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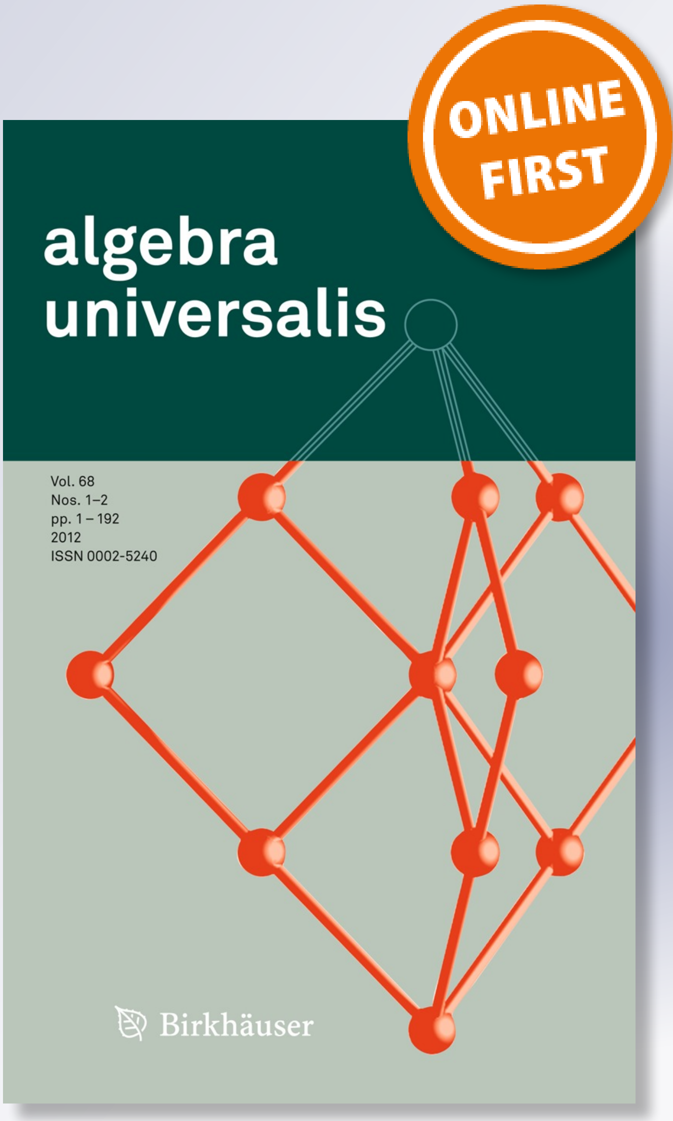

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A Boolean topological orthomodular poset

John Harding

Abstract. Wilce introduced the notion of a topological orthomodular poset and proved any compact topological orthomodular poset whose underlying orthomodular poset is a Boolean algebra is a topological Boolean algebra in the usual sense. Wilce asked whether the compactness assumption was necessary for this result. We provide an example to show the compactness assumption is necessary.

Several authors [2, 7, 8] have considered topological orthomodular lattices as the obvious generalization of topological lattices [1], and topological Boolean algebras [3]. Specifically, they consider an orthomodular lattice L with a Hausdorff topology that makes the operations of join, meet, and orthocomplementation, continuous. While they develop a very nice theory, these studies unfortunately miss the prime example from quantum logic. The orthomodular lattice that motivates the area of quantum logic is the orthomodular lattice $L(\mathcal{H})$ of projection operators on a Hilbert space $\mathcal{H}$. This orthomodular lattice $L(\mathcal{H})$ carries natural topologies, the norm and the strong operator topology (which agrees with the weak operator topology here), but these topologies do not make $L(\mathcal{H})$ into a topological orthomodular lattice in this sense.

Wilce [9, 10, 11] proposed a more subtle definition for topological orthomodular structures that does include the motivating example of the projection operators $L(\mathcal{H})$ of a Hilbert space with either norm or the strong operator topology. His definition is in the spirit of topological posets [5] rather than topological lattices. This fits with the study of quantum logic where the natural setting is that of orthomodular posets or orthoalgebras, rather than orthomodular lattices, and the joins of interest are only the orthogonal ones. Directing the reader to [9] for the definition, we recall only that orthoalgebras are certain structures with a partial ordering, orthocomplementation $'$, orthogonality relation $\perp$ given by $x \leq y'$, and a partial binary operation $\oplus$ defined on orthogonal pairs. Examples of orthoalgebras include orthomodular posets and orthomodular lattices [6].

Definition 1 (Wilce [9])**.** A *topological orthoalgebra* is an orthoalgebra L with a topology τ such that

(1) $\perp$ is a closed subset of $L \times L$,

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(2) $\oplus\colon \perp \to L$ and $'\colon L \to L$ are continuous.

As first observations, Wilce shows [9] the order $\leq$ of a topological orthoalgebra is closed in the product topology, the topology of a topological orthoalgebra is Hausdorff, and the topological orthomodular lattices and topological Boolean algebras considered in [2, 3, 7, 8] are topological orthoalgebras. In developing the deeper theory, particularly that of compact topological orthoalgebras, Wilce [9] gives the following.

Theorem 2. *A compact topological orthoalgebra whose underlying orthoalgebra is a Boolean algebra is a topological Boolean algebra in the usual sense.*

Wilce asks [9, Question 3.10] whether this result holds without the compactness assumption. As every orthomodular poset and orthomodular lattice is constructed by pasting its maximal Boolean subalgebras, this question has importance in developing further the theory of non-compact topological orthoalgebras.

