an equational axiomatization of this variety, although this problem remains open.

Keywords: Type-2 fuzzy set; Variety; Locally finite; Equational base

1. Introduction

The algebra of truth values of type-2 fuzzy sets was introduced by Zadeh in 1975, generalizing the truth-value algebras of ordinary fuzzy sets, and of interval-valued fuzzy sets [27]. This algebra is quite a complicated object. Many fundamental mathematical properties of this algebra have been developed, but many basic questions remain open. See, for example [5–7,12–15,17–21,23–25]. See also [9–11,22] for background on the truth-value algebras for ordinary and interval-valued fuzzy sets.

This paper addresses some questions about the variety generated by the truth-value algebra of type-2 fuzzy sets, and our principal result is that it is generated by a finite algebra. As consequences of this result, we obtain that the variety is locally finite, and we obtain an algorithm for determining when an identity holds in this truth-value algebra. These results also shed light on determining an equational axiomatization of this variety, but this problem remains open.

The algebra is $\mathbb{M} = (M, \sqcup, \sqcap, ^*, \bar{0}, \bar{1})$ where M is the set $[0, 1]^{[0,1]}$ of all functions from $[0, 1]$ to $[0, 1]$, and $\sqcup, \sqcap, ^*, \bar{0}, \bar{1}$ are certain convolutions of the operations $\vee, \wedge, \neg, 0, 1$ on $[0, 1]$. We begin in Section 2 by defining the operations of this algebra, and listing some of its known properties that we need in our development.

We show in Section 3 that the truth-value algebra $\mathbb{M}$ of type-2 fuzzy sets is locally finite and that the list of equations satisfied by that algebra provided in Section 2 is not an equational basis for the variety that $\mathbb{M}$ generates. Similar results hold for $\mathbb{M}$ without its negation *. We give a syntactic algorithm in Section 4 for deciding whether an equation holds in $(M, \sqcup, \sqcap)$.

In Section 5, we show that the variety $\mathcal{V}(\mathbb{M})$ generated by $\mathbb{M}$ is equal to the variety $\mathcal{V}(\mathbb{E})$ generated by the subalgebra $\mathbb{E} = (\{0, 1\}^{[0,1]}, \sqcup, \sqcap, ^*, \bar{0}, \bar{1})$ of "sets" in $\mathbb{M}$. This result is key in showing, in later sections, that $\mathcal{V}(\mathbb{M})$ is generated by a finite algebra.

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In Section 6, we observe that $\mathbb{E}$ is isomorphic to the complex algebra of the unit interval with its usual negation $\neg x = 1 - x$. This enables us to show in Section 7 that $\mathcal{V}(\mathbb{E})$ is generated by an algebra with 2^5 elements, and the variety generated by $\mathbb{E}$ without * is generated by an algebra with 2^3 elements.

Section 8 is devoted to examining congruences on the two finite algebras to show that the varieties they generate are generated by smaller algebras. The main result in this regard is that $\mathcal{V}(\mathbb{M})$ is generated by a 12-element algebra. We also show that the variety generated by $\mathbb{M}$ without * is generated by a five-element algebra and the variety generated by $(M, \sqcup, \sqcap)$ is generated by a four-element algebra. These results yield semantic algorithms to determine when an equation holds in these varieties, much as one uses truth tables for classical logic.

As indicated, some fundamental problems remain, and some of these are discussed in the concluding remarks, Section 9.

2. The algebra of truth values

The underlying set of the algebra of truth values of type-2 fuzzy sets is the set $M = Map([0,1],[0,1])$ of all functions from the unit interval into itself. The operations imposed are certain convolutions of the operations on [0, 1]. These are the binary operations $\sqcup$ and $\sqcap$, the unary operation *, and the nullary operations $\bar{1}$ and $\bar{0}$ as spelled out below:

$$(f \sqcup g)(x) = \sup\{f(y) \wedge g(z) | y \vee z = x\}$$
$$(f \sqcap g)(x) = \sup\{f(y) \wedge g(z) | y \wedge z = x\}$$
$$f^*(x) = \sup\{f(y) | 1 - y = x\} = f(\neg x) = f(1-x)$$
$$\bar{1}(x) = \begin{cases} 0 & \text{if } x \neq 1 \\ 1 & \text{if } x = 1 \end{cases} \quad \bar{0}(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases} \tag{1}$$

Note that the operation $\sqcup$ is the convolution of the join operation on the unit interval, $\sqcap$ is the convolution of the meet operation, $*$ is the convolution of the negation $\neg x = 1 - x$, and $\bar{1}, \bar{0}$ are convolutions of the constants 0,1 of the unit interval. (For background on the notion of convolution in this context, see Section I of [23].)

Definition 1. The algebra $\mathbb{M} = (M, \sqcup, \sqcap, ^*, \bar{1}, \bar{0})$ is the *algebra of truth values* for fuzzy sets of type-2.

There are several other operations of natural interest when considering the set *M*. The pointwise join and meet operations $\vee, \wedge$ on *M* induce the structure of a complete distributive lattice on *M*. The constant functions 0,1 are the bounds of this lattice. Two additional operations, L and R described below, also play a fundamental role.

Definition 2. For $f \in M$, let f^L and f^R be the elements of *M* defined by

$$f^L(x) = \bigvee_{y \le x} f(y)$$
$$f^R(x) = \bigvee_{y \ge x} f(y) \tag{2}$$

One can easily see that f^L is the pointwise smallest increasing function lying above *f*, and f^R is the pointwise smallest decreasing function lying above *f*. Part of the utility of these operations is that $\sqcap$ and $\sqcup$ can be expressed in terms of L, R, $\wedge$, and $\vee$ as follows [23].

Proposition 3. *The following hold for all* $f, g \in M$:

$$f \sqcup g = (f \wedge g^L) \vee (f^L \wedge g) = (f \vee g) \wedge (f^L \wedge g^L)$$
$$f \sqcap g = (f \wedge g^R) \vee (f^R \wedge g) = (f \vee g) \wedge (f^R \wedge g^R)$$

Using these auxiliary operations, it is fairly routine to verify the following properties of the algebra $\mathbb{M}$.

Proposition 4. *Let $f, g, h \in M$.*

1. $f \sqcup f = f$; $f \sqcap f = f$;
2. $f \sqcup g = g \sqcup f$; $f \sqcap g = g \sqcap f$;
3. $f \sqcup (g \sqcup h) = (f \sqcup g) \sqcup h$; $f \sqcap (g \sqcap h) = (f \sqcap g) \sqcap h$;
4. $f \sqcup (f \sqcap g) = f \sqcap (f \sqcup g)$;
5. $\bar{1} \sqcap f = f$; $\bar{0} \sqcup f = f$;
6. $f^{**} = f$;
7. $(f \sqcup g)^* = f^* \sqcap g^*$; $(f \sqcap g)^* = f^* \sqcup g^*$.

Some additional equations of interest, that are consequences of Eqs. (1)–(7), are

8. $\bar{1}^* = \bar{0}$; $\bar{0}^* = \bar{1}$;
9. $(f \sqcup g) \sqcap (f \sqcap g) = f \sqcap g$; $(f \sqcup g) \sqcup (f \sqcap g) = f \sqcup g$.

Proof. The first seven identities are established in [23] using Proposition 3. To see (8), $\bar{0} = (\bar{0}^*)^* = (\bar{1} \sqcap \bar{0}^*)^* = (\bar{1}^* \sqcup \bar{0}) = \bar{1}^*$, and the other part follows similarly. The final two identities are consequences of the first four. To see this, using (1)–(3) we have $(f \sqcup g) \sqcap (f \sqcap g) = f \sqcap (f \sqcup g) \sqcap (f \sqcap g)$. Using (4), this equals $(f \sqcup (f \sqcap g)) \sqcap (f \sqcap g)$, and applying (4) again, this equals $(f \sqcap f \sqcap g) \sqcup (f \sqcap g)$. This simplifies to $f \sqcap g$. The final identity is similar. □

Easy examples show that $\mathbb{M}$ is not a lattice since it does not satisfy either of the absorption laws $x \sqcap (x \sqcup y) = x$ or $x \sqcup (x \sqcap y) = x$. However, $\mathbb{M}$ is a type of algebra known as a *De Morgan bisemilattice* [3,15,16], as we describe below.

