MODAL OPERATORS ON COMPACT REGULAR FRAMES AND DE VRIES ALGEBRAS

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Abstract. In [5] we introduced the category MKHaus of modal compact Hausdorff spaces, and showed these were concrete realizations of coalgebras for the Vietoris functor on compact Hausdorff spaces, much as modal spaces are coalgebras for the Vietoris functor on Stone spaces. Also in [5] we introduced the categories MKRFrm and MDV of modal compact regular frames, and modal de Vries algebras as algebraic counterparts to modal compact Hausdorff spaces, much as modal algebras are algebraic counterparts to modal spaces. In [5], MKRFrm and MDV were shown to be dually equivalent to MKHaus, hence equivalent to one another.

Here we provide a direct, choice-free proof of the equivalence of MKRFrm and MDV. We also detail connections between modal compact regular frames and the Vietoris construction for frames [11, 12], describe a Vietoris construction for de Vries algebras, and show how modal de Vries algebras are linked to this construction. Also described is an alternative approach to Isbell duality between compact regular frames and compact Hausdorff spaces obtained by using de Vries algebras as an intermediary.

1. Introduction

In [5] we began a program of lifting structures and techniques of modal logic, based fundamentally on Stone spaces and Boolean algebras, to the setting of compact Hausdorff spaces, de Vries algebras, and compact regular frames. Here, we consider aspects of this work more closely linked to the study of point-free topology than to modal logic. While we briefly recall some important facts from [5], the reader would benefit from having access to this paper when reading this note.

A modal space, or descriptive frame, (X, R) is a Stone space X with binary relation R satisfying certain properties equivalent to requiring the associated map from X into its Vietoris space $\mathcal{V}(X)$ be continuous. With the so-called p-morphisms between them, the category MS of modal spaces is isomorphic to the category of coalgebras for the Vietoris functor on Stone spaces. This lies at the heart of the coalgebraic treatment of modal logic. A modal algebra $(B, \Diamond)$ is a Boolean algebra with unary operation $\Diamond$ that preserves finite joins. The category MA of modal algebras and the homomorphisms between them is dually equivalent to MS via a lifting of Stone duality. These equivalences and dual equivalences tie the coalgebraic, algebraic, and relational treatments of modal logic.

In [5] the situation was lifted from the setting of Stone spaces to compact Hausdorff spaces. We defined a modal compact Hausdorff space (X, R) to be a compact Hausdorff space with binary relation R satisfying conditions equivalent to having the associated map from X to its Vietoris space $\mathcal{V}(X)$ be continuous. Then with morphisms again being p-morphisms, we showed the category MKHaus of modal compact Hausdorff spaces is isomorphic to the category of coalgebras for the Vietoris functor on KHaus. For algebraic counterparts to modal compact Hausdorff spaces, we lifted Isbell duality between KHaus and compact regular frames, and de Vries duality between KHaus and de Vries algebras, obtaining categories MKRFrm of modal compact regular frames, and MDV of modal de Vries algebras, each dually equivalent to MKHaus. For various reasons, the category MDV was a bit poorly behaved. We defined two full subcategories of MDV, the categories LMDV and UMDV of lower and upper continuous modal de Vries algebras, that were better behaved, and showed both were equivalent to MDV. The situation is summarized in Figure 1 below.

The functors in Figure 1 are described in [5]. Those between MKRFrm and MKHaus lift the usual point and open set functors between compact regular frames and compact Hausdorff spaces, and those between MDV and MKHaus lift the usual end and regular open set functors between de Vries algebras and compact Hausdorff spaces. As such, they require the axiom of choice. The composite of these functors then gives

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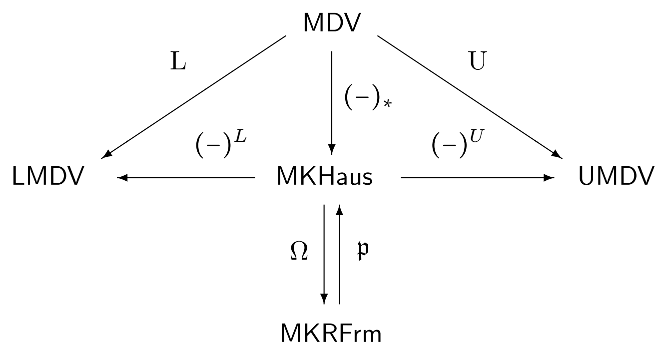

Figure 1

an equivalence between MKRFrm and MDV, but again, this requires the axiom of choice. The equivalences between MDV and its subcategories LMDV and UMDV are choice-free.

A primary purpose here is to give a direct, choice-free proof of the equivalence of MKRFrm and each of MDV, LMDV and UMDV. To do so, we construct functors $\mathfrak{L}$: MKRFrm → LMDV and $\mathfrak{U}$: MKRFrm → UMDV that lift the Booleanization functor in two ways, and a functor $\mathfrak{R}$: MDV → MKRFrm that lifts the round ideal functor. After the preliminaries in Section 2, this equivalence is established in Section 3.

The direct equivalence between MKRFrm and MDV when cut down gives a direct equivalence between compact regular frames and de Vries algebras. When composed with usual de Vries duality, this gives an alternative to the usual Isbell duality between compact regular frames and compact Hausdorff spaces that is of interest. Details of this alternative to Isbell duality are given in Section 4.

The definition of modal compact regular frames involves identities for the modal operators that appear in Johnstone's construction of Vietoris frames [11, 12]. This is not surprising as modal compact regular frames arise as algebraic counterparts of coalgebras for the Vietoris functor on compact Hausdorff spaces. The details of this connection are given in Section 5. In this section we also provide a counterpart of the Vietoris construction for de Vries algebras, and explain its connection to the axioms of modal de Vries algebras.

2. Preliminaries

We briefly recall the primary definitions. The reader should consult [5] for complete details.

Definition 2.1. *A frame is a complete lattice L where finite meets distribute over infinite joins, and a frame homomorphism is a map between frames preserving finite meets and infinite joins. A frame is compact if $\bigvee S = 1$ implies there is a finite subset $S' \subseteq S$ with $\bigvee S' = 1$. Using $\neg a$ for the pseudocomplement of an element a, we say a is well inside b, and write $a \prec b$, if $\neg a \vee b = 1$. A frame L is a compact regular frame if it is compact and for each $b \in L$ we have $b = \bigvee\{a : a \prec b\}$. The category of compact regular frames and the frame homomorphisms between them is denoted* KRFrm*.*

There is an extensive literature on compact regular frames (see, e.g., [10, 2, 11]).

Definition 2.2. *A* modal compact regular frame *(abbreviated:* MKR*-frame) is a triple $\mathcal{L} = (L, \Box, \Diamond)$ where L is a compact regular frame, and $\Box, \Diamond$ are unary operations on L satisfying the following conditions.*

1. *$\Box$ preserves finite meets, so $\Box 1 = 1$ and $\Box(a \wedge b) = \Box a \wedge \Box b$.*
2. *$\Diamond$ preserves finite joins, so $\Diamond 0 = 0$ and $\Diamond(a \vee b) = \Diamond a \vee \Diamond b$.*
3. *$\Box(a \vee b) \leq \Box a \vee \Diamond b$ and $\Box a \wedge \Diamond b \leq \Diamond(a \wedge b)$.*
4. *$\Box, \Diamond$ preserve directed joins, so $\Diamond \bigvee S = \bigvee\{\Diamond s : s \in S\}$, $\Box \bigvee S = \bigvee\{\Box s : s \in S\}$ for any up-directed S.*

An MKR*-morphism is a frame homomorphism h that satisfies $h(\Box a) = \Box h(a)$ and $h(\Diamond a) = \Diamond h(a)$. The category of modal compact regular frames and their morphisms, composed by ordinary function composition, is denoted* MKRFrm*.*

We next describe de Vries algebras. For further details on this topic see [8, 3, 5].

