Categories with Fuzzy Sets and Relations

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Abstract

We define a 2-category whose objects are fuzzy sets and whose maps are relations subject to certain natural conditions. We enrich this category with additional monoidal and involutive structure coming from t-norms and negations on the unit interval. We develop the basic properties of this category and consider its relation to other familiar categories. A discussion is made of extending these results to the setting of type-2 fuzzy sets.

1 Introduction

A fuzzy set is a map $A : X \to \mathbb{I}$ from a set X to the unit interval $\mathbb{I}$. Several authors [2, 6, 7, 20, 22] have considered fuzzy sets as a category, which we will call FSet, where a morphism from $A : X \to \mathbb{I}$ to $B : Y \to \mathbb{I}$ is a function $f : X \to Y$ that satisfies $A(x) \leq (B \circ f)(x)$ for each $x \in X$. Here we continue this path of investigation.

The underlying idea is to lift additional structure from the unit interval, such as t-norms, t-conorms, and negations, to provide additional structure on the category. Our eventual aim is to provide a setting where processes used in fuzzy control can be abstractly studied, much in the spirit of recent categorical approaches to processes used in quantum computation [1].

Order preserving structure on $\mathbb{I}$, such as t-norms and conorms, lifts to provide additional covariant structure on FSet. In fact, each t-norm T lifts to provide a symmetric monoidal tensor $\otimes_T$ on FSet. However, it is problematic to lift order inverting structure on $\mathbb{I}$, such as a negation $\neg$, to FSet. For this reason, and its inherent interest, we widen the category

FSet of fuzzy sets and functions to the category FRel of fuzzy sets and relations. Here the objects are fuzzy sets as before, but a morphism between fuzzy sets $A : X \to \mathbb{I}$ and $B : Y \to \mathbb{I}$ is a relation R from X to Y that satisfies $xRy \Rightarrow A(x) \leq B(y)$. Order inverting structure on $\mathbb{I}$ then lifts to contravariant structure on FRel making use of the converse of a relation.

The categories FSet and FRel parallel the familiar categories Set, of sets and the functions between them, and Rel, of sets and the relations between them. Just as Rel differs from Set in an essential way, behaving much more like a category of vector spaces than the category of sets, so also does FRel differ from FSet.

In particular, FRel has finite biproducts, hence a semiadditive structure on its homsets. Lifting a negation and t-norm from the unit interval $\mathbb{I}$ equips FRel with an involution $\ddagger$ and a symmetric monoidal structure $\otimes$. Additionally, Rel naturally carries a 2-category structure, where morphisms between relations are given by set inclusion. This is the case with FRel as well.

Our purpose here is to outline the basic properties of the categories FSet, FRel, and their relationships to each other and to the categories Set and Rel. We develop such basic properties as biproducts, monomorphisms, epimorphisms, injectives, projectives, as well as properties related to lifting structure from $\mathbb{I}$ to these categories.

We consider the matter of extending the categorical setting to interval-valued and type-2 fuzzy sets. The idea is to replace $\mathbb{I}$ with a the appropriate truth value algebra. Following Zadeh [23], the truth value algebra for type-2 fuzzy sets is the algebra $\mathbf{M}$ of all functions from $\mathbb{I}$ to itself. This carries not one, but two natural orders. Here we propose the intersection of these orders as the basic one. A later paper considers this order in more detail [12].

Finally, we briefly compare the category FRel to some other categorical generalizations of Rel.

2 The Category FRel of Fuzzy Sets and Relations

In this section, we define the categories of interest in this note, and give some notation. We begin with the category of primary interest, FRel.

Definition 2.1 *The category* FRel *of fuzzy sets and relations is defined as follows:*

1. *An* ***object*** *is a map A from a set X to the unit interval $\mathbb{I}$. (We will write either $A : X \to \mathbb{I}$ or simply (X, A) for an object).*
2. *A* ***morphism*** *from (X, A) to (Y, B) is a relation $R \subseteq X \times Y$ satisfying $A(x) \le B(y)$ for all $(x, y) \in R$ (sometimes written as xRy).*
3. ***Composition*** *of morphisms is the usual composition of relations: if $R \subseteq X \times Y$ and $S \subseteq Y \times Z$, then*

   $S \circ R = \{(x, z) :$ *there exists* $y \in Y$ *such that* $(x, y) \in R$, $(y, z) \in S\}$.