Definition 5. An algebra $(A, \sqcap, \sqcup)$ with two binary operations is called a *bisemilattice* if it satisfies Eqs. (1)–(3) above. A bisemilattice is called *distributive* if it satisfies the distributive laws $f \sqcup (g \sqcap h) = (f \sqcup g) \sqcap (f \sqcup h)$ and $f \sqcap (g \sqcup h) = (f \sqcap g) \sqcup (f \sqcap h)$. A bisemilattice with a unary operation $*$ is called a *De Morgan bisemilattice* if it satisfies Eqs. (6) and (7).

Bisemilattices in general, and distributive bisemilattices in particular, have a fairly extensive literature [16]. While $\mathbb{M}$ does not satisfy either of the distributive laws, it does satisfy Eq. (4) which is a restriction of these laws to the case that $f = h$, and this equation ties the operations $\sqcap, \sqcup$ in an interesting way.

It is natural to consider whether the identities (1)–(7) above are sufficient to imply all identities that hold in $\mathbb{M}$. In the following section we will see this is not the case.

3. The variety generated by $\mathbb{M}$ is locally finite

Our primary interest lies in the algebra $\mathbb{M} = (M, \sqcap, \sqcup, *, \bar{0}, \bar{1})$, but it is useful to also consider the operations L, R of Definition 2 and the pointwise meet and join $\wedge, \vee$. Recall that $\sqcap$ and $\sqcup$ can be expressed in terms of these other operations, and that M is a distributive lattice under $\wedge, \vee$.

Lemma 6. *For $S \subseteq M$, the subalgebra of $\mathbb{M}$ generated by S is contained in the sublattice of $(M, \wedge, \vee)$ generated by S' where*

$$S' = \{f, f^*, f^L, f^R, (f^*)^L, (f^*)^R, f^{LR} \mid f \in S\} \cup \{\bar{0}, \bar{1}, \bar{0}^L\}.$$

Proof. We show more, namely, that the closure of S under the operations $\sqcap, \sqcup, *, \bar{0}, \bar{1}$, L, and R is contained in the sublattice of $(M, \wedge, \vee)$ generated by S'. Observe first that S' is the closure of S under the operations $*, \bar{0}, \bar{1}$, L, R. This follows from the facts $f^{**} = f$, $(f^L)^* = (f^*)^R$, $(f^R)^* = (f^*)^L$, $\bar{0}^* = \bar{1}$, $f^{LL} = f^L$, $f^{RR} = f^R$, $\bar{0}^R = \bar{0}$, $\bar{1}^L = \bar{1}$, $\bar{0}^L = \bar{1}^R$ and $f^{LR} = f^{RL} = f^{LR*}$ found in [23].

We next prove that the result of any expression obtained from elements of S using the operations $\sqcap, \sqcup, *, \bar{0}, \bar{1}$, L, R belongs to the sublattice of $(M, \wedge, \vee)$ generated by S'. The proof is by induction on the number n of occurrences of the operations $\sqcap, \sqcup$ in this expression. If $n = 0$, then as noted above, the result of this expression belongs to S'.

Suppose $n > 0$. Then using the following facts $(f \sqcap g)^* = f^* \sqcup g^*$, $(f \sqcup g)^* = f^* \sqcap g^*$, $(f \sqcap g)^L = f^L \sqcap g^L$, $(f \sqcap g)^R = f^R \sqcap g^R$, $(f \sqcup g)^L = f^L \sqcup g^L$, and $(f \sqcup g)^R = f^R \sqcup g^R$ [23], any occurrences of the operations $*$, L, R in the expression may be moved inward until they are next to the arguments from S or the constants. So the expression may be equivalently written as an expression involving only the operations $\sqcap, \sqcup$ applied to elements of S', and further, this new expression will have the same number n of occurrences of $\sqcap, \sqcup$ as the old.

Since $n > 0$ this new expression is either of the form $t_1 \sqcap t_2$ or $t_1 \sqcup t_2$ for some expressions t_1, t_2 with fewer than n occurrences of $\sqcap, \sqcup$. In the first case, we use the fact that $t_1 \sqcap t_2 = (t_1 \vee t_2) \wedge t_1^R \wedge t_2^R$ from Proposition 3, and apply the inductive hypothesis to t_1, t_2, t_1^R, t_2^R to obtain the result. The second case is handled similarly using $t_1 \sqcup t_2 = (t_1 \vee t_2) \wedge t_1^L \wedge t_2^L$. □

This result has a number of interesting consequences. To describe these, we recall some standard terminology from universal algebra. The reader should consult [4] for further details.

Definition 7. For an algebra $\mathbb{A}$, the *variety* $\mathcal{V}(\mathbb{A})$ generated by $\mathbb{A}$ is the class of all algebras that satisfy the same equations as $\mathbb{A}$. We say $\mathbb{A}$ is *locally finite* if each finite subset of the underlying set of $\mathbb{A}$ generates a finite subalgebra of $\mathbb{A}$, and that $\mathcal{V}(\mathbb{A})$ is *locally finite* if each algebra in this variety is locally finite.

Theorem 8. $\mathbb{M} = (M, \sqcap, \sqcup, *, \bar{0}, \bar{1})$ *is locally finite with a uniform upper bound on the size of a subalgebra in terms of the size of a generating set.*

Proof. Let S be a subset of M with n elements. By Lemma 6 the subalgebra of $\mathbb{M}$ generated by S is contained in the sublattice of $(M, \wedge, \vee)$ generated by $S' = \{f, f^*, f^L, f^R, (f^*)^L, (f^*)^R, f^{LR} \mid f \in S\} \cup \{\bar{0}, \bar{1}, \bar{0}^L\}$. Since S' has at most $7n + 3$ elements, the sublattice of the distributive lattice $(M, \wedge, \vee)$ generated by S', has at most $2^{2^{7n+3}}$ elements. □

Corollary 9. *The variety* $\mathcal{V}(\mathbb{M})$ *generated by* $\mathbb{M}$ *is locally finite.*

Proof. This is an immediate consequence of the previous theorem since $\mathbb{M}$ is locally finite with a uniform upper bound on the size of a subalgebra in terms of the size of its generating set [1]. □

Corollary 10. *There are equations satisfied by* $\mathbb{M}$ *that are not consequences of Eqs.* (1)–(7) *of Proposition* 4, *and there are equations satisfied by* $(M, \sqcap, \sqcup)$ *that are not consequences of Eqs.* (1)–(4) *of Proposition* 4.

Proof. If all equations satisfied by $\mathbb{M}$ were consequences of these equations, then $\mathcal{V}(\mathbb{M})$ would consist of all the algebras satisfying these equations. Since this variety is locally finite, it would follow that any algebra satisfying Eqs. (1)–(7) would be locally finite. But ortholattices satisfy these equations, and not all ortholattices are locally finite [2]. A similar argument holds when we restrict to the operations $\sqcup, \sqcap$ since the variety $\mathcal{V}((M, \sqcap, \sqcup))$ must also be locally finite, and not all the algebras satisfying Eqs. (1)–(4) are locally finite, as all lattices satisfy (1)–(4), but not all lattices are locally finite. □

Remark 11. We have shown that Eqs. (1)–(7) do not define the variety $\mathcal{V}(\mathbb{M})$. It would be of interest to extend this set of identities to one that does define this variety. But a word of caution is in order. There is no a priori reason that there should exist a finite set of identities that defines this variety. A first step would be to produce even one specific identity valid in $\mathbb{M}$ and not a consequence of these seven.