Definition 2.3. *A de Vries algebra is a pair $(A, \prec)$ where A is a complete Boolean algebra and $\prec$ is a binary relation on A satisfying*

(1) $1 \prec 1$.
(2) $a \prec b$ *implies* $a \le b$.
(3) $a \le b \prec c \le d$ *implies* $a \prec d$.
(4) $a \prec b, c$ *implies* $a \prec b \wedge c$.
(5) $a \prec b$ *implies* $\neg b \prec \neg a$.
(6) $a \prec b$ *implies there exists* c *with* $a \prec c \prec b$.
(7) $a \ne 0$ *implies there exists* $b \ne 0$ *with* $b \prec a$.

A morphism between de Vries algebras is a function α that satisfies (i) $\alpha(0) = 0$, (ii) $\alpha(a \wedge b) = \alpha(a) \wedge \alpha(b)$, (iii) $a \prec b$ implies $\neg\alpha(\neg a) \prec \alpha(b)$, and (iv) $\alpha(a) = \bigvee\{\alpha(b) : b \prec a\}$.

The motivating example of a de Vries algebra is the complete Boolean algebra $\mathcal{RO}X$ of regular open sets of a compact Hausdorff space X with relation $\prec$ on $\mathcal{RO}X$ defined by $S \prec T$ if $\mathbf{C}S \subseteq T$ where $\mathbf{C}$ is usual topological closure. A continuous map $f : X \to Y$ between compact Hausdorff spaces gives a de Vries morphism $\mathbf{IC}f^{-1}[-]$ from $\mathcal{RO}Y$ to $\mathcal{RO}X$ where $\mathbf{I}$ is usual topological interior. In this setting, one can see that the ordinary function composite of de Vries morphisms need not be a de Vries morphism.

Definition 2.4. *For de Vries morphisms α and β, define their composite to be $\beta \star \alpha$ where*

$$(\beta \star \alpha)(a) = \bigvee\{\beta\alpha(b) : b \prec a\}.$$

Let DeV *be the category of de Vries algebras and their morphisms under this $\star$ composition.*

Definition 2.5. *A modal de Vries algebra (abbreviated:* MDV*-algebra) is a triple $\mathfrak{A} = (A, \prec, \Diamond)$ where $(A, \prec)$ is a de Vries algebra and $\Diamond$ is a unary operation on A that satisfies the following conditions.*

(1) $\Diamond 0 = 0$.
(2) $a_1 \prec b_1$ *and* $a_2 \prec b_2$ *imply* $\Diamond(a_1 \vee a_2) \prec \Diamond b_1 \vee \Diamond b_2$.

A morphism between modal de Vries algebras is a de Vries morphism α for which $a \prec b$ implies both $\alpha(\Diamond a) \prec \Diamond\alpha(b)$ and $\Diamond\alpha(a) \prec \alpha(\Diamond b)$. Let MDV *be the category of modal de Vries algebras and morphisms with composition being the $\star$ composition of Definition 2.4.*

Two full subcategories of MDV play an important role in [5], and also in our considerations here.

Definition 2.6. *An* MDV*-algebra $(A, \prec, \Diamond)$ is called lower continuous if $\Diamond a = \bigvee\{\Diamond b : b \prec a\}$ and upper continuous if $\Diamond a = \bigwedge\{\Diamond b : a \prec b\}$. Let* LMDV *and* UMDV *be the full subcategories of* MDV *consisting of all lower, respectively upper, continuous* MDV*-algebras.*

We recall that in [5, Sec. 4.3] it was shown that each member of MDV is isomorphic to a member of LMDV and to a member of UMDV, this despite the fact that a modal de Vries algebra need be neither lower nor upper continuous. This somewhat counter intuitive situation is due to the fact that composition in MDV is not function composition, and isomorphisms are not structure preserving bijections.

3. Equivalence of MKRFrm, MDV, LMDV, and UMDV

In this section we provide direct equivalences between MKRFrm and each of MDV, LMDV, and UMDV. These proofs do not rely on the axiom of choice, as did ones in [5].

Definition 3.1. *For a de Vries algebra $(A, \prec)$ and $S \subseteq A$, define $\downarrow S = \{a : a \le s \text{ for some } a \in S\}$, and $\twoheaddownarrow S = \{a : a \prec s \text{ for some } s \in S\}$. An ideal I of A is called* round *if $I = \twoheaddownarrow I$.*

It is known (see, e.g., [1, Lem. 2] or [6, Prop. 4.6]) that the collection $\mathfrak{R}A$ of all round ideals of A is a subframe of the frame of all ideals of A.

Definition 3.2. *For $\mathfrak{A} = (A, \prec, \Diamond)$ an* MDV*-algebra, define $\Box$ on A by setting $\Box a = \neg \Diamond \neg a$ for all $a \in A$.*

Lemma 3.3. *Let $\mathfrak{A} = (A, \prec, \Diamond)$ be an* MDV*-algebra and $a \prec b$, $a_1 \prec b_1$, $a_2 \prec b_2$. Then*

(1) $\Diamond a \prec \Diamond b$ *and* $\Box a \prec \Box b$.
(2) $\Diamond(a_1 \vee a_2) \prec \Diamond b_1 \vee \Diamond b_2$ *and* $\Box a_1 \wedge \Box a_2 \prec \Box(b_1 \wedge b_2)$.
(3) $\Box(a_1 \vee a_2) \prec \Box b_1 \vee \Diamond b_2$ *and* $\Box a_1 \wedge \Diamond a_2 \prec \Diamond(b_1 \wedge b_2)$.

Proof. The definition of an MDV-algebra gives $\Diamond a \prec \Diamond b$ and $\Diamond(a_1 \vee a_2) \prec \Diamond b_1 \vee \Diamond b_2$. In any de Vries algebra we have $a \prec b$ iff $\neg b \prec \neg a$. This gives $\Box a \prec \Box b$ and $\Box a_1 \wedge \Box a_2 \prec \Box(b_1 \wedge b_2)$. So (1) and (2) are established. For (3) use interpolation to find $a_1 \prec c_1 \prec d_1 \prec b_1$ and $a_2 \prec c_2 \prec d_2 \prec b_2$. Then $a_1 \vee a_2 \prec c_1 \vee c_2$ and $\neg d_2 \prec \neg c_2$, so by (2), $\Box(a_1 \vee a_2) \wedge \Box\neg d_2 \prec \Box((c_1 \vee c_2) \wedge \neg c_2)$. As $(c_1 \vee c_2) \wedge \neg c_2 \leq c_1 \prec d_1$, applying (1) gives $\Box(a_1 \vee a_2) \wedge \Box\neg d_2 \prec \Box d_1$, hence $\Box(a_1 \vee a_2) \leq \Box d_1 \vee \neg \Box \neg d_2 = \Box d_1 \vee \Diamond d_2$. Finally use (1) once again to obtain $\Box d_1 \vee \Diamond d_2 \prec \Box b_1 \vee \Diamond b_2$. This gives the first statement in (3). Using that $x \prec y$ iff $\neg y \prec \neg x$, the second statement in (3) is equivalent to $\Box(\neg b_1 \vee \neg b_2) \prec \Diamond\neg a_1 \vee \Box\neg a_2$, which is equivalent to the first. □

