Projective Bichains

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Basics

Definition A bisemilattice is an algebra $(B, \cdot, +)$ where $\cdot$ and $+$ are commutative, associative, and idempotent. A Birkhoff system is a bisemilattice that satisfies the weak absorption law

$$x \cdot (x + y) \; = \; x + (x \cdot y).$$

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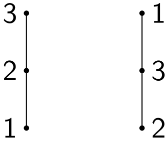
Basics

Studied since the 70's by Balbes, Dudek, McKenzie, Plonka, Padmannabhan, Romanowska, Tepavcevic, etc.

Not a congruence distributive variety.

Basics

Definition A bichain is a bisemilattice where the the two partial orderings given by $\cdot$ and $+$ are chains.

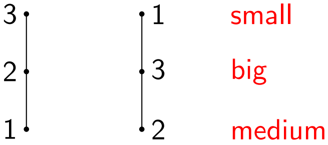


N

Proposition Every bichain is a Birkhoff system.
Proposition Up to iso, there are $n!$ n-element bichains.

Basics

This particular bichain will play a key role. We remember it as medium, big, small.



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Main Theorem

Theorem For a finite bichain C these are equivalent.

1. C is weakly projective in $\textsc{Birk}$.
2. C does not contain a subalgebra isomorphic to N.

Note Weakly projective means wrt onto homomorphisms.

Corollary A recursive description of all finite weakly projective bichains, all finite s.i. weakly projective (hence splitting) bichains.

Example

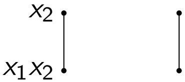
Show The two-element bichain C is weakly projective in $\mathrm{B}\textsc{irk}$.

C

Say x_1, x_2 are generators of a free Birkhoff system F and they are mapped to $1, 2$ respectively. We must build a copy of C in F that is mapped isomorphically onto C.

Example

Fix the $\cdot$ operation (the left).



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x_2 \bullet \arrow[d, dash] & \bullet\, x_1x_2 \arrow[d, dash] \\
x_1x_2 \bullet & \bullet\, x_2
\end{tikzcd}$$

Now fix the $+$ operation (the right).

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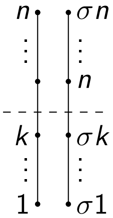
Now fix the $+$ operation (the right). This affects the left.

We don't have to fix the left again because we can prove it is okay. Indeed, $x_2(x_2 + x_1x_2) = x_2 + x_2x_1x_2 = x_2 + x_1x_2$.

Sketch of Main Proof

Say C is an n-element bichain on the elements $1, \ldots, n$.

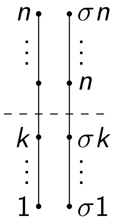


No N as a subalgebra, means no medium, big, small on right.

So below dashed line permutes $1, \ldots, k$, above $k+1, \ldots, n$.

Sketch of main proof

When we fix left, then the right, then left, etc. we can show ...



After a certain number of steps the terms on the top of the dashed line do not occur when fixing the bottom, and the terms on the bottom of the dashed line do not occur when fixing the top.

Induction!

Notes on the main proof

All this involves a horrible amount of axiomatics.

We used PROVER 9 on medium sized examples (8 elements) to guide us to conjectures of what general identities might be true, then proved our conjectures by hand.

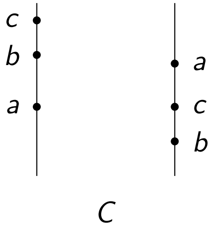
PROVER 9 could not have done this without us, and we could not have done this without it. An interesting use of machine.

It would be nice if Prover9 made more readable proofs.

The main proof — other direction

Theorem If a bichain C contains N it is not weakly projective.

Proof. Choose the right copy of N in C (tricky).



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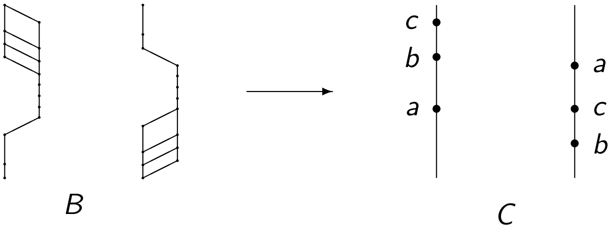
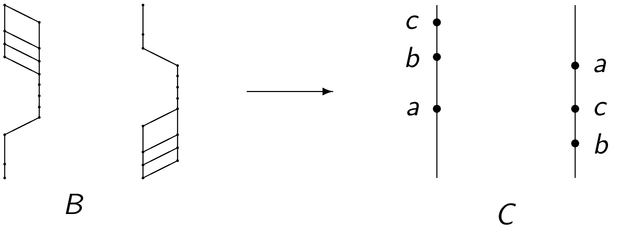


The main proof — other direction

Theorem If a bichain C contains N it is not weakly projective.

Proof. Choose the right copy of N in C (tricky). Use this to create a Birkhoff system B from $C - \{a\}$ (also tricky). Then C is a quotient of B but not a subalgebra.



Remarks

Recursively construct all n-element projective bichains by stacking an m-element projective on a k-element one when $m + k = n$.

Theorem The number of n-element projective bichains is the n^{th} Catalan number.

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A slightly more complex recursion gives all s.i. projective bichains. These are of interest as they are splitting algebras.

Theorem The number of n-element s.i. projective bichains is the n^{th} Fibonacci number.

The origins of this work

The basic object of type-II fuzzy sets is the following algebra M defined using certain convolutions of functions.

Definition On the set M of all functions from $[0,1]$ to itself define

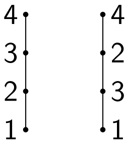
$$\begin{aligned}(f \cdot g)(x) &= \sup\{\min\{f(y), g(z)\} : \min\{y,z\} = x\} \\ (f + g)(x) &= \sup\{\min\{f(y), g(z)\} : \max\{y,z\} = x\}\end{aligned}$$

Ambition Find a decision procedure and axiomatic basis for the equational theory of M.

The origins of this work

Theorem Each of the following generate the same variety as M.

1. The complex algebra of the the real unit interval.
2. The complex algebra of the 3-element chain.
3. The 4-element bichain below



Corollary The equational theory of M is decidable.

Towards an equational basis

We still don't know an equational basis, but have a guess. Here we let $\textsc{Bichains}$ be the variety generated by all bichains.

Conjecture M is defined by the equations defining $\textsc{Bichains}$ (whatever those are) and the generalized distributive law

$$(x(y+z))(xy+xy) = (x(y+z)) + (xy+xz)$$

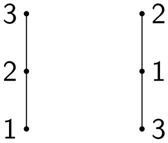
Note This is $p \cdot q = p + q$ where $p = q$ is the distributive law.

Towards an equational basis

The following lends credibility to the conjecture.

Theorem For a bichain C these are equivalent.

1. C belongs to $V(M)$.
2. C satisfies the generalized distributive law.
3. C does not contain the splitting bichain below.



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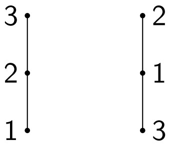

Proving this bichain projective started our work.

Thank you for listening.

Papers at www.math.nmsu.edu/$\sim$jharding

BLAST at New Mexico State next year!