4. The word problem for $(M, \sqcap, \sqcup)$

In this section, we use the results of the previous section to give a syntactic algorithm to determine whether an equation $s \approx t$ holds in $(M, \sqcap, \sqcup)$. Here, we assume s, t are expressions formed over a set V of variables using $\sqcap, \sqcup$. We let $V' = \{x, x^L, x^R, x^{LR} \mid x \in V\}$, and call the elements of V' *literals*.

The proof of Lemma 6 gives a recursive algorithm to produce equivalent expressions s', t' where each expression is obtained by successive applications of the pointwise meet $\wedge$ and pointwise join $\vee$ to literals. Since $(M, \wedge, \vee)$ is a distributive lattice, s' can be rewritten as an expression s'' that is a finite join of finite meets of literals, and t' can be rewritten as an expression t'' that is a finite meet of finite joins of literals. To determine if $s'' \leq t''$, where $\leq$ is the pointwise ordering, it is sufficient to determine if $m \leq j$ when m is a meet of literals and j is a join of literals. Determining whether both $s'' \leq t''$ and $t'' \leq s''$ hold is equivalent to determining if $s'' \approx t''$, hence whether $s \approx t$.

So the task reduces to determining when $m \leq j$ for m a meet of literals and j a join of literals. As $x^L \vee x^R = x^{LR}$ holds in M, we assume without loss of generality that j does not contain both x^L and x^R.

Proposition 12. *Suppose m is a meet of literals and j is a join of literals as above. Then $m \leq j$ if and only if there is a variable $x \in V$ for which at least one of the following holds.*

1. *x occurs in m and one of x, x^L, x^R, x^{LR} occurs in j.*
2. *x^L occurs in m and one of x^L, x^{LR} occurs in j.*
3. *x^R occurs in m and one of x^R, x^{LR} occurs in j.*
4. *x^{LR} occurs in m and x^{LR} occurs in j.*

Proof. From the definitions of L and R, we see that $x \leq x^L$, $x^R \leq x^{LR}$ hold in M. So any one of these conditions is sufficient to ensure $m \leq j$.

For the converse, suppose that for each variable x, none of these conditions apply. Then for each variable x, we produce an element $f_x \in M$ so that when the variables are assigned so that x is interpreted as f_x, we have m evaluates to a function taking value 1 at $\frac{1}{2}$ and j evaluates to a function taking value 0 at $\frac{1}{2}$. Thus, we define a map $V \to M : x \mapsto f_x$ as follows:

- If x occurs in m, set $f_x = \chi_{\{1/2\}}$, the characteristic function of the singleton $\{1/2\}$. (In this case none of x, x^L, x^R, x^{LR} occur in j.)
- If x does not occur in m and both x^L, x^R do occur in m, set $f_x = \chi_{\{1/4,3/4\}}$. (In this case, none of x^L, x^R, x^{LR} can occur in j, but perhaps x does occur in j.)
- If neither x, x^R occurs in m and x^L occurs in m, set $f_x = \chi_{\{1/4\}}$. (In this case at most x, x^R occur in j.)
- If neither x, x^L occurs in m and x^R occurs in m, set $f_x = \chi_{\{3/4\}}$. (In this case at most x, x^L occur in j.)
- If none of x, x^L, x^R occur in m, and x^{LR} does occur in m, and x^L does not occur in j, set $f_x = \chi_{\{1/4\}}$. (In this case, x^{LR} cannot occur in j.)
- If none of x, x^L, x^R occur in m, and x^{LR} does occur in m, and x^R does not occur in j, set $f_x = \chi_{\{3/4\}}$. (In this case, x^{LR} cannot occur in j. Note also that if x^{LR} occurs in m, at most one of x^L, x^R occurs in j because we agreed to replace $x^L \vee x^R$ with x^{LR} in j.)
- Finally, if none of x, x^L, x^R, x^{LR} occur in m, let f_x be the constant function 0.

It is now straightforward to check that when the variables are assigned so that x is interpreted as f_x, we have m evaluates to a function taking value 1 at $\frac{1}{2}$ and j evaluates to a function taking value 0 at $\frac{1}{2}$. Thus, if none of conditions (1)–(4) apply, we have $m \not\leq j$. □

Theorem 13. *There is an algorithm to decide when an equation $s \approx t$ holds in $(M, \sqcap, \sqcup)$.*

Proof. One checks whether $s \leq t$ and $t \leq s$ as described at the beginning of this section, using Proposition 12. □

Remark 14. The above techniques can likely be adapted to give a syntactic algorithm to determine when an equation $s \approx t$ holds in $\mathbb{M}$. Set the literals to be $V' = \{x, x^L, x^R, x^*, x^{*L}, x^{*R}, x^{LR} \mid x \in V\} \cup \{\bar{0}, \bar{1}, \bar{0}^L\}$. Lemma 6 shows s can be written as a join of meets of literals, and t can be written as a meet of joins of literals. So it suffices to determine when $m \leq j$ for m a meet of literals and j a join of literals. However, as the supply of literals is now quite rich, this leads to a sizable number of cases.

Remark 15. Semantic algorithms to determine when equations hold in $\mathbb{M}$ and $(M, \sqcap, \sqcup)$ are given in a later section. However, the algorithm described above may be more practically useful to verify small equations.

5. The algebra $\mathbb{E}$ of sets in $\mathbb{M}$

In this section we consider the collection E of all characteristic functions of subsets of [0, 1]. It was shown in [23] that this forms a subalgebra $\mathbb{E}$ of the algebra $\mathbb{M}$. Here we show that the algebras $\mathbb{E}$ and $\mathbb{M}$ generate the same variety. This will be employed in the following sections to show that the variety $\mathcal{V}(\mathbb{M})$ is generated by a single finite algebra. Again, it is often convenient, and instructive, to consider also the operations L, R, $\wedge$, $\vee$.

Definition 16. Let E be the set of all elements of M taking values in the two-element set $\{0, 1\}$. In other words, E is the set of all characteristic functions of subsets of [0, 1].

Proposition 17. *The set E is closed under the operations $\sqcap, \sqcup, *, \bar{0}, \bar{1}, \wedge, \vee$, L, R of M. In particular, $\mathbb{E} = (E, \sqcap, \sqcup, *, \bar{0}, \bar{1})$ is a subalgebra of $\mathbb{M}$.*

Proof. This was first observed in [23], and is easily verified directly. □

Definition 18. For each a with $0 \le a < 1$ define $\varphi_a : M \to M$ by

$$\varphi_a(f)(x) = \begin{cases} 1 & \text{if } a < f(x) \\ 0 & \text{otherwise} \end{cases}$$

Proposition 19. *For $0 \le a < 1$ the map φ_a is a homomorphism with respect to all of the operations $\sqcap, \sqcup, *, \bar{0}, \bar{1}, \wedge, \vee$, L, R.*

Proof. We abbreviate φ_a to φ. Clearly $\varphi(\bar{0}) = \bar{0}$ and $\varphi(\bar{1}) = \bar{1}$. Next, note that the following are equivalent

$$\begin{aligned} \varphi(f \wedge g)(x) &= 1 \\ f(x) \wedge g(x) &> a \\ f(x) > a \quad &\text{and} \quad g(x) > a \\ \varphi(f)(x) = 1 \quad &\text{and} \quad \varphi(g)(x) = 1 \\ (\varphi(f) \wedge \varphi(g))(x) &= 1 \end{aligned}$$

It follows that $\varphi(f \wedge g) = \varphi(f) \wedge \varphi(g)$. A similar argument considering when $\varphi(f \vee g)(x) = 0$ shows that $\varphi(f \vee g) = \varphi(f) \vee \varphi(g)$.