The condition in (2) will be indicated by diagrams of the form

$$\begin{array}{ccc} X & \xrightarrow{A} & \mathbb{I} \\ R \downarrow & \le & \| \\ Y & \xrightarrow[B]{} & \mathbb{I} \end{array}$$

Lemma 2.2 *If $R : (X, A) \to (Y, B)$ and $S : (Y, B) \to (Z, C)$ are morphisms in* FRel*, then so is $S \circ R$.*

Proof. Note $x (S \circ R) z$ if and only if there exists $y \in Y$ with xRy and ySz. In this case, $A(x) \le B(y)$ and $B(y) \le C(z)$ so that $A(x) \le C(z)$. Thus $S \circ R$ is a morphism in FRel. ■

Since composition of relations is associative, the same is true for morphisms in FRel. The diagonal morphism 1_X, where $1_X = \{(x, x) : x \in X\}$ is the identity relation on (X, A). It satisfies $R \circ 1_X = R$ and $1_X \circ S = S$ for R and S for which the compositions are defined. This establishes that FRel is a category.

Definition 2.3 *The category* FSet *of fuzzy sets and functions is the subcategory of* FRel *whose objects are the same as in* FRel *but whose morphisms are those morphisms in* FRel *that are actually functions.*

Note that when R is a function, condition (2) of Definition 2.1 is equivalent to $A(x) \le B(R(x))$. That is, $A \le B \circ R$, in the case R is a function.

Notation 2.4 *We often use (X, A) for an object in either* FSet *or* FRel*, and $R : (X, A) \to (Y, B)$ for a morphism in either category.*

3 Categorical Properties of FRel

In this section, we point to some basic properties of the category FRel. Largely, these parallel results for the category Rel. While these results for Rel are well known in many circles, we had difficulty finding them in print.

Definition 3.1 [13, p. 47] *Let Z be an object in a category $\mathcal{C}$. We call Z* ***initial*** *if for each object A there is exactly one morphism from Z to A; we call Z* ***terminal*** *if for each object A there is exactly one morphism from A to Z; and we call Z a* ***zero object*** *if it is both initial and terminal.*

For objects A, B in a category with zero object Z, we use $0_{A,B}$ for the unique morphism $A \to Z \to B$.

Proposition 3.2 *The empty set (with the empty function into $\mathbb{I}$) is a zero object in* FRel*.*

Proof. Given any object (X, A) in FRel there are unique morphisms

$$
\begin{array}{ccc}
X & \xrightarrow{\;A\;} & \mathbb{I} \\
{\scriptstyle \emptyset}\downarrow & & \| \\
\emptyset & \xrightarrow{\;\emptyset\;} & \mathbb{I}
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccc}
\emptyset & \xrightarrow{\;\emptyset\;} & \mathbb{I} \\
{\scriptstyle \emptyset}\downarrow & & \| \\
X & \xrightarrow{\;A\;} & \mathbb{I}
\end{array}
$$

The inequalities are satisfied by default. Thus the sets $\mathrm{FRel}\left(\left(X, A\right), \left(\varnothing, \varnothing\right)\right)$ and $\mathrm{FRel}\left(\left(\varnothing, \varnothing\right), \left(X, A\right)\right)$ each contain exactly one morphism. ■

Definition 3.3 [13, p. 306] *A category C with zero has* ***biproducts*** *if for each family $(A_i)_I$ of objects there is an object $\bigoplus_I A_i$, together with families of morphisms $\mu_i : A_i \to \bigoplus_I A_j$ and $\pi_i : \bigoplus_I A_j \to A_i$, such that*

1. *The morphisms $\mu_i : A_i \to \bigoplus_I A_j$ are a coproduct of the family $(A_i)_I$.*
2. *The morphisms $\pi_i : \bigoplus_I A_j \to A_i$ are a product of the family $(A_i)_I$.*
3. *$\pi_i \circ \mu_j = \delta_{ij}$ for each $i, j \in I$.*

Here δ_{ij} is the identity map 1_{A_i} if $i = j$ and the zero map $0_{A_i,A_j}$ if $i \neq j$.