We note that in the natural example of $L(\mathcal{H})$, the restriction of the norm and strong operator topologies to Boolean subalgebras does yield topological Boolean algebras. This is because the operations of addition and multiplication are continuous with respect to these topologies (for the strong operator topology, the continuity of multiplication is due to the restriction to operators of norm one [4, p. 116]) and join and meet in a Boolean algebra of projections are constructed via addition and multiplication. So it is natural to hope that Boolean subalgebras of all topological orthoalgebras would be well behaved in this regard.

It is our aim here to show that Wilce's question has a negative answer in general. In particular, we give an example to show the following.

Proposition 3. *There is a topological orthoalgebra whose underlying orthoalgebra is Boolean that is not a topological Boolean algebra.*

The Boolean algebra used to construct the example is $\mathbf{2}^\omega$. To describe the topology, we begin as follows.

Definition 4. Let $\mathbf{C}$ be the Cantor set, and define $d\colon \mathbf{C}\times\mathbf{C} \to \mathbb{R}$ by setting

$$d(x,y) = \begin{cases} |x-y| & \text{if } x-y \text{ is rational,} \\ 2 & \text{if } x-y \text{ is irrational.} \end{cases}$$

Clearly d is symmetric and positive definite, and as we are constrained to one dimension, it satisfies the triangle inequality. Thus d is a metric on $\mathbf{C}$, so induces a topology on $\mathbf{C}$. It is well known that the points in $\mathbf{C}$ are exactly those that have an infinite ternary expansion using only the digits 0 and 2, and that each point in $\mathbf{C}$ has a unique infinite ternary expansion using just 0 and 2. So $\mathbf{C}$ is in bijective correspondence with the set $\{0,2\}^\omega$. Considering $\mathbf{2} = \{0,2\}$ as a 2-element Boolean algebra with bottom 0 and top 2, the metric d then induces a topology τ on the Boolean algebra $\mathbf{2}^\omega$. Basic properties of geometric sequences yield the following.

Lemma 5. *Let $x, y \in \mathbf{C} = \mathbf{2}^\omega$.*

(1) *If the ternary expansions for x, y agree in the first n spots after the decimal, then $|x, y| \leq 3^{-n}$.*
(2) *If $|x - y| < 3^{-n}$, then the ternary expansions for x, y agree in the first n spots after the decimal.*

For $x \in \mathbf{C}$ and $\epsilon > 0$, we use the standard notation $\mathcal{B}(x, \epsilon)$ for the open ball of d-radius ϵ centered at x.

Proposition 6. *The Boolean algebra $\mathbf{C} = \mathbf{2}^\omega$ with topology τ is a topological orthoalgebra.*

Proof. We first show the relation $\perp$ is closed in the product topology. Note that $x \perp y$ iff x and y never have 2 s in the same spot of their ternary expansions. Suppose $x \not\perp y$, and let n be such that both have a 2 in the n^{th} spot of their ternary expansions. Then $\mathcal{B}(x, 3^{-n}) \times \mathcal{B}(y, 3^{-1})$ is an open neighborhood of (x, y). If $w \in \mathcal{B}(x, 3^{-n})$, then w and x agree in their first n spots, so w has a 2 in the n^{th} spot of its ternary expansion, and similarly if $z \in \mathcal{B}(y, 3^{-n})$, then z has a 2 in the n^{th} spot of its ternary expansion, so $w \not\perp z$. Thus, this is an open neighborhood of (x, y) disjoint from $\perp$, showing $\perp$ is closed.

For $x \in \mathbf{C}$, the orthocomplement x' is given by $1 - x$, ordinary subtraction of real numbers. Clearly $d(x, y) = d(1-x, 1-y)$, and it follows that orthocomplementation on $\mathbf{C} = \mathbf{2}^\omega$ is continuous. The join of two orthogonal elements is given by the usual sum of real numbers. Suppose $x \perp y$ and $x \oplus y = z$. Suppose $\epsilon > 0$ and $(u, v) \in \perp \cap (\mathcal{B}(x, \epsilon/2) \times \mathcal{B}(y, \epsilon/2))$. Then $|x - u| < \epsilon/2$ and $|y - v| < \epsilon/2$, and both are rational. So $|(x + y) - (u + v)| < \epsilon$ and is rational. So this neighborhood of (x, y) in $\perp$ is mapped by $\oplus$ into $\mathcal{B}(z, \epsilon)$, showing $\oplus\colon \perp \times \perp \to \mathbf{C}$ is continuous. □

Proposition 7. *The join operation of the Boolean algebra $\mathbf{C} = \mathbf{2}^\omega$ is not continuous with respect to the topology τ. So with this topology, $\mathbf{C} = \mathbf{2}^\omega$ is a topological orthoalgebra, but not a topological Boolean algebra in the usual sense.*

Proof. Note that an element of $\mathbf{C}$ is rational iff its ternary expansion is repeating. The proof of this fact is identical to the familiar result for decimal expansions of real numbers. Choose an irrational $x \in \mathbf{C}$. For join to be continuous, there must be some $\epsilon > 0$ with join mapping $\mathcal{B}(x, \epsilon) \times \mathcal{B}(1, \epsilon)$ into the open neighborhood $\mathcal{B}(1, 2)$ of 1. We note that $\mathcal{B}(1, 2)$ consists of all rational members of $\mathbf{C}$. Given $\epsilon > 0$, we can find n with $3^{-n} < \epsilon$. Surely x belongs to $\mathcal{B}(x, \epsilon)$ and the number y whose ternary expansion has 2's in the first n places followed by an infinite tail of 0's belongs to $\mathcal{B}(1, \epsilon)$. But $x \vee y$ has 2's in its first n places, then agrees with x in the remaining tail. As x is irrational, this tail is non-repeating, showing $x \vee y$ is irrational. So $x \vee y$ does not belong to $\mathcal{B}(1, 2)$. □

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