Now, note that the following are equivalent:

$$\begin{aligned} \varphi(f^*)(x) &= 1 \\ f^*(x) &> a \\ f(1-x) &> a \\ \varphi(f)(1-x) &= 1 \\ (\varphi(f))^*(x) &= 1 \end{aligned}$$

thus $\varphi(f^*) = \varphi(f)^*$.

Next, note that the following are equivalent:

$$\begin{aligned} \varphi(f^L)(x) &= 1 \\ f^L(x) &> a \\ \sup\{f(y) \mid y \le x\} &> a \\ f(y) > a \quad &\text{for some } y \le x \\ \varphi(f)(y) = 1 \quad &\text{for some } y \le x \\ \varphi(f)^L(x) &= 1 \end{aligned}$$

So $\varphi(f^L) = \varphi(f)^L$. A similar argument shows that $\varphi(f^R) = \varphi(f)^R$.

Finally, since $\sqcap$ and $\sqcup$ can be expressed in terms of L, R, $\wedge$, $\vee$, it follows that φ preserves these as well. □

We then have the following result, first observed in [26] for the case $a = 0$.

Corollary 20. *For each* $0 \leq a < 1$, φ_a *is a homomorphism from* $\mathbb{M}$ *onto* $\mathbb{E}$. *In fact, it is a retraction; that is, it is an idempotent endomorphism.*

Proof. The above result shows φ_a is a homomorphism. By definition, for any $f \in M$, we have $\varphi_a(f)$ takes only the values 0, 1, so belongs to E. It is routine to show that if $f \in E$, then $\varphi_a(f) = f$. So φ_a is a retraction, and hence is onto E. □

Theorem 21. *The algebras* $\mathbb{M}$ *and* $\mathbb{E}$ *generate the same variety.*

Proof. As $\mathbb{E}$ is a subalgebra of $\mathbb{M}$, we have $\mathcal{V}(\mathbb{E}) \subseteq \mathcal{V}(\mathbb{M})$. For the other containment, consider the product map,

$$\prod_{a\in[0,1)} \varphi_a : \mathbb{M} \to \prod_{a\in[0,1)} \mathbb{E}$$

By general considerations, this map is a homomorphism. Suppose that $f, g \in M$ with $f \neq g$. Let x be such that $f(x) \neq g(x)$ and pick a strictly between $f(x)$ and $g(x)$. Then $\varphi_a(f)(x) \neq \varphi_a(g)(x)$. It follows that the product map is an embedding. So $\mathbb{M}$ is isomorphic to a subalgebra of a power of $\mathbb{E}$, showing $\mathcal{V}(\mathbb{M}) \subseteq \mathcal{V}(\mathbb{E})$. □

6. Complex algebras of chains

In this section, we give an alternate way to view the algebra $\mathbb{E}$. The idea is standard, and comes from the complex algebra $2^{\mathbb{G}}$ of a group $\mathbb{G}$. This consists of all subsets of G with the operations on these subsets given by: $A \circ B = \{a \circ b \mid a \in A, b \in B\}$, $A^{-1} = \{a^{-1} \mid a \in A\}$ and $e = \{e\}$, where e is the identity of the group.

Definition 22. For a bounded chain $\mathbb{C} = (C, \wedge, \vee, 0, 1)$, the *complex algebra* $2^{\mathbb{C}}$ consists of all subsets of C with constants $\bar{0} = \{0\}$, $\bar{1} = \{1\}$, and binary operations $\sqcap, \sqcup$ given by

$$A \sqcap B = \{a \wedge b \mid a \in A, b \in B\}$$
$$A \sqcup B = \{a \vee b \mid a \in A, b \in B\}$$

Definition 23. An *involution* on a chain is an order-reversing map from the chain to itself of period two.

The involution $\neg$ on [0, 1] defined by $\neg x = 1 - x$ will be of primary interest here.

Definition 24. For a bounded chain $\mathbb{C}$ with involution, its complex algebra $2^{\mathbb{C}}$ is as above with an additional unary operation $*$ given by $A^* = \{\neg a \mid a \in A\}$.

Just as it is fruitful to consider auxiliary operations on $\mathbb{M}$, it is useful also to consider additional operations on these complex algebras of chains.

Definition 25. For $\mathbb{C}$ a bounded chain with, or without, involution, define the binary operations $\wedge, \vee$ on its complex algebra $2^{\mathbb{C}}$ to be set intersection and set union, and define the unary operations L, R on this complex algebra to be upset and downset, respectively. Specifically,

$$A^L = \{c \mid a \leq c \text{ for some } a \in A\}$$
$$A^R = \{c \mid c \leq a \text{ for some } a \in A\}$$

The following result clarifies the situation. Its proof is not difficult, but is quite long, and is left to the reader.

Proposition 26. *The complex algebra of a bounded chain with involution satisfies all the equations of Propositions 3 and 4, and Lemma 6.*

As a first step to link these complex algebras with our earlier considerations, note the following.

Proposition 27. *The subalgebra $\mathbb{E}$ of $\mathbb{M}$ of all characteristic functions of sets, is isomorphic to the complex algebra of the unit interval* $[0, 1]$ *with the standard negation* $\neg x = 1 - x$.

Proof. The map sending the set A to its characteristic function is a bijection from $2^{[0,1]}$ to E that clearly preserves $\wedge, \vee$. Since $\chi_{\{0\}}$ and $\chi_{\{1\}}$ are the constants of $\mathbb{E}$, this bijection preserves the constants as well. Since $(\chi_A)^*(x) = \chi_A(1-x)$, it follows that $(\chi_A)^* = \chi_{A^*}$, so $*$ is preserved. Since $(\chi_A)^L$ is the least increasing function above χ_A and A^L is the upset of A, it follows that $(\chi_A)^L = \chi_{A^L}$. Similarly $(\chi_A)^R = \chi_{A^R}$. As this bijection preserves L, R, $\wedge$, $\vee$ and $\sqcap$, $\sqcup$ are expressed in terms of these operations in both the complex algebra and in $\mathbb{E}$, it follows that these operations are preserved. $\square$

One more link is needed to connect more fully the algebra $\mathbb{E}$ to complex algebras of chains.

Proposition 28. *If $\mathbb{C}$ and $\mathbb{D}$ are bounded chains, or bounded chains with involution, and $\varphi : \mathbb{C} \to \mathbb{D}$ is a homomorphism, then $\varphi[\cdot] : 2^{\mathbb{C}} \to 2^{\mathbb{D}}$ is a homomorphism where $\varphi[A]$ is the image of the set A under the map φ.*

Proof. For $A, B \subseteq C$ we have $\varphi[A \sqcap B] = \{\varphi(a \wedge b) \mid a \in A, b \in B\}$. Since φ is a homomorphism, this equals $\{\varphi(a) \wedge \varphi(b) \mid a \in A, b \in B\}$, which is $\varphi[A] \sqcap \varphi[B]$. The arguments for the operations $\sqcup, *, \bar{0}, \bar{1}$ are similar. $\square$

Remark 29. The result above extends to show that a homomorphism between any two algebras leads to a homomorphism between their complex algebras. But one must take care with the additional operations L, R, $\wedge$, $\vee$. It is easy to see that $\varphi[\cdot]$ will preserve upset and downset L, R if the map φ is onto. Also, as $\varphi[A \cup B] = \varphi[A] \cup \varphi[B]$, we have $\varphi[\cdot]$ preserves $\vee$, but it does not usually preserve $\wedge$. Also, one sometimes considers the additional operation of set complementation on complex algebras. The map $\varphi[\cdot]$ will not usually be compatible with this operation either.