Definition 3.4. *For $\mathfrak{A} = (A, \prec, \Diamond)$ an* MDV*-algebra, define $\mathfrak{R}\mathfrak{A} = (\mathfrak{R}A, \overline{\Box}, \overline{\Diamond})$ where $\mathfrak{R}A$ is the frame of round ideals of A and $\overline{\Box}, \overline{\Diamond}$ are given by $\overline{\Box}(I) = \twoheaddownarrow \Box[I]$ and $\overline{\Diamond}(I) = \twoheaddownarrow \Diamond[I]$.*

Proposition 3.5. *If $\mathfrak{A}$ is an* MDV*-algebra, then $\mathfrak{R}\mathfrak{A}$ is an* MKR*-frame.*

Proof. It is well-known that $\mathfrak{R}A$ is a subframe of the ideal frame of A that is compact regular (see, e.g., [1] or [4]). It is easy to see that $\overline{\Box}(I)$ and $\overline{\Diamond}(I)$ are round ideals so $\overline{\Box}, \overline{\Diamond}$ are well defined. By Lemma 3.3.1, both $\Box, \Diamond$ are proximity preserving on A, so we can alternately describe $\overline{\Box}(I) = \downarrow \Box[I]$ and $\overline{\Diamond}(I) = \downarrow \Diamond[I]$.

We must verify the conditions of Definition 2.2. As $\Diamond 0 = 0$ and $\Box 1 = 1$, we have $\overline{\Diamond} 0 = 0$ and $\overline{\Box} 1 = 1$. Clearly $\overline{\Box}$ and $\overline{\Diamond}$ are order-preserving, so $\overline{\Diamond}(I) \vee \overline{\Diamond}(J) \subseteq \overline{\Diamond}(I \vee J)$ and $\overline{\Box}(I \wedge J) \subseteq \overline{\Box}(I) \wedge \overline{\Box}(J)$. If $a_1 \in I$ and $a_2 \in J$, then roundness gives $b_1 \in I$ and $b_2 \in J$ with $a_1 \prec b_1$ and $a_2 \prec b_2$. Then Lemma 3.3.2 gives $\Diamond(a_1 \vee a_2) \prec \Diamond b_1 \vee \Diamond b_2$, showing $\overline{\Diamond}(I \vee J) \subseteq \overline{\Diamond}(I) \vee \overline{\Diamond}(J)$, and $\Box a_1 \wedge \Box a_2 \prec \Box(b_1 \wedge b_2)$, showing $\overline{\Box}(I \wedge J) \subseteq \overline{\Box}(I) \wedge \overline{\Box}(J)$. Thus $\overline{\Diamond}$ is finitely additive and $\overline{\Box}$ is finitely multiplicative. Also, Lemma 3.3.3 gives $\Box(a_1 \vee a_2) \prec \Box b_2 \vee \Diamond b_2$ and $\Box a_1 \wedge \Diamond a_2 \prec \Diamond(b_1 \wedge b_2)$, showing $\overline{\Box}(I \vee J) \subseteq \overline{\Box}(I) \vee \overline{\Diamond}(J)$ and $\overline{\Box}(I) \wedge \overline{\Diamond}(J) \subseteq \overline{\Diamond}(I \wedge J)$. Finally, directed joins in $\mathfrak{R}A$ are given by unions, and it follows easily that both $\overline{\Box}$ and $\overline{\Diamond}$ preserve directed joins. □

Theorem 3.6. *The assignment $\mathfrak{A} \mapsto \mathfrak{R}\mathfrak{A}$ can be extended to a functor $\mathfrak{R} : \mathsf{MDV} \to \mathsf{MKRFrm}$ by setting $\mathfrak{R}\alpha = \twoheaddownarrow \alpha[\cdot]$ for an* MDV*-morphism $\alpha : \mathfrak{A} \to \mathfrak{B}$.*

Proof. It is known [4, Rem. 3.10] that the "restriction" of $\mathfrak{R}$ gives a functor $\mathfrak{R} : \mathsf{DeV} \to \mathsf{KRFrm}$, so it remains only to show that the frame homomorphism $\mathfrak{R}\alpha$ is an MKR-morphism. This means we must show $(\mathfrak{R}\alpha)(\overline{\Diamond} I) = \overline{\Diamond}((\mathfrak{R}\alpha)I)$ and $(\mathfrak{R}\alpha)(\overline{\Box} I) = \overline{\Box}((\mathfrak{R}\alpha)I)$ for each round ideal I of $\mathfrak{A}$. This follows directly once we show $a \prec b$ implies (i) $\alpha(\Diamond a) \prec \Diamond\alpha(b)$, (ii) $\Diamond\alpha(a) \prec \alpha(\Diamond b)$, (iii) $\alpha(\Box a) \prec \Box\alpha(b)$, and (iv) $\Box\alpha(a) \prec \alpha(\Box b)$.

Items (i) and (ii) are part of the definition of an MDV-morphism. For (iii), use interpolation to find $a \prec c \prec d \prec b$ and recall that an MDV-morphism also satisfies $x \prec y$ implies $\alpha(\neg y) \prec \neg\alpha(x)$ and $\neg\alpha(y) \prec \alpha(\neg x)$. Then as $\Diamond\neg c \prec \Diamond\neg a$ we have $\alpha(\Box a) = \alpha(\neg\Diamond\neg a) \prec \neg\alpha(\Diamond\neg c)$, and as $\neg d \prec \neg c$ we have $\Diamond\alpha(\neg d) \prec \alpha(\Diamond\neg c)$, hence $\alpha(\Box a) \prec \neg\alpha(\Diamond\neg c) \prec \neg\Diamond\alpha(\neg d)$. But $d \prec b$ gives $\neg\alpha(b) \prec \alpha(\neg d)$, hence $\alpha(\Box a) \prec \neg\Diamond\alpha(\neg d) \prec \neg\Diamond\neg\alpha(b) = \Box\alpha(b)$. This gives (iii), and a similar calculation provides (iv). □

Next we construct a functor from MKRFrm to MDV. In fact, we will construct two functors, one will have image in LMDV and the other in UMDV.

Lemma 3.7 ([5, Lem. 3.6])**.** *Let $\mathcal{L} = (L, \Box, \Diamond)$ be an* MKR*-frame and $a, b \in L$. Then*

1. *$\Diamond a \leq \neg\Box\neg a$ and $\Box a \leq \neg\Diamond\neg a$.*
2. *If $a \prec b$, then $\Diamond a \prec \Diamond b$ and $\Box a \prec \Box b$.*
3. *If $a \prec b$, then $\neg\Box\neg a \prec \Diamond b$ and $\neg\Diamond\neg a \prec \Box b$.*
4. *If $a \prec b$, then $\Box a \prec \neg\Diamond\neg b$ and $\Diamond a \prec \neg\Box\neg b$.*

Recall that for a compact regular frame L, the operation $\neg\neg$ is a closure operator on L whose fixed points $\mathfrak{B}L$ are a de Vries algebra with proximity given by the restriction of the well inside relation $\prec$ on L [4, Lem. 3.1]. Meets in $\mathfrak{B}L$ agree with those in L, joins are given by applying the closure operator $\neg\neg$ to the join in L. We use $\sqcup$ for finite joins in $\mathfrak{B}L$ and $\bigsqcup$ for infinite joins.