The category Rel has biproducts. For a family of sets $(X_i)_I$ let X be their disjoint union $\bigsqcup_I X_i = \{(x,i) : x \in X_i \text{ for some } i \in I\}$ and define relations μ_i from X_i to X and π_i from X to X_i by setting $\mu_i = \{(x,(x,i) : x \in X_i\}$ and $\pi_i = \{((x,i),x) : x \in X_i\}$. Then the disjoint union X with morphisms μ_i and π_i is a biproduct of the family $(X_i)_I$ (see for example [1]).

Proposition 3.4 *The category* FRel *has biproducts given by disjoint unions. In more detail, for objects (X_i, A_i) with $(i \in I)$, let $X = \bigsqcup_I X_i$ be the disjoint union of the X_i, define a map $A : X \to \mathbb{I}$ by setting $A(x,i) = A_i(x)$, and define relations μ_i from X_i to X and π_i from X to X_i by setting*

$$\mu_i = \{(x,(x,i)) : x \in X_i\}$$
$$\pi_i = \{((x,i),x) : x \in X_i\}$$

Then (X, A) with the morphisms μ_i, π_i for $i \in I$ is a biproduct of the (X_i, A_i).

Proof. To see the μ_i are morphisms in FRel, take an element $(x,(x,i))$ in μ_i and note that by definition $A_i(x) = A(x,i)$, hence $A_i(x) \leq A(x,i)$, showing μ_i is a morphism from (X_i, A_i) to (X, A). Similarly, for $((x,i),x)$ in π_i we have $A(x,i) = A_i(x)$, showing π_i is a morphism from (X, A) to (X_i, A_i).

Suppose $R_i : (X_i, A_i) \to (Y, B)$ for each $i \in I$. Define a relation R from X to Y by setting $(x,i)\, R\, y$ iff $x\, R_i\, y$. It is easy to see R is a morphism from (X, A) to (Y, B) and is the unique such morphism in FRel with $R \circ \mu_i = R_i$ for each $i \in I$. Thus the μ_i are the morphisms for a coproduct.

Suppose $S_i : (Y, B) \to (X_i, A_i)$ for each $i \in I$. Define a relation S from Y to X by setting $y\, S\, (x,i)$ iff $y\, S_i\, x$. It is easy to see S is a morphism from (Y, B) to (X, A) and is the unique such morphism in FRel with $\pi_i \circ S = S_i$ for each $i \in I$. Thus the π_i are morphisms for a product.

Finally, a calculation gives $\pi_i \circ \mu_j$ is the identical relation on X_i if $i = j$ and is the empty relation from X_j to X_i if $i \neq j$. Thus $\pi_i \circ \mu_j = \delta_{ij}$. ■

Corollary 3.5 *Each object in* FRel *is isomorphic to a biproduct of singleton sets: $(X, A) = \bigoplus_{x \in X} (\{x\}, A_x)$, where $A_x(x) = A(x)$.*

Examples of categories with finite biproducts include abelian groups, and vector spaces over a given field K. In such categories there is an additive structure on homsets and a type of matrix mechanics for working with morphisms. These notions lift to any category with biproducts. We briefly describe some aspects for FRel, see [13, Chapter XI] for a complete account.

Definition 3.6 *A **semiadditive category** is a category $\mathcal{C}$ where each hom-set $\mathcal{C}(B,C)$ is equipped with the structure of a commutative monoid with operation $+$ such that for any $f : A \to B$, $g, h : B \to C$, and $k : C \to D$*

$$(g+h) \circ f = (g \circ f) + (h \circ f)$$
$$k \circ (g+h) = (k \circ g) + (k \circ h).$$

Any category with biproducts carries a unique semiadditive structure [13, p. 310] that can be defined via biproducts. In Rel, this semiadditive structure on Rel (X,Y) is given by letting $R+S$ be the union of the relations R and S from X to Y. The empty relation serves as additive identity. This is known, and easily verified by checking that union does give a semiadditive structure that distributes over composition.