7. Varieties and complex algebras of chains

In this section, we show that for any bounded chain $\mathbb{C}$ with at least three elements, the variety generated by $2^{\mathbb{C}}$ is generated by $2^{\mathbf{3}}$ where $\mathbf{3}$ is the three-element chain. We also show that for $\mathbb{C}$ a bounded chain with involution, and having at least five elements, the variety generated by $2^{\mathbb{C}}$ is generated by $2^{\mathbf{5}}$ where $\mathbf{5}$ is the five-element chain. As a consequence, we will obtain that the variety $\mathcal{V}(\mathbb{M})$ is generated by the finite algebra $2^{\mathbf{5}}$.

Definition 30. Let $\mathbf{3}$ be the three-element bounded chain $0 < u < 1$.

Theorem 31. *For $\mathbb{C}$ any bounded chain with at least three elements, the variety $\mathcal{V}(2^{\mathbb{C}})$ generated by the complex algebra of $\mathbb{C}$ equals the variety $\mathcal{V}(2^{\mathbf{3}})$ generated by the complex algebra of $\mathbf{3}$.*

Proof. Suppose $\mathbb{C}$ is a bounded chain with at least three elements and bounds 0, 1. Clearly there is an embedding $\varphi : \mathbf{3} \to \mathbb{C}$. By Proposition 28 $\varphi[\cdot] : 2^{\mathbf{3}} \to 2^{\mathbb{C}}$ is a homomorphism, and as φ is an embedding, so is this map. This shows $\mathcal{V}(2^{\mathbf{3}}) \subseteq \mathcal{V}(2^{\mathbb{C}})$.

For each $c \in C$, define a map $\varphi_c : C \to 3$ as follows. If $c \neq 0, 1$ set

$$\varphi_c(x) = \begin{cases} 1 & \text{if } x > c \\ u & \text{if } x = c \\ 0 & \text{if } x < c \end{cases}$$

If c is either of the bounds 0,1 set

$$\varphi_c(x) = \begin{cases} 1 & \text{if } x = 1 \\ u & \text{if } 0 < x < 1 \\ 0 & \text{if } x = 0 \end{cases}$$

Then each $\varphi_c : \mathbb{C} \to \mathbf{3}$ is a homomorphism of bounded chains. By Proposition 28 it follows that each $\varphi_c[\cdot] : 2^{\mathbb{C}} \to 2^{\mathbf{3}}$ is a homomorphism, so the product map $\prod_C \varphi_c[\cdot] : 2^{\mathbb{C}} \to \prod_C 2^{\mathbf{3}}$ is a homomorphism. We show this map is an embedding. Suppose A, B are subsets of C with $A \neq B$. We may assume some c belongs to A and not to B. If $c \neq 0, 1$

we have u belongs to $\varphi_c[A]$ and not to $\varphi_c[B]$, if $c = 0$ then 0 belongs to $\varphi_c[A]$ and not to $\varphi_c[B]$, and if $c = 1$ then 1 belongs to $\varphi_c[A]$ and not to $\varphi_c[B]$. So this product map is an embedding, and this shows $\mathcal{V}(2^{\mathbb{C}}) \subseteq \mathcal{V}(2^{\mathbf{3}})$. □

Theorem 32. *The varieties* $\mathcal{V}((M, \sqcap, \sqcup, \bar{0}, \bar{1}))$, $\mathcal{V}((E, \sqcap, \sqcup, \bar{0}, \bar{1}))$, *and* $\mathcal{V}(2^{\mathbf{3}})$ *are equal, and they are equal to the variety* $\mathcal{V}(2^{\mathbb{C}})$ *where* $\mathbb{C}$ *is any bounded chain with at least three elements.*

Proof. Note that $(E, \sqcap, \sqcup, \bar{0}, \bar{1})$ is the algebra $\mathbb{E}$ without the operation $*$, and $(M, \sqcap, \sqcup, \bar{0}, \bar{1})$ is the algebra $\mathbb{M}$ without $*$. Theorem 21 gives $\mathcal{V}(\mathbb{M}) = \mathcal{V}(\mathbb{E})$. This means $\mathbb{M}$ and $\mathbb{E}$ satisfy the same equations in the operations $\sqcap, \sqcup, *, \bar{0}, \bar{1}$, so they satisfy the same equations in the operations $\sqcap, \sqcup, \bar{0}, \bar{1}$. Thus $\mathcal{V}((E, \sqcap, \sqcup, \bar{0}, \bar{1})) = \mathcal{V}((M, \sqcap, \sqcup, \bar{0}, \bar{1}))$. In Proposition 27 we showed that $\mathbb{E}$ is isomorphic to $2^{[0,1]}$ where $[0, 1]$ is the unit interval considered as a bounded chain with the standard negation. So $(E, \sqcap, \sqcup, \bar{0}, \bar{1})$ is isomorphic to $2^{[0,1]}$ where $[0, 1]$ is considered as just a bounded chain. The remainder of the theorem then follows immediately from Theorem 31. □

We turn our attention to the matter of chains with involutions.

Definition 33. Let **5** be the five-element bounded chain $0 < p < q < r < 1$ with the only possible involution $*$.

Theorem 34. *For* $\mathbb{C}$ *any bounded chain with involution having at least five elements, the variety* $\mathcal{V}(2^{\mathbb{C}})$ *equals the variety* $\mathcal{V}(2^{\mathbf{5}})$.

Proof. Suppose $\mathbb{C}$ is a bounded chain with involution $*$ having at least five elements and bounds 0, 1. There can be at most one element in C that is fixed under $*$, so we can find an element z with $z < z^*$, and this z can be chosen to be different from 0,1. Using this, we can find $x, y \in C$ with $0 < x < y < x^*$. Indeed, we can either find an element w lying between z and z^*, and then use $0 < z < w < z^* < 1$, or an element w lying between 0 and z, and then use $0 < w < z < w^* < 1$. Having found $0 < x < y < x^* < 1$, we can find a homomorphism φ mapping $\mathbb{C}$ onto **5**. Define φ by

$$\varphi(c) = \begin{cases} 0 & \text{if } c < x \\ p & \text{if } c = x \\ q & \text{if } x < c < x^* \\ r & \text{if } c = x^* \\ 1 & \text{if } c > x^* \end{cases}$$

Then by Proposition 28, $\varphi[\cdot] : 2^{\mathbb{C}} \to 2^{\mathbf{5}}$, and since φ is onto, this homomorphism is onto. This shows $\mathcal{V}(2^{\mathbf{5}}) \subseteq \mathcal{V}(2^{\mathbb{C}})$.