Definition 3.8. *For $\mathcal{L} = (L, \Box, \Diamond)$ an* MKR*-frame, define $\Diamond^L$, $\Diamond^U$ on $\mathfrak{B}L$ by $\Diamond^L a = \neg\neg\Diamond a$ and $\Diamond^U a = \neg\Box\neg a$, and following our convention, define $\Box^L = \neg\Diamond^L\neg$ and $\Box^U = \neg\Diamond^U\neg$.*

Proposition 3.9. *For $\mathcal{L} = (L, \Box, \Diamond)$ an* MKR*-frame, $\mathfrak{L}\mathcal{L} = (\mathfrak{B}L, \Diamond^L)$ is a lower continuous* MDV*-algebra, and $\mathfrak{U}\mathcal{L} = (\mathfrak{B}L, \Diamond^U)$ is an upper continuous* MDV*-algebra.*

Proof. Clearly $\Diamond^L 0 = 0$ and $\Diamond^U 0 = 0$. Let $a_1, a_2, b_1, b_2 \in \mathfrak{B}L$ with $a_1 \prec b_1$ and $a_2 \prec b_2$. Then $a_1 \vee a_2 \prec b_1 \vee b_2$. As $x \prec y$ implies $\neg\neg x \prec y$ we have $a_1 \sqcup a_2 = \neg\neg(a_1 \vee a_2) \prec b_1 \vee b_2$. Lemma 3.7.2 and the additivity of $\Diamond$

give $\Diamond(a_1 \sqcup a_2) \prec \Diamond b_1 \vee \Diamond b_2 \leq \neg\neg\Diamond b_1 \sqcup \neg\neg\Diamond b_2$, so $\Diamond^L(a_1 \sqcup a_2) \prec \Diamond^L b_1 \sqcup \Diamond^L b_2$. This shows $\Diamond^L$ is de Vries additive, so $\mathfrak{L}\mathcal{L}$ is an MDV-algebra. For de Vries additivity of $\Diamond^U$, we have $a_1 \sqcup a_2 \prec b_1 \sqcup b_2$, and as $x \prec y$ iff $\neg y \prec \neg x$ in any de Vries algebra, $\neg(b_1 \sqcup b_2) \prec \neg(a_1 \sqcup a_2)$. Then Lemma 3.7.2 gives $\Box\neg(b_1 \sqcup b_2) \prec \Box\neg(a_1 \sqcup a_2)$, hence $\neg\Box\neg(a_1 \sqcup a_2) \prec \neg\Box\neg(b_1 \sqcup b_2)$. Using DeMorgan's law and the fact that $\Box$ is multiplicative, this gives $\Diamond^U(a_1 \sqcup a_2) \prec \Diamond^U b_2 \sqcup \Diamond^U b_2$, and shows $\mathfrak{U}\mathcal{L}$ is an MDV-algebra.

To see $\mathfrak{L}\mathcal{L}$ is lower continuous, let $a \in \mathfrak{B}L$. Recall we use $\bigvee$ for joins in L and $\bigsqcup$ for joins in $\mathfrak{B}L$. As L is compact regular, $a = \bigvee\{b : b \in L \text{ and } b \prec a\}$. As $b \prec a$ implies $\neg\neg b \prec a$, we have $a = \bigvee\{c : c \in \mathfrak{B}L \text{ and } c \prec a\}$. As $\Diamond$ is additive, $\Diamond a = \bigvee\{\Diamond c : c \in \mathfrak{B}L \text{ and } c \prec a\}$, and it follows that $\neg\neg\Diamond a = \neg\neg\bigvee\{\neg\neg\Diamond c : c \in \mathfrak{B}L \text{ and } c \prec a\}$. Thus $\Diamond^L a = \bigsqcup\{\Diamond^L c : c \prec a\}$, showing $\mathfrak{L}\mathcal{L}$ is lower continuous.

To see $\mathfrak{U}\mathcal{L}$ is upper continuous, recall meets in $\mathfrak{B}L$ agree with those in L. For $a \in \mathfrak{B}L$, we have $\neg a \in \mathfrak{B}L$, so as above, $\neg a = \bigvee\{c : c \in \mathfrak{B}L \text{ and } c \prec \neg a\}$. Noting that the $c \in \mathfrak{B}L$ with $c \prec \neg a$ are exactly the $\neg b$ with $b \in \mathfrak{B}L$ and $a \prec b$, we have $\neg a = \bigvee\{\neg b : b \in \mathfrak{B}L \text{ and } a \prec b\}$. As $\Box$ preserves directed joins, $\Box\neg a = \bigvee\{\Box\neg b : b \in \mathfrak{B}L \text{ and } a \prec b\}$. Then as $\neg\bigvee x_i = \bigwedge\neg x_i$ in any frame, and $\Diamond^U = \neg\Box\neg$, we have $\Diamond^U a = \bigwedge\{\Diamond^U b : b \in \mathfrak{B}L \text{ and } a \prec b\}$. Thus $\mathfrak{U}\mathcal{L}$ is upper continuous. □

Theorem 3.10. *The assignments $\mathcal{L} \mapsto \mathfrak{L}\mathcal{L}$ and $\mathcal{L} \mapsto \mathfrak{U}\mathcal{L}$ can be extended to functors $\mathfrak{L} : \mathsf{MKRFrm} \to \mathsf{LMDV}$ and $\mathfrak{U} : \mathsf{MKRFrm} \to \mathsf{UMDV}$ by setting $\mathfrak{L}h = \mathfrak{U}h = \neg\neg h$ for an* MKR*-morphism $h : \mathcal{L} \to \mathcal{M}$.*

Proof. The "restrictions" of $\mathfrak{L}, \mathfrak{U}$ to KRFrm are known [4, Lem. 3.4] to give a functor $\mathfrak{B} : \mathsf{KRFrm} \to \mathsf{DeV}$. It remains to show the de Vries morphisms $\mathfrak{L}h : \mathfrak{L}\mathcal{L} \to \mathfrak{L}\mathcal{M}$ and $\mathfrak{U}h : \mathfrak{U}\mathcal{L} \to \mathfrak{U}\mathcal{M}$ are modal de Vries morphisms. This means we must show that $a \prec b$ in $\mathfrak{B}L$ implies (i) $\neg\neg h(\Diamond^L a) \prec \Diamond^L \neg\neg h(b)$, (ii) $\Diamond^L \neg\neg h(a) \prec \neg\neg h(\Diamond^L b)$, (iii) $\neg\neg h(\Diamond^U a) \prec \Diamond^U \neg\neg h(b)$, and (iv) $\Diamond^U \neg\neg h(a) \prec \neg\neg h(\Diamond^U b)$.

Before proving these items, we collect some facts. As h is a frame homomorphism, it preserves proximity and order, and satisfies $h(\neg x) \leq \neg h(x)$; and as h is an MKR-morphism, $x \prec y$ implies $h(\Diamond x) \prec \Diamond h(y)$, $\Diamond h(x) \prec h(\Diamond y)$, $h(\Box x) \prec \Box h(y)$, and $\Box h(x) \prec h(\Box y)$. Lemma 3.7 shows $\Diamond, \Box$ preserve proximity. Finally, in any frame, $x \prec y$ iff $\neg\neg x \prec y$.