Proposition 3.7 *The semiadditive structure on homsets in* FRel *is given by taking $R+S$ to be the union of the relations $R \cup S$. Here, the empty relation serves as the additive identity.*

Proof. Suppose R and S are morphisms from (X,A) to (Y,B). To see $R \cup S$ is a morphism from (X,A) to (Y,B), suppose $x(R \cup S)y$. Then either $x\,R\,y$ or $x\,S\,y$. In the first case, R being a morphism in FRel gives $A(x) \leq B(y)$; and in the second case S being a morphism in FRel gives $A(x) \leq B(y)$. So $R \cup S$ is a morphism in FRel. Clearly $\cup$ gives a commutative monoid structure on FRel $((X,A),(Y,B))$ with the empty relation as identity, and composition distributes over union. ■

In addition to carrying a semiadditive structure, homsets in Rel also carry a complete lattice structure where the ordering of two relations is ordinary set inclusion. In fact, the homset Rel (X,Y) is the complete Boolean algebra of all subsets of $X \times Y$. In this way Rel is a 2-category where the 0-cells are sets, the 1-cells are relations between sets, and there is a unique 2-cell between relations R, S from X to Y precisely when $R \subseteq S$.

Proposition 3.8 *Homsets in* FRel *carry the structure of complete Boolean algebras when ordered by set inclusion. In fact,* FRel $((X,A),(Y,B))$ *is the complete Boolean algebra of all subsets of $\{(x,y) : A(x) \leq B(y)\}$. In this way,* FRel *is a 2-category.*

Proof. Being a subset of $\{(x,y) : A(x) \leq B(y)\}$ is equivalent to being a relation R from X to Y that satisfies $x\,R\,y$ implies $A(x) \leq B(y)$. ■

An involution on a category $\mathcal{C}$ is a contravariant functor from $\mathcal{C}$ to itself of period two. An involution † that is the identity on objects is called a dagger, and a dagger category is a category with a dagger [18]. Rel is a natural example of a dagger category where $X^\dagger = X$ and for a morphism $R : X \to Y$ we define $R^\dagger : Y \to X$ to be the converse relation $R^\smile$.

Proposition 3.9 *There is an involution ‡ on* FRel *defined as follows. For an object (X, A) and morphism $R : (X, A) \to (Y, B)$ set*

1. $(X, A)^\ddagger = (X, 1 - A)$ *where* $(1 - A)(x) = 1 - A(x)$.
2. $R^\ddagger : (Y, B)^\ddagger \to (X, A)^\ddagger$ *is the converse relation* $R^\smile$.

Note that ‡ is an involution, but is not the identity on objects.

Proof. Suppose $y\, R^\ddagger x$. As $R^\ddagger$ is the converse of R, this means $x\, R\, y$. As R is a morphism we have $A(x) \leq B(y)$, hence $1 - B(y) \leq 1 - A(x)$. So $R^\ddagger$ is a morphism from $(Y, B)^\ddagger$ to $(X, A)^\ddagger$. As $(R \circ S)^\smile = S^\smile \circ R^\smile$ it follows that ‡ is compatible with composition, and clearly ‡ takes the identity map on (X, A), the identical relation on X, to the identity map on $(X, A)^\ddagger$. So ‡ is a contravariant functor that is obviously period two. ■

We note that this involution ‡ gives a bijective mapping from a homset FRel $((X, A), (Y, B))$ to FRel $((Y, 1 - B), (X, 1 - A))$. This isomorphism preserves both the commutative monoid structure and the Boolean algebra structure on these homsets, so is both a commutative monoid isomorphism and a Boolean algebra isomorphism. This involution also provides a duality that is of use in establishing further properties of FRel.