Showing the other containment is similar to the case without involution. For each $c \in C$ we define a map $\varphi_c : C \to 5$ as follows. If $c \in \{0, 1\}$ define

$$\varphi_c(x) = \begin{cases} 0 & \text{if } x = 0 \\ q & \text{if } 0 < x < 1 \\ 1 & \text{if } x = 1 \end{cases}$$

If $0 < c < c^*$ define

$$\varphi_c(x) = \begin{cases} 0 & \text{if } x < c \\ p & \text{if } x = c \\ q & \text{if } c < x < c^* \\ r & \text{if } x = c^* \\ 1 & \text{if } c^* < x \end{cases}$$

If $c = c^*$ define

$$\varphi_c(x) = \begin{cases} 0 & \text{if } x < c \\ q & \text{if } x = c \\ 1 & \text{if } c < x \end{cases}$$

Finally, if $c^* < c < 1$ define

$$\varphi_c(x) = \begin{cases} 0 & \text{if } x < c^* \\ p & \text{if } x = c^* \\ q & \text{if } c^* < x < c \\ r & \text{if } x = c \\ 1 & \text{if } c < x \end{cases}$$

For each $c \in C$ the map $\varphi_c : \mathbb{C} \to \mathbf{5}$ is a homomorphism and further has the property that $\varphi_c(c)$ does not belong to $\varphi_c[C \setminus \{c\}]$. This shows that the associated product map from $2^{\mathbb{C}}$ to $\prod_C 2^{\mathbf{5}}$ is an embedding, hence $\mathcal{V}(2^{\mathbb{C}}) \subseteq \mathcal{V}(2^{\mathbf{5}})$. □

Theorem 35. *The varieties $\mathcal{V}(\mathbb{M})$, $\mathcal{V}(\mathbb{E})$, and $\mathcal{V}(2^{\mathbf{5}})$ are equal, and they are equal to the variety $\mathcal{V}(2^{\mathbb{C}})$ where $\mathbb{C}$ is any bounded chain with involution having at least five elements.*

Proof. This is similar to the proof of Theorem 32. □

Corollary 36. *The variety $\mathcal{V}(\mathbb{M})$ is generated by a single finite algebra and therefore is locally finite. The same holds for the variety $\mathcal{V}((M, \sqcap, \sqcup, \bar{0}, \bar{1}))$.*

Remark 37. Since the algebra $2^{\mathbf{5}}$ generates the variety $\mathcal{V}(\mathbb{M})$, the algebras $2^{\mathbf{5}}$ and $\mathbb{M}$ satisfy exactly the same identities. Given an identity, there is an obvious method to see if it holds in $2^{\mathbf{5}}$: try all possible combinations of elements of $2^{\mathbf{5}}$ for the variables of the equation and see if they all hold. This is much like the process of truth tables for classical propositional logic where the two-element Boolean algebra 2 generates the variety of all Boolean algebras. While the method of truth tables is practical in verifying small identities for classical propositions, it would be quite unwieldy to try to verify an identity for $\mathbb{M}$ by actually trying all possible combinations of elements of the 32-element algebra $2^{\mathbf{5}}$ for the variables. If one wanted a practical algorithm for small identities, finding a syntactic algorithm, along the lines outlined in Remark 14, would seem a better route.

8. The algebras $2^{\mathbf{3}}$ and $2^{\mathbf{5}}$ revisited

We consider more closely the algebras $2^{\mathbf{3}}$ and $2^{\mathbf{5}}$. By doing so, we are able to show that the varieties they generate, and hence the varieties $\mathcal{V}((M, \sqcap, \sqcup, \bar{0}, \bar{1}))$ and $\mathcal{V}(\mathbb{M})$, are generated by smaller, more manageable algebras. We begin with the depiction of the algebra $2^{\mathbf{3}}$ in Fig. 1.

To understand this figure, note that the algebra $2^{\mathbf{3}}$ has operations $\sqcap, \sqcup, \bar{0}, \bar{1}$. Each of the operations $\sqcap, \sqcup$ is a semilattice operation, so can be drawn as a poset. Above, $\sqcap$ is to be interpreted as the meet operation in the poset at left, $\sqcup$ as the join operation of the poset at right. The constants $\bar{0}, \bar{1}$ are shown on the figure. The elements of $2^{\mathbf{3}}$ are the subsets of the three-element set $0 < u < 1$. We first represent a subset such as $\{0, u\}$ by a triple giving its characteristic function 110. Here a 1 on the spot farthest left means 0 is in the set, a 1 in the middle spot means u is in the set, and a 1 in the rightmost spot means 1 is in the set. This represents the eight elements of $2^{\mathbf{3}}$ as the binary representations of the numbers $0, \ldots, 7$. We convert these binary numbers to their decimal counterparts to label the figure. So the subset $\{0, u\}$ is converted to 110, and finally to 6.

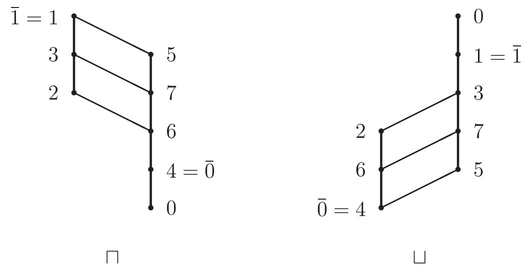

Fig. 1. The algebra $2^{\mathbf{3}}$.

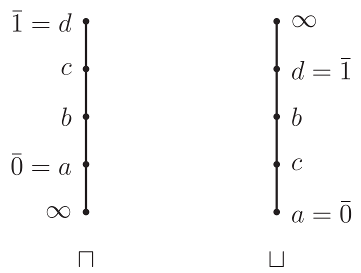

Fig. 2. A five-element bisemichain.

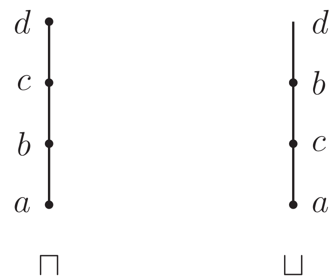

Fig. 3. A four-element bisemichain.

Lemma 38. *Consider the equivalence relations* $\theta_1, \theta_2, \theta$ *on* $2^{\mathbf{3}}$ *with associated partitions*:

$\theta_1 : \{0\}, \{1, 2, 3\}, \{4, 5, 6, 7\}$

$\theta_2 : \{0\}, \{1, 3, 5, 7\}, \{2, 4, 6\}$

$\theta : \{0\}, \{1\}, \{2, 3, 6, 7\}, \{4\}, \{5\}$

Then $\theta_1, \theta_2, \theta$ *are congruences and intersect to the diagonal.*

This lemma was checked by hand, and with the "Universal Algebra calculator" [8] written by Freese and Kiss. The algebra $2^{\mathbf{3}}/\theta$ is isomorphic to the one given in Fig. 2 with ∞ corresponding to the equivalence class $\{0\}$, a to $\{4\}$, b to $\{2, 3, 6, 7\}$, c to $\{5\}$, and d to $\{1\}$. The element ∞ satisfies $\infty \sqcap x = \infty$ and $\infty \sqcup x = \infty$, and we call it the *absorbing* element. We call the algebra in Fig. 2 a *bisemichain*, as it is a chain under each of the operations $\sqcap$ and $\sqcup$.

Theorem 39. *The variety* $\mathcal{V}((M, \sqcap, \sqcup, \bar{0}, \bar{1}))$ *is generated by the five-element algebra in Fig.* 2.

Proof. By Theorem 32, $\mathcal{V}((M, \sqcap, \sqcup, \bar{0}, \bar{1})) = \mathcal{V}(2^{\mathbf{3}})$. Let $\mathbb{A}$ be the algebra in Fig. 2. As $\mathbb{A}$ is a quotient of $2^{\mathbf{3}}$ we have $\mathcal{V}(\mathbb{A}) \subseteq \mathcal{V}(2^{\mathbf{3}})$. Lemma 38 shows $2^{\mathbf{3}} \leq 2^{\mathbf{3}}/\theta_1 \times 2^{\mathbf{3}}/\theta_2 \times 2^{\mathbf{3}}/\theta$. One notes that $2^{\mathbf{3}}/\theta_1$ and $2^{\mathbf{3}}/\theta_2$ are both isomorphic to the three-element subalgebra $\{\infty, \bar{0}, \bar{1}\}$ of $\mathbb{A}$. It follows that $2^{\mathbf{3}}$ is isomorphic to a subalgebra of a power of $\mathbb{A}$, hence $\mathcal{V}(2^{\mathbf{3}}) \subseteq \mathcal{V}(\mathbb{A})$. □

We next consider the situation when the constants $\bar{0}, \bar{1}$ are no longer considered as basic operations of the algebra. Our aim is to find a generator for the variety $\mathcal{V}((M, \sqcap, \sqcup))$. We consider the algebra in Fig. 3. It is obtained by removing the absorbing element from the algebra in Fig. 2, and by no longer considering the constants $\bar{0}, \bar{1}$ to be basic operations.