As $a \prec b$, we have $\Diamond^L a = \neg\neg\Diamond a \prec \Diamond b$. So $h(\Diamond^L a) \prec h(\Diamond b) = \Diamond h(b) \leq \Diamond^L \neg\neg h(b)$. From this, (i) follows. Also $a \prec b$ implies $\neg\neg h(a) \prec h(b)$, hence $\Diamond\neg\neg h(a) \prec \Diamond h(b) = h(\Diamond b) \leq \neg\neg h(\Diamond^L b)$, and from this (ii) follows. As $a \prec b$, we have $\neg a \vee b = 1$. Thus $\Box(\neg a \vee b) = 1$, and the definition of an MKR-frame gives $\Box\neg a \vee \Diamond b = 1$. Then, by Lemma 3.7.1, $1 = h(\Box\neg a \vee \Diamond b) \leq h(\neg\neg\Box\neg a) \vee \Diamond h(b) \leq \neg h(\neg\Box\neg a) \vee \neg\Box\neg h(b) = \neg h(\Diamond^U a) \vee \Diamond^U h(b)$, giving $h(\Diamond^U a) \prec \Diamond^U h(b)$, and (iii) follows. Finally, $a \prec b$ gives $h(a) \prec h(c)$, and as in (iii), $\Box\neg h(a) \vee \Diamond h(c) = 1$. So $\neg\neg\Box\neg\neg\neg h(a) \vee h(\Diamond b) = 1$, giving $\Diamond^U(\neg\neg h(a)) \prec h(\Diamond b) \leq \neg\neg h(\Diamond^U b)$. □

Theorem 3.11. *There is an equivalence between* MKRFrm *and* LMDV *given by $\mathfrak{L}$ and the restriction of $\mathfrak{R}$ to* LMDV*; and an equivalence between* MKRFrm *and* UMDV *given by $\mathfrak{U}$ and the restriction of $\mathfrak{R}$ to* UMDV.

Proof. Suppose $\mathcal{L} = (L, \Box, \Diamond)$ is an MKR-frame, $\mathfrak{A} = (A, \prec, \Diamond)$ is a lower continuous MDV-algebra, and $\mathfrak{C} = (C, \prec, \Diamond)$ is an upper continuous MDV-algebra. Define $h : \mathfrak{R}\mathfrak{L}\mathcal{L} \to \mathcal{L}$ and $k : \mathfrak{R}\mathfrak{U}\mathcal{L} \to \mathcal{L}$ by $h(I) = k(I) = \bigvee I$. Also, define $\alpha : \mathfrak{A} \to \mathfrak{L}\mathfrak{R}\mathfrak{A}$ and $\beta : \mathfrak{C} \to \mathfrak{U}\mathfrak{R}\mathfrak{C}$ by $\alpha(a) = \twoheaddownarrow a$ and $\beta(c) = \twoheaddownarrow c$. It is known [4, Section 3] that on the level of compact regular frames and de Vries algebras h, k and α, β are natural isomorphisms. It remains only to show h, k are MKR-isomorphisms and α, β are MDV-isomorphisms.

To show h is an MKR-isomorphism, we must show $h(\overline{\Diamond^L}(I)) = \Diamond h(I)$ and $h(\overline{\Box^L}(I)) = \Box h(I)$ for I a round ideal of the regular elements of L. In Proposition 3.5 we noted $\overline{\Diamond^L}(I) = \downarrow\Diamond^L[I]$ and $\overline{\Box^L}(I) = \downarrow\Box^L[I]$. Then $h(\overline{\Diamond^L}(I)) = \bigvee\Diamond^L[I]$ and $h(\overline{\Box^L}(I)) = \bigvee\Box^L[I]$. Also, as $\Diamond$ and $\Box$ preserve directed joins, $\Diamond h(I) = \bigvee\Diamond[I]$ and $\Box h(I) = \bigvee\Box[I]$. So to show h is an isomorphism, we must show $\bigvee\Diamond^L[I] = \bigvee\Diamond[I]$ and $\bigvee\Box^L[I] = \bigvee\Box[I]$. Similarly, to show k is an isomorphism, we must show $\bigvee\Diamond^U[I] = \bigvee\Diamond[I]$ and $\bigvee\Box^U[I] = \bigvee\Box[I]$. But for $a \in L$ regular, Definition 3.8 gives $\Diamond^L a = \neg\neg\Diamond a$, $\Box^L a = \neg\Diamond\neg a$, $\Diamond^U a = \neg\Box\neg a$, and $\Box^U a = \neg\neg\Box a$. So if a, b are regular with $a \prec b$, Lemma 3.7 gives $\Diamond a \leq \Diamond^L a \leq \Diamond^U a \prec \Diamond b$ and $\Box a \leq \Box^U a \leq \Box^L a \prec \Box b$. The required equalities of the above joins follow easily from these inequalities and the roundness of I.

To show α is an MDV-isomorphism, we must show $\alpha(\Diamond a) = (\overline{\Diamond})^L\alpha(a)$. Recall $(\overline{\Diamond})^L I = \neg\neg\Diamond[I]$ where pseudocomplement $\neg$ in the frame of round ideals is given by $\neg I = \twoheaddownarrow\neg\bigvee I$, hence $\neg\neg I = \twoheaddownarrow\bigvee I$ [4, Lem. 3.5]. We then have $(\overline{\Diamond})^L\alpha(a) = \neg\neg\Diamond[\twoheaddownarrow a] = \twoheaddownarrow\bigvee\Diamond[\twoheaddownarrow a]$. As $\mathfrak{A}$ is lower continuous, $\bigvee\Diamond[\twoheaddownarrow a] = \Diamond a$, and the result follows. To show β is an isomorphism we must show $\beta(\Diamond c) = (\overline{\Diamond})^U\beta(c)$. Recall $(\overline{\Diamond})^U I = \neg\overline{\Box}\neg I$ where $\overline{\Box} I = \twoheaddownarrow\Box[I]$ and $\Box = \neg\Diamond\neg$. So $(\overline{\Diamond})^U\beta(c) = \neg\overline{\Box}\neg\twoheaddownarrow c = \neg\overline{\Box}\twoheaddownarrow\neg c = \twoheaddownarrow\neg\bigvee\Box[\twoheaddownarrow\neg c] = \twoheaddownarrow\neg\bigvee\{\Box b : b \prec \neg c\}$. Using the infinite DeMorgan law in a Boolean algebra, the fact that in an MDV-algebra $\neg\Box b = \Diamond\neg b$ and $b \prec \neg c$ iff $c \prec \neg b$, we have $(\overline{\Diamond})^U\beta(c) = \twoheaddownarrow\bigwedge\{\Diamond a : c \prec a\}$. Then as $\mathfrak{C}$ is upper continuous, this is $\twoheaddownarrow\Diamond c$, giving the result. □

Corollary 3.12. *Without choice, the categories* MKRFrm*,* LMDV*,* UMDV*, and* MDV *are equivalent; and with choice, they are all dually equivalent to* MKHaus*.*

Proof. We have just proved without choice that MKRFrm, LMDV, and UMDV are equivalent, and in [5, Sec. 4.3] we proved without choice that LMDV and UMDV are equivalent to MDV. In [5, Sec. 3], using choice, we proved MKRFrm and MKHaus are equivalent. □

4. An alternative approach to Isbell duality

In this section we describe an alternate approach to Isbell duality via round filters and ideals that in many ways closely resembles the familiar Stone duality. We first recall a few basics.

Definition 4.1. *A point of a frame L is a frame homomorphism $p : L \to 2$ into the 2-element frame. A filter F of L is called prime if $a \vee b \in F$ implies $a \in F$ or $b \in F$ for each $a, b \in L$, and F is completely prime if $\bigvee S \in F$ implies $S \cap F \neq \varnothing$ for each subset $S \subseteq L$. A meet prime element of L is an element $m \in L$ where $a \wedge b \leq m$ implies $a \leq m$ or $b \leq m$ for each $a, b \in L$.*

There are bijective correspondences between points of L, completely prime filters of L, and meet prime elements of L (see, e.g., [11, Sec. II.1]). For a point p, the set $p^{-1}(1)$ is a completely prime filter and $\bigvee p^{-1}(0)$ is a meet prime element of L. The set of points of L is topologized by $\{\varphi(a) : a \in L\}$ where $\varphi(a) = \{p : p(a) = 1\}$. This topological space is denoted $\mathfrak{p}L$. Homeomorphic spaces can be constructed from the completely prime filters or meet prime elements of L in an obvious way. The functor $\mathfrak{p} : \mathsf{KRFrm} \to \mathsf{KHaus}$ takes a frame L to its space of points, and a frame homomorphism $h : L \to M$ to the continuous map $\mathfrak{p}h : \mathfrak{p}M \to \mathfrak{p}L$ where $\mathfrak{p}h(q) = q \circ h$. The functor $\mathfrak{p}$ with the open set functor $\Omega : \mathsf{KHaus} \to \mathsf{KRFrm}$ provide Isbell duality.