Definition 3.10 *A morphism f is **monic** if $f \circ g = f \circ h$ implies $g = h$; and **epic** if $g \circ f = h \circ f$ implies $g = h$.*

To describe monic morphisms in Rel, note a relation R from X to Y gives a map $R[\,\cdot\,] : 2^X \to 2^Y$ from the power set of X to the power set of Y taking a subset $Z \subseteq X$ to the set $R[Z] = \{y \in Y : z\, R\, y \text{ for some } z \in Z\}$. In [19] it was shown that the monic morphisms in Rel are those relations R where $R[\,\cdot\,]$ is one-one. This lifts to FRel.

Proposition 3.11 *For $R : (X, A) \to (Y, B)$ in* FRel*, these are equivalent.*

1. *R is monic.*
2. *$R[\,\cdot\,] : 2^X \to 2^Y$ is one-one.*
3. *For each $x \in X$ there is $y \in Y$ with x the only element related to y.*

Proof. $(1 \Rightarrow 2)$ Let $U, V \subseteq X$ with $R[U] = R[V]$. Take a singleton $\{*\}$ and let $0 : \{*\} \to [0, 1]$ be the map sending $*$ to 0. Define relations S, T from $\{*\}$ to X by letting $S = \{(*, u) : u \in U\}$ and $T = \{(*, v) : v \in V\}$. Our inequality is satisfied so $S, T : (\{*\}, 0) \to (X, A)$. Since $R[U] = R[V]$, it follows that $R \circ S = R \circ T$, and as R is monic, that $S = T$. Thus $U = V$. $(2 \Rightarrow 3)$ $R[X - \{x\}] \neq R[X]$. $(3 \Rightarrow 1)$ Suppose $S, T : (Z, C) \to (X, A)$ and $S \neq T$. We may assume there is $z\, S\, x$ with $z \not{T} x$. Choose y with $x'\, R\, y \Leftrightarrow x' = x$. Then $z(R \circ S)y$ but not $z(R \circ T)y$. So $R \circ S \neq R \circ T$. So R is monic. ∎

Corollary 3.12 *For $R : (X, A) \to (Y, B)$ in* FRel*, these are equivalent.*

1. *R is epic.*
2. *$R^{\smile}[\,\cdot\,] : 2^Y \to 2^X$ is one-one.*
3. *For each $y \in Y$ there is $x \in X$ with y the only element related to x.*

Proof. As ‡ is an involution, R is epic if and only if $R^{\ddagger}$ is monic. ∎

Definition 3.13 *A morphism $f : A \to B$ is an* ***isomorphism*** *if there is a morphism $g : B \to A$ with $g \circ f = 1_A$ and $f \circ g = 1_B$. Such g, if it exists, is unique, and is called the* ***inverse*** *of f.*

Isomorphisms are always both monic and epic. A category is called balanced if every morphism that is both monic and epic is an isomorphism. The category Rel is balanced since the morphisms that are monic and epic are exactly the bijective correspondences, and for these their converse is their inverse. This does not hold in FRel since the identity relation from $(X, 0)$ to $(X, 1)$, where $0, 1$ are the obvious constant functions, is both monic and epic by the above results, yet has no inverse in the category FRel.

Proposition 3.14 *A morphism $R : (X, A) \to (Y, B)$ in* FRel *is an isomorphism if and only if R is a bijection and $A = B \circ R$. In this case, its inverse is its converse $R^{\smile}$ viewed as a morphism from (Y, B) to (X, A).*

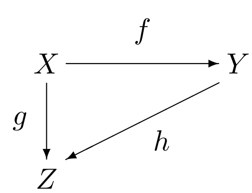
Proof. Suppose R is an isomorphism. By Proposition 3.11 and Corollary 3.12, for each $x \in X$ there is $y \in Y$ with x the only element related to y, and for each $y \in Y$ there is $x' \in X$ with y the only element related to x'. This implies R is a bijection. In Rel, a bijection is an isomorphism with its converse being its inverse, and by the uniqueness of inverses, the converse of R is the only relation S from X to Y with $S \circ R = 1_X$ and $R \circ S = 1_Y$. As R has an inverse in FRel, it follows that this inverse must be its converse $R^{\smile}$ viewed as a morphism from (Y, B) to (X, A). As both $R : (X, A) \to (Y, B)$ and $R^{\smile} : (Y, B) \to (X, A)$ are morphisms in FRel, if $x \, R \, y$ then $A(x) \leq B(y)$ and $B(y) \leq A(x)$, hence $A(x) = B(y)$. It follows that $A = B \circ R$. Showing the other direction, that a bijection R satisfying $A = B \circ R$ is an isomorphism amounts to the trivial observation that in this case $R^{\smile} : (Y, B) \to (X, A)$ is a morphism in FRel. ■