Theorem 40. *Removing the constants* $\bar{0}, \bar{1}$ *from consideration, the variety* $\mathcal{V}((M, \sqcap, \sqcup))$ *is generated by the four-element algebra in Fig.* 3.

Proof. Let $\mathbb{A}$ be the algebra in Fig. 2, $\mathbb{A}'$ be the algebra obtained from $\mathbb{A}$ by not considering $\bar{0}, \bar{1}$ to be basic operations, and $\mathbb{B}$ be the algebra of Fig. 3. Theorem 39 gives $\mathcal{V}((M, \sqcap, \sqcup, \bar{0}, \bar{1})) = \mathcal{V}(\mathbb{A})$, so by the same argument as in

$0^* = 0$
$1^* = 16$
$2^* = 8$
$3^* = 24$
$4^* = 4$
$5^* = 20$
$6^* = 12$
$7^* = 28$
$8^* = 2$
$9^* = 18$
$10^* = 10$
$11^* = 26$
$12^* = 6$
$13^* = 22$
$14^* = 14$
$15^* = 30$
$16^* = 1$
$17^* = 17$
$18^* = 9$
$19^* = 25$
$20^* = 5$
$21^* = 21$
$22^* = 13$
$23^* = 29$
$24^* = 3$
$25^* = 19$
$26^* = 11$
$27^* = 27$
$28^* = 7$
$29^* = 23$
$30^* = 15$
$31^* = 31$

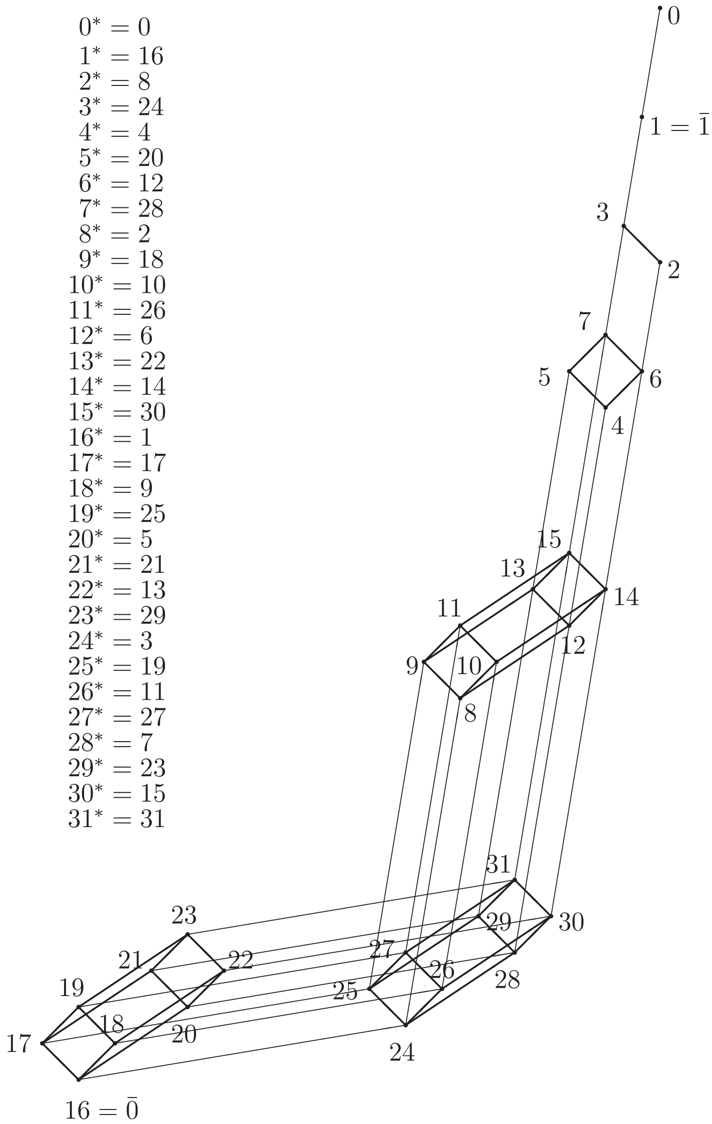

Fig. 4. The algebra $2^{\mathbf{5}}$.

Theorem 32, $\mathcal{V}((M, \sqcap, \sqcup)) = \mathcal{V}(\mathbb{A}')$. We show $\mathcal{V}(\mathbb{A}') = \mathcal{V}(\mathbb{B})$. As $\mathbb{B}$ is a subalgebra of $\mathbb{A}'$, $\mathcal{V}(\mathbb{B}) \subseteq \mathcal{V}(\mathbb{A}')$. For the other containment, let $\mathbb{C}$ be the subalgebra of $\mathbb{B}$ consisting of the elements $\{b, c\}$. Consider the algebra $\mathbb{B} \times \mathbb{C}$, and note that there is a congruence θ on this algebra with $\{(a, b), (b, b), (c, b), (d, b)\}$ as its only non-trivial block. Then $(\mathbb{B} \times \mathbb{C})/\theta$ is isomorphic to $\mathbb{A}'$, giving $\mathcal{V}(\mathbb{A}') \subseteq \mathcal{V}(\mathbb{B})$. □

Remark 41. One might notice that if one were to define constant operations $\bar{0}$, $\bar{1}$ on the four-element bisemichain in Fig. 3 by setting $\bar{0} = a$ and $\bar{1} = d$, this algebra would be a subalgebra of the algebra $\mathbb{A}$ in Fig. 2. Without constants, the above proof shows the four-element bisemichain and $\mathbb{A}$ generate the same variety. One might ask whether this holds also when we do include the constants. It does not, as is seen by observing that with constants, the four-element bisemichain satisfies $x \sqcap \bar{0} = 0$, while $\mathbb{A}$ does not.

We turn our attention to the case where we consider not only the operations $\sqcap$, $\sqcup$, $\bar{0}$, $\bar{1}$, but also the operation $*$. Recall that $*$ is defined to be part of the type of the algebra $2^{\mathbf{5}}$, and that $\mathcal{V}(\mathbb{M}) = \mathcal{V}(2^{\mathbf{5}})$. We consider more closely this algebra $2^{\mathbf{5}}$, and the variety it generates. Fig. 4 depicts $2^{\mathbf{5}}$ in much the same way Fig. 1 depicted $2^{\mathbf{3}}$. The nodes are labelled 0 through 31, where each such decimal number, when converted to binary, represents a characteristic function of a

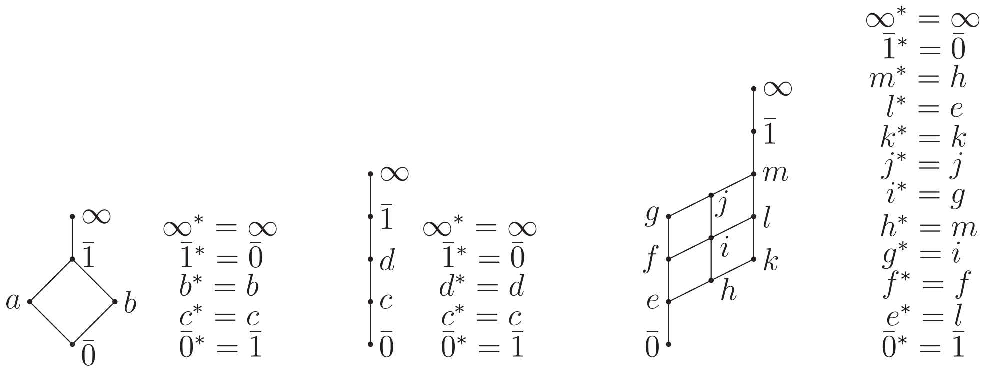

Fig. 5. The quotients of $2^{\mathbf{5}}$ by θ_1, θ_2, θ.

subset of $0 < p < q < r < 1$. For instance, 19 in binary is 10011, and represents the subset $\{0, r, 1\}$. One difference between this figure and Fig. 1 is that we describe the operations $\sqcup, *, \bar{0}, \bar{1}$. We do not describe $\sqcap$ because it is obtained by $x \sqcap y = (x^* \sqcup y^*)^*$.