Definition 4.2. *For L a compact regular frame and $S \subseteq L$, let $\Downarrow S = \{a : a \prec s \text{ for some } s \in S\}$ and $\Uparrow S = \{a : s \prec a \text{ for some } s \in S\}$. Here $\prec$ is well-inside relation. We say an ideal I is round if $I = \Downarrow I$ and a filter F is round if $F = \Uparrow F$.*

Lemma 4.3. *For L a compact regular frame and F a filter of L, these are equivalent.*

1. *F is a completely prime filter.*
2. *F is a prime filter that is round.*
3. *$F = \Uparrow G$ for some prime filter G.*
4. *F is a meet-prime element of the lattice of round filters of L ordered by set inclusion $\subseteq$.*

Proof. (1) ⇒ (2) A completely prime filter is prime. To see F is round, suppose $a \in F$. Then by the definition of a compact regular frame, $a = \bigvee\{b : b \prec a\}$, and as F is completely prime, there is some $b \in F$ with $b \prec a$. (2) ⇒ (3) $F = \Uparrow F$. (3) ⇒ (4) Note that meet in lattice of round filters is given by intersection. Suppose P and Q are round filters with neither contained in F. Then there are $p \in P - F$ and $q \in Q - F$. As P and Q are round, there are $p' \in P$ and $q' \in Q$ with $p' \prec p$ and $q' \prec q$. Then neither p' nor q' is in G. As G is prime, $p' \vee q'$ is not in G, and as $p' \vee q' \prec p \vee q$, we have $p \vee q$ is not in F. But $p \vee q \in P \cap Q$, so $P \cap Q \nsubseteq F$. (4) ⇒ (1) Let $\bigvee S \in F$. As F is round, there exists $a \in F$ such that $a \prec \bigvee S$. Therefore, $\neg a \vee \bigvee S = 1$. By compactness there are $s_1, \dots s_n \in S$ with $\neg a \vee s_1 \vee \cdots \vee s_n = 1$. As each s_i is the join of the elements well inside it, compactness yields $\neg a \vee t_1 \vee \cdots \vee t_n = 1$ for some $t_i \prec s_i$. Therefore, $a \prec t_1 \vee \cdots \vee t_n$, and so $t_1 \vee \cdots \vee t_n \in F$. Thus, $\Uparrow t_1 \cap \cdots \cap \Uparrow t_n = \Uparrow(t_1 \vee \cdots \vee t_n) \subseteq F$. As F is meet prime in the lattice of round filters, $\Uparrow t_i \subseteq F$ for some t_i, giving some $s_i \in F$. So F is completely prime. □

In Isbell duality, the space of points of a compact regular frame L is homeomorphic to the space whose points are the completely prime filters of L topologized by the sets $\varphi(a) = \{F : a \in F\}$. This is nothing more than the above mentioned correspondence between points and completely prime filters. Using the term *prime round filter* to mean either a prime filter that is round, or a meet-prime element of the lattice of round filters, the above lemma shows the space of completely prime filters is literally equal to the space of prime round filters topologized by the sets $\varphi(a) = \{F : a \in F\}$. This closely parallels the construction of a dual space in Stone duality.

Corollary 4.4. *For a compact regular frame L, its space of points $\mathfrak{p}L$ is homeomorphic to the space of its prime round filters topologized by the sets $\varphi(a) = \{F : a \in F\}$.*

Stone duality is often realized via prime ideals rather than prime filters. In the setting of compact regular frames, there is a similar path using round ideals, but some care is required. For L the ideal lattice of the Boolean algebra of finite and cofinite subsets of the natural numbers, the ideal of finite subsets is a meet-prime element of the lattice of round ideals of L, but is not a prime ideal. We define the term *prime round ideal* to mean a meet-prime element of the lattice of round ideals, and note that this in general is different from being a prime ideal that is round. An argument similar to that in Lemma 4.3 shows any prime ideal that is round is a prime round ideal, but not conversely.

Lemma 4.5. *For L a compact regular frame and I an ideal of L, these are equivalent.*

(1) *I is a prime round ideal.*
(2) *$I = \twoheaddownarrow m$ for some meet-prime element m.*
(3) *$I = \twoheaddownarrow J$ for some prime ideal J.*

Proof. (1) $\Rightarrow$ (2) Set $m = \bigvee I$. Then $I = \twoheaddownarrow I \subseteq \twoheaddownarrow m$. If $b \prec m$, then $\neg b \vee \bigvee I = 1$, and by compactness $\neg b \vee a = 1$ for some $a \in I$, giving $b \prec a$, hence $b \in I$. Thus $I = \twoheaddownarrow m$. To see m is meet-prime, suppose $a \wedge b \leq m$. Then $\twoheaddownarrow a \cap \twoheaddownarrow b = \twoheaddownarrow (a \wedge b) \subseteq I$. As I is meet-prime in the lattice of round ideals, $\twoheaddownarrow a \subseteq I$ or $\twoheaddownarrow b \subseteq I$. So $a = \bigvee \twoheaddownarrow a \leq \bigvee I$ or $b = \bigvee \twoheaddownarrow b \leq \bigvee I$, giving $a \leq m$ or $b \leq m$. (2) $\Rightarrow$ (3) Let $J = \downarrow m$, the principal ideal generated by m. (3) $\Rightarrow$ (1) This is dual to the proof of (3) $\Rightarrow$ (4) in Lemma 4.3. □

The above lemma shows there is a map from the meet-prime elements of L to the prime round ideals of L given by $m \rightsquigarrow \twoheaddownarrow m$, and further, that this map is onto. For any element m of a compact regular frame, we have $m = \bigvee \twoheaddownarrow m$, so this map is also one-one. Thus there is a bijection between the meet-prime elements of L and the prime round ideals of L. As the meet-prime elements of L are in bijective correspondence to its points, we have the following analog of a familiar result from Stone duality.

Corollary 4.6. *The space of points of L is homeomorphic to the space of prime round ideals of L topologized by the sets $\varphi(a) = \{I : a \notin I\}$.*

If I is a prime round ideal, then there is a meet-prime element m with $I = \twoheaddownarrow m$. Then $F = L - \uparrow m$ is a completely prime filter, hence a prime round filter, and is the largest round filter that is disjoint from I. Conversely, if F is a prime round filter, then $L - F = \downarrow m$ for some meet-prime element m. Then $I = \twoheaddownarrow m$ is a prime round ideal, and is the largest round ideal disjoint from F. So, we have the following analog of the familiar result that the complement of a prime ideal in a distributive lattice is a prime filter, and conversely.

Corollary 4.7. *Let L be a compact regular frame, I be a prime round ideal, and F be a prime round filter.*

(1) *There is a largest round filter disjoint from I and this is a prime round filter.*
(2) *There is a largest round ideal disjoint from F and this is a prime round ideal.*

This gives the following analog of the *Prime Ideal Theorem.*

Corollary 4.8. *Let L be a compact regular frame, with I a round ideal, F a round filter, and $I \cap F = \varnothing$. Then there is a prime round ideal P containing I and a prime round filter Q containing F with $P \cap Q = \varnothing$.*

There are a number of directions to take these observations. We collect these in the remarks below.