In a category with involution ‡, an isomorphism f is called unitary if $f^{\ddagger}$ is its inverse. It is easily seen that in Rel each isomorphism is unitary. The situation is different in FRel.

Corollary 3.15 *In* FRel*, an isomorphism $R : (X, A) \to (Y, B)$ is unitary if and only if A and B are constant functions taking value $\frac{1}{2}$.*

Proof. The previous result shows the inverse of R is the converse $R^{\smile}$ viewed as a map from (Y, B) to (X, A), and $R^{\ddagger}$ is the converse of R viewed as a map from $(Y, 1 - B)$ to $(X, 1 - A)$. Thus $R^{-1} = R^{\ddagger}$ precisely when $A = 1 - A$ and $B = 1 - B$, and this occurs when A and B are constants taking value $\frac{1}{2}$. ■

Definition 3.16 *In a category $\mathcal{C}$, an object Z is* ***injective*** *if for each monic $f : X \to Y$ and each $g : X \to Z$, there is an $h : Y \to Z$ with $g = h \circ f$.*



We say $e : X \to Z$ is an ***injective hull*** *of X if e is monic, Z is injective, and for any $k : Z \to V$ we have $k \circ e$ being monic implies k is monic.*

Proposition 3.17 *In* FRel *the injectives are the objects* $(Z, 1)$ *where* 1 *is the constant function on* Z *taking value* 1*. For each object* (X, A)*, the identical embedding* $\Delta_X : (X, A) \to (X, 1)$ *is an injective hull.*

Proof. Suppose $R : (X, A) \to (Y, B)$ is monic and $S : (X, A) \to (Z, 1)$. Let $Y_1 = \{y \in Y : \text{there is exactly one } x \text{ with } x\,R\,y\}$. By Proposition 3.11, for each $x \in X$ there is $y \in Y_1$ with $x\,R\,y$. Define $T : (Y, B) \to (Z, 1)$ to be $T = \{(y, z) : y \in Y_1 \text{ and } x\,S\,z \text{ for some } x\,R\,y\}$. Trivially T is a morphism in FRel and $T \circ R = S$. So $(Z, 1)$ is injective.

Suppose (Z, A) is an object and $z_0 \in Z$ has $A(z_0) < 1$. Pick a singleton set $\{*\}$, define $S : (\{*\}, 0) \to (Z, A)$ to be $S = \{(*, z_0)\}$ and let $R : (\{*\}, 0) \to (\{*\}, 1)$ be the identical relation. Surely R is monic. There can be no morphism $T : (\{*\}, 1) \to (Z, A)$ with $T \circ R = S$ since such a T would have $*\,T z_0$ and $1 \not\leq A(z_0)$. So (Z, A) is not injective.

To see $\Delta_X : (X, A) \to (X, 1)$ is an injective hull, note Proposition 3.11 gives that Δ_X is monic, and we have shown $(X, 1)$ is injective. Suppose $R : (X, 1) \to (Y, B)$ and that $R \circ \Delta_X$ is monic. As a relation rather than a morphism, $R \circ \Delta_X = R$, then Proposition 3.11 gives that R is monic. ■

The notion of a projective object in a category is dual to that of an injective object, and the notion of a projective cover is dual to that of an injective hull. Here the direction of the morphisms is reversed, and monics are replaced by epics (see [13] for details).