Lemma 42. *Consider the equivalence relations $\theta_1, \theta_2, \theta$ on $2^{\mathbf{5}}$ with associated partitions*:

$$\begin{aligned}
\theta_1 &: \{0\}, \{1, 3, 5, 7, 9, 11, 13, 15\}, \{2, 4, 6, 8, 10, 12, 14\}\\
&\quad \{16, 18, 20, 22, 24, 26, 28, 30\}, \{17, 19, 21, 23, 25, 27, 29, 31\}\\
\theta_2 &: \{0\}, \{1, 2, 3\}, \{4, 5, 6, 7, 12, 13, 14, 15, 20, 21, 22, 23, 28, 29, 30, 31\}\\
&\quad \{8, 16, 24\}, \{9, 10, 11, 17, 18, 19, 25, 26, 27\}\\
\theta &: \{0\}, \{1\}, \{2, 3, 6, 7\}, \{4\}, \{5\}, \{8, 12, 24, 28\}, \{9, 13, 25, 29\}\\
&\quad \{10, 11, 14, 15, 26, 27, 30, 31\}, \{16\}, \{17, 21\}, \{18, 19, 22, 23\}, \{20\}
\end{aligned}$$

Then $\theta_1, \theta_2, \theta$ are congruences and intersect to the diagonal.

Again, the proof can be checked using the "Universal Algebra Calculator" as well as by hand. The three quotients are depicted in Fig. 5 with $2^{\mathbf{5}}/\theta_1$ at left, $2^{\mathbf{5}}/\theta_2$ in the middle, and $2^{\mathbf{5}}/\theta$ at right. In each case, the operation $\sqcup$ is shown with the constants $\bar{0}$, $\bar{1}$ labelled, and the operation $*$ is given explicitly. The element ∞ is again an absorbing element. The algebra $2^{\mathbf{5}}/\theta_1$ is a four-element De Morgan algebra with an absorbing element added, and $2^{\mathbf{5}}/\theta_2$ is the five-element bisemichain from Fig. 2 equipped with an extra operation $*$.

Theorem 43. *The variety $\mathcal{V}(\mathbb{M})$ is generated by the twelve-element algebra shown at right in Fig. 5.*

Proof. Theorem 35 gives that the variety $\mathcal{V}(\mathbb{M})$ equals $\mathcal{V}(2^{\mathbf{5}})$, so it is enough to show that $\mathcal{V}(2^{\mathbf{5}})$ equals $\mathcal{V}(2^{\mathbf{5}}/\theta)$. Surely $\mathcal{V}(2^{\mathbf{5}}/\theta) \subseteq \mathcal{V}(2^{\mathbf{5}})$. For the other containment, Lemma 42 gives $2^{\mathbf{5}} \leq 2^{\mathbf{5}}/\theta_1 \times 2^{\mathbf{5}}/\theta_2 \times 2^{\mathbf{5}}/\theta$. We note that the subalgebra $\{\bar{0}, f, j, \bar{1}, \infty\}$ of $2^{\mathbf{5}}/\theta$ is isomorphic to $2^{\mathbf{5}}/\theta_2$. Also, if we consider the subalgebra $\{\bar{0}, e, f, k, l, \bar{1}, \infty\}$ of $2^{\mathbf{5}}$, then the quotient of this subalgebra that collapses $\bar{0}$, e and l, $\bar{1}$ is isomorphic to $2^{\mathbf{5}}/\theta_1$. Therefore $2^{\mathbf{5}}$ is isomorphic to a subalgebra of a power of $2^{\mathbf{5}}/\theta$, hence $\mathcal{V}(2^{\mathbf{5}}) \subseteq \mathcal{V}(2^{\mathbf{5}}/\theta)$. $\square$

Remark 44. These results, showing that $\mathcal{V}(\mathbb{M})$ is generated by the above 12-element algebra, that $\mathcal{V}((M, \sqcap, \sqcup, \bar{0}, \bar{1}))$ is generated by the five-element bisemichain, and that $\mathcal{V}((M, \sqcap, \sqcup))$ is generated by the four-element bisemichain give improved methods to determine if certain kinds of equations hold. For an equation $s \approx t$ involving only the operations $\sqcap, \sqcup$, it is enough to determine if it holds in the four-element bisemichain. This requires testing every possible combination of these four elements for the variables. For an equation using the operations $\sqcap, \sqcup$ and the constants $\bar{0}, \bar{1}$, we must test it in the five-element bisemichain, and for one using $*$ as well, we must test it in the 12-element algebra. Even for small equations, this could be a tiring job to try by hand. The algorithm from Theorem 13 may be preferable for equations using only the operations $\sqcap, \sqcup$.

9. Concluding remarks

The results obtained here answer a number of questions about the variety generated by the algebra $\mathbb{M}$ of truth values of type-2 fuzzy sets. But there remain a number of questions of interest. One question is to find a finite set of equations, called a *basis*, defining the variety $\mathcal{V}(\mathbb{M})$. A similar question could be asked for the varieties obtained from $\mathbb{M}$ by discarding the unary operation $*$ or the constants $\bar{0}$, $\bar{1}$. We have discussed this question to an extent in Remark 11. We should comment also that we have shown $\mathcal{V}(\mathbb{M})$ is generated by a finite algebra. If this variety were congruence distributive that would imply a finite basis exists. Unfortunately we have no reason to believe this variety is congruence distributive. So there might not be a finite basis for any of these varieties.

This leads directly to determining which of the varieties we have discussed are congruence distributive. We know that the variety $\mathcal{V}((M, \sqcap, \sqcup))$ is not congruence distributive. This follows as this variety is generated by the five-element bisemichain of Fig. 2 when the constants are removed from this algebra, and is also generated by the four-element bisemichain of Fig. 3. As this five-element bisemichain is subdirectly irreducible, the variety $\mathcal{V}((M, \sqcap, \sqcup))$ cannot be congruence distributive by Jónsson's Lemma [4]. It remains a possibility that with more operations, the variety $\mathcal{V}(\mathbb{M})$ is congruence distributive, but this seems unlikely.

In the previous section we obtained a number of results saying that a particular variety is generated by a single finite algebra of relatively small size. In each case, the algebra obtained is subdirectly irreducible. However, in the absence of congruence distributivity, it may be possible that these varieties are generated by smaller algebras. It would be of interest to know whether the results in the previous section can be sharpened to provide smaller algebras generating the varieties of interest.

Whether or not one is able to improve the results of the previous section to find smaller algebras generating the varieties of interest, the most practical algorithm to determine if a small equation is valid in a given variety may be a syntactic one along the lines described in Remark 14. It may be worthwhile to follow this path to its conclusion and produce such an algorithm for the variety generated by $\mathbb{M}$.

Finally, as a matter of basic curiosity, it would be of interest to explore various techniques used here in a more general setting. The very first step in defining the algebra $\mathbb{M}$ is as a "convolution" algebra of the unit interval with standard negation. Surely convolutions of other algebras can be considered, and may have some general theory. Another direction is provided by our study of the algebra $\mathbb{E}$. This algebra is naturally realized as the complex algebra of the unit interval with negation. A more thorough study of complex algebras of lattices with additional operations may be of interest.

Acknowledgments

We thank Keith Kearnes with suggestions related to Theorem 40.

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