Remark 4.9. The point functor $\mathfrak{p}$ used in Isbell duality may be replaced by a functor associating to a compact regular frame L its space of prime round filters, or its space of prime round ideals. In the first case, a frame homomorphism $h : L \to M$ is taken to the continuous map h^{-1} between the associated spaces of prime round filters, in the second case, h is taken to the continuous map $\twoheaddownarrow h^{-1}$ between the spaces of prime round ideals.

Remark 4.10. In a compact regular frame, $a \prec b$ is easily seen to imply $\neg\neg a \prec b$. From this, it is easily seen that each round ideal and round filter is determined by the regular elements it contains. Recall that the de Vries algebra associated to a compact regular frame L is the Boolean algebra $\mathfrak{B}L$ of its regular elements with proximity being the restriction of the well-inside relation. It is not difficult to see that the ends of $\mathfrak{B}L$, which are its prime round filters, are exactly the intersection of the prime round filters of L with these regular elements. This provides a nice way to show the point functor $\mathfrak{p}$ is naturally isomorphic to the composite $(-)_* \circ \mathfrak{B}$ of the Booleanization functor and the space of ends functor (see [5] for further details).

Remark 4.11. In [5] the duality between MKRFrm and MKHaus was established by lifting the usual functors Ω and $\mathfrak{p}$ used in Isbell duality to the modal setting. The above method of viewing the point functor $\mathfrak{p}$ in terms of prime round filters provides an alternate route that is closer to Stone duality, and in the modal setting, closer to the familiar duality between modal algebras and modal spaces (descriptive frames). For a MKR-frame $\mathcal{L} = (L, \Box, \Diamond)$ a relation R is defined on its space of points by $p\,R\,q$ iff $q(a) = 1$ implies $p(\Diamond a) = 1$ for each $a \in L$ [5, Def. 3.11]. Viewing the space of points of L via its prime round filters, this amounts to defining a relation R on the prime round filters by $P\,R\,Q$ iff $Q \subseteq \Diamond^{-1}P$. This is the approach most commonly taken in defining a relation on the dual space of a modal algebra.

There is more to say about the definition of the relation R on the dual space of an MKR-frame. In modal logic, the $\Box$ and $\Diamond$ operators are definable from each other, and the relation R on the dual space of a modal algebra may be defined either by setting $P\,R\,Q$ iff $Q \subseteq \Diamond^{-1}P$ or by setting $P\,R\,Q$ iff $\Box^{-1}P \subseteq Q$. For an MKR-frame, the operators $\Box$ and $\Diamond$ are also definable from each other [5, Rem. 3.7]. The following lemma shows that the relation R on its dual space of prime round filters of an MKR-frame may also be equivalently be defined by either approach.

Lemma 4.12. *Let $\mathcal{L} = (L, \Box, \Diamond)$ be an* M*KR-frame and let P and Q be prime round filters of L. The following are equivalent.*

(1) $Q \subseteq \Diamond^{-1}P$.
(2) $\Box^{-1}P \subseteq Q$.

Proof. (1) $\Rightarrow$ (2) Let $\Box a \in P$. By [5, Rem. 3.7] we have $\Box a = \bigvee\{\neg \Diamond \neg c : c \prec a\}$, and as P is completely prime, there is $c \prec a$ with $\neg \Diamond \neg c \in P$. Then $\Diamond \neg c$ is not in P, and as $Q \subseteq \Diamond^{-1}P$, we have $\neg c$ is not in Q. But $c \prec a$ gives $\neg c \vee a = 1$, so $\neg c$ not being in Q implies $a \in Q$. Thus $\Box a \in P$ implies $a \in Q$, so $\Box^{-1}P \subseteq Q$. (2) $\Rightarrow$ (1) Let $a \in Q$. As $a = \bigvee\{c : c \prec a\}$ we have $c \in Q$ for some $c \prec a$, and by interpolation there is b with $c \prec b \prec a$. Then $\neg c$ is not in Q, and $\Box^{-1}P \subseteq Q$ gives $\Box \neg c$ is not in P. Note $c \prec b$ gives $\Box \neg b \prec \Box \neg c$, hence $\neg \Box \neg b \vee \Box \neg c = 1$. Therefore, as $\Box \neg c$ is not in P, we have $\neg \Box \neg b \in P$. By [5, Rem. 3.7] we have $\Diamond a = \bigvee\{\neg \Box \neg b : b \prec a\}$, so $\Diamond a \in P$. Thus, $a \in Q$ implies $\Diamond a \in P$, so $Q \subseteq \Diamond^{-1}P$. $\square$

Remark 4.13. This shows that for an MKR-frame, the relation on its dual space may be defined either through $\Diamond$ by $P\,R\,Q$ iff $Q \subseteq \Diamond^{-1}P$ or via $\Box$ by setting $P\,R\,Q$ iff $\Box^{-1}P \subseteq Q$. This seems linked to the fact that the operations $\Box$ and $\Diamond$ are definable from one another. This is perhaps a bit unexpected. These MKR-frames are examples of positive modal algebras [7], algebras consisting of bounded distributive lattices with operators $\Box$ and $\Diamond$ satisfying the first three conditions of Definition 2.2. Dual spaces of positive modal logics are constructed through their prime filters, and relations defined by the above conditions are considered, but in general are not equal. Also in this setting of positive modal algebras, the operations $\Box$ and $\Diamond$ are not in general definable from one another.

We conclude this section with a final remark connecting round ideals and filters to the topology of the dual space of a compact regular frame. Recall the basic fact that in Stone duality, open sets of the dual space X of a Boolean algebra B correspond to ideals of B, and closed sets of X correspond to filters of B.

Remark 4.14. For a compact regular frame L, the open sets of its dual space are the $\varphi(a)$ where $a \in L$. For any round ideal I, we have $I = \twoheaddownarrow \bigvee I$, so there is a bijection between round ideals of L and elements of L, so open sets of the dual space correspond to round ideals of L. Similarly, closed sets of the dual space correspond to round filters of L. Here the underlying point is that each closed set in a compact Hausdorff space is the intersection of the open sets that contain it. As round filters of a compact regular frame are exactly Scott open filters and closed subsets of a compact Hausdorff space are exactly compact saturated subsets, this correspondence between round filters and closed sets of the dual space amounts to the Hofmann-Mislove theorem [9] for compact regular frames.

5. Connections to the Vietoris construction

In this section we relate MKR-frames to Johnstone's construction of the Vietoris functor on frames. We begin with a brief summary of Johnstone's results ([11, Sec. III.4], [12]).

Definition 5.1. *For a frame L, let L^* be the set of all formal symbols $L^* = \{\Box_a, \Diamond_a : a \in L\}$ and $F(L^*)$ be the free frame over L^*. Let θ be the frame congruence on $F(L^*)$ generated by the following:*

(1) $\Box_{a\wedge b} = \Box_a \wedge \Box_b$ *and* $\Box_1 = 1$ *where* $a,b \in L$.
(2) $\Diamond_{a\vee b} = \Diamond_a \vee \Diamond_b$ *and* $\Diamond_0 = 0$ *where* $a,b \in L$.
(3) $\Box_{a\vee b} \le \Box_a \vee \Diamond_b$ *and* $\Box_a \wedge \Diamond_b \le \Diamond_{a\wedge b}$ *where* $a,b \in L$.
(4) $\Box_{\bigvee S} = \bigvee\{\Box_s : s \in S\}$ *and* $\Diamond_{\bigvee S} = \bigvee\{\Diamond_s : s \in S\}$ *where* $S \subseteq L$ *is directed.*

Then set $\mathcal{W}(L) = F(L^*)/\theta$ *and call this the Vietoris frame of* L.

This construction on objects extends to give a functor $\mathcal{W}$, called the Vietoris frame functor, from the category of frames to itself. A frame homomorphism $g : L \to M$ lifts to $\mathcal{W}(g) : \mathcal{W}(L) \to \mathcal{W}(M)$ that maps the generator $\Box_a/\theta$ to $\Box_{ga}/\theta$ and the generator $\Diamond_a/\theta$ to $\Diamond_{ga}/\theta$. The following specializes Johnstone's results on this functor to our setting of compact regular frames.

Theorem 5.2 (Johnstone)**.** *The Vietoris frame functor* $\mathcal{W}$ *restricts to a functor on* KRFrm*. Here, if* L *is a compact regular frame isomorphic to the frame of open sets of the compact Hausdorff space* X*, then* $\mathcal{W}(L)$ *is isomorphic to the frame of open sets of the Vietoris space of* X*. Further, for* $\mathcal{V}$ *the Vietoris functor on* KHaus *and* $\Omega, \mathfrak{p}$ *the open set and point functors providing a dual equivalence between* KHaus *and* KRFrm*, we have* $\mathcal{W}$ *is naturally isomorphic to* $\Omega \circ \mathcal{V} \circ \mathfrak{p}$.

We now come to the key result relating MKR-frames and the Vietoris frame functor.

Proposition 5.3. *If* L *is a compact regular frame, then each frame homomorphism* $h : \mathcal{W}(L) \to L$ *gives a* M*KR-frame structure* $\mathcal{L}_h = (L, \Box_h, \Diamond_h)$ *on* L *where* $\Box_h a = h(\Box_a/\theta)$ *and* $\Diamond_h a = h(\Diamond_a/\theta)$ *for each* $a \in L$*. This provides a bijective correspondence between frame homomorphisms* $h : \mathcal{W}(L) \to L$ *and* M*KR-frames having underlying frame* L.

Proof. For a frame homomorphism $h : \mathcal{W}(L) \to L$, the operations $\Box_h$ and $\Diamond_h$ on L are obviously well-defined. We must show they satisfy the conditions of Definition 2.2. For $a, b \in L$, we have $\Box_h(a \wedge b) = h(\Box_{a\wedge b}/\theta) = h((\Box_a \wedge \Box_b)/\theta) = h(\Box_a/\theta \wedge \Box_b/\theta) = h(\Box_a/\theta) \wedge h(\Box_b/\theta) = \Box_h a \wedge \Box_h b$. Here, the second equality follows from the definition of θ. Also $\Box_h 1 = h(\Box_1/\theta) = h(1/\theta) = 1$, establishing the first condition of Definition 2.2. The second condition is similar. For the third condition, $\Box_h(a \vee b) = h(\Box_{a\vee b}/\theta) \le h((\Box_a \vee \Diamond_b)/\theta) = h(\Box_a/\theta) \vee h(\Diamond_b/\theta) = \Box_h a \vee \Diamond_h b$, with the other item in the third condition similar. For the final condition, if $S \subseteq L$ is up-directed, then $\Box_h(\bigvee S) = h(\Box_{\bigvee S}/\theta) = h(\bigvee\{\Box_s : s \in S\}/\theta) = \bigvee\{h(\Box_s/\theta) : s \in S\} = \bigvee\{\Box_h s : s \in S\}$. Here we have used that frame congruences and frame homomorphisms preserve arbitrary joins. The other item in the fourth condition is obviously similar.

The above paragraph shows each frame homomorphism $h : \mathcal{W}(L) \to L$ induces an MRK-structure on L as indicated. If $h, h' : \mathcal{W}(L) \to L$ are frame homomorphisms that induce the same structure, then $h(\Box_a/\theta) = h'(\Box_a/\theta)$ and $h(\Diamond_a/\theta) = h'(\Diamond_a/\theta)$ for each $a \in L$. So h and h' agree on a generating set of $\mathcal{W}(L)$, hence are equal.

It remains to show each MKR-frame structure on L is induced by a frame homomorphism $h : \mathcal{W}(L) \to L$. Suppose $\mathcal{L} = (L, \Box, \Diamond)$ is an MKR-frame. Define $g : L^* \to L$ by $g(\Box_a) = \Box a$ and $g(\Diamond_a) = \Diamond a$ for each $a \in L$. As $F(L^*)$ is the free frame over the set L^*, the map g extends to a frame homomorphism $\overline{g} : F(L^*) \to L$. We claim the kernel of $\overline{g}$ contains θ. Indeed, if $a, b \in L$, then as the MKR-frame $\mathcal{L}$ satisfies $\Box(a \wedge b) = \Box a \wedge \Box b$, we have $\overline{g}(\Box_{a\wedge b}) = \overline{g}(\Box_a) \wedge \overline{g}(\Box_b) = \overline{g}(\Box_a \wedge \Box_b)$, showing the pair $\Box_{a\wedge b}$ and $\Box_a \wedge \Box_b$ belongs to the kernel of $\overline{g}$. Similar arguments show all pairs in the generating set of θ belong to the kernel of $\overline{g}$, showing θ is contained in the kernel of $\overline{g}$. Thus, there is a frame homomorphism $h : F(L^*)/\theta \to L$ with $h \circ \kappa = g$ where κ is the canonical homomorphism $\kappa : F(L^*) \to F(L^*)/\theta$. Then $h(\Box_a/\theta) = \overline{g}(\Box_a) = \Box a$ and $h(\Diamond_a/\theta) = \overline{g}(\Diamond_a) = \Diamond a$ for all $a \in L$, showing h induces the structure $\mathcal{L}$ on L. □

Algebras for the Vietoris frame functor on KRFrm are morphisms $h : \mathcal{W}(L) \to L$. So the above result shows these algebras are concretely realized by MKR-frames. The algebras for $\mathcal{W}$ form a category where a morphism between algebras $h : \mathcal{W}(L) \to L$ and $h' : \mathcal{W}(M) \to M$ is a frame homomorphism $g : L \to M$ where the square formed from h, h', g and $\mathcal{W}(g)$ commutes. Then $g(\Box a) = gh(\Box_a/\theta) = h'\mathcal{W}(g)(\Box_a/\theta) = h'(\Box_{ga}/\theta) = \Box ga$, with a similar calculation showing $g(\Diamond a) = \Diamond ga$. This provides the following.

Theorem 5.4. *The category of algebras for the Vietoris frame functor* $\mathcal{W}$ *on* KRFrm *is isomorphic to the category* MKRFrm *of modal compact regular frames.*

From a general categorical argument, it follows that the category of algebras for the Vietoris frame functor $\mathcal{W}$ on KRFrm is dually isomorphic to the category of coalgebras for the Vietoris functor $\mathcal{V}$ on KHaus. These coalgebras are morphisms $f : X \to \mathcal{V}(X)$ from a compact Hausdorff space X into its Vietoris space. In [5] we showed that the category MKHaus of modal compact Hausdorff spaces was isomorphic to the category of coalgebras for $\mathcal{V}$. This provides an alternate proof to the following result established directly in [5].

Theorem 5.5. *The categories* MKRFrm *and* MKHaus *are dually isomorphic.*

It would be desirable to have an analog of the Vietoris functor for de Vries algebras, and to realize modal de Vries algebras as algebras for this functor, as we have done with MKR-frames. Of course, one can use the equivalence of DeV and KRFrm to transfer $\mathcal{W}$ to this setting, but a more direct construction of a Vietoris de Vries algebra functor would be desirable.

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