Proposition 3.18 *In* FRel *the projectives are the objects* $(Z, 0)$ *where* 0 *is the constant function on* Z *taking value* 0*. For each object* (X, A)*, the identical embedding* $\Delta_X : (X, 0) \to (X, A)$ *is a projective cover.*

Proof. This follows as the involution ‡ takes injectives to projectives, projectives to injectives, and interchanges monics and epics. ■

Properties of four categories are summarized in the following table, from which it can be seen that Set and FSet have very similar properties, as do Rel and FRel.

	Set	FSet	Rel	FRel
Term. Obj.	$\{*\}$	$(\{*\}, 1)$	$\varnothing$	$(\varnothing, \varnothing)$
Init. Obj.	$\varnothing$	$(\varnothing, \varnothing)$	$\varnothing$	$(\varnothing, \varnothing)$
Zero Obj.	no	no	$\varnothing$	$(\varnothing, \varnothing)$
Product	$X \times Y$	$(X \times Y, A \wedge B)$	$X \sqcup Y$	$X \sqcup Y$
Coproduct	$X \sqcup Y$	$X \sqcup Y$	$X \sqcup Y$	$X \sqcup Y$
Biproduct	no	no	$X \sqcup Y$	$X \sqcup Y$
Monics	1-1	1-1	$2^X \to 2^Y$ 1-1	$2^X \to 2^Y$ 1-1
Epics	onto	onto	$2^Y \to 2^X$ 1-1	$2^Y \to 2^X$ 1-1
Isom's	bijection	bijection $A = B \circ f$	$2^X \to 2^Y$ bij.	$2^X \to 2^Y$ bij. $A = B \circ R$
Injectives	sets $\neq \varnothing$	$(X, 1)$, $X \neq \varnothing$	sets	$(X, 1)$
Projectives	sets	$(X, 0)$	sets	$(X, 0)$

4 Related Categories

In this section we consider the relationships among the categories Set, Rel, FSet and FRel. In particular, we show there are a number of adjunctions between these categories. The situation is illustrated below.

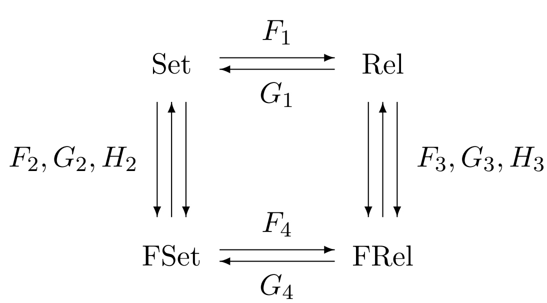


Figure 1: Functors relating categories

Definition 4.1 *For categories $\mathcal{C}$ and $\mathcal{D}$ and functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$, we say (F, G) is an* ***adjoint situation*** *if F is left adjoint to G and G is right adjoint to F. This implies that for objects $X \in \mathcal{C}$ and $Y \in \mathcal{D}$, there is a natural isomorphism between the homsets*

$$\mathcal{C}(X, G(Y)) \approx \mathcal{D}(F(X), Y)).$$

For a full account of adjoint functors, adjoint situations, and their properties with respect to composition and preservation of limits and colimits, see [15]. We next describe the functors indicated above.

Definition 4.2 *Let* $F_1 : \mathrm{Set} \to \mathrm{Rel}$ *be the inclusion functor and define* $G_1 : \mathrm{Rel} \to \mathrm{Set}$ *for an object* X *and morphism* $R : X \to Y$ *by setting*

1. $G_1(X)$ *to be the power set* 2^X.
2. $G_1(R)$ *to be the relational image function* $R[\,\cdot\,] : 2^X \to 2^Y$.

Theorem 4.3 *The pair* (F_1, G_1) *is an adjoint situation.*

Proof. We supply only the main point. For sets X and Y there is a bijection between relations from X to Y and functions from X to the power set 2^Y. Here a relation R is taken to the function f with $f(x)$ being the set of all elements related to x. This provides a natural isomorphism from $\mathrm{Rel}(X, Y)$ to $\mathrm{Set}(X, 2^Y)$. Thus $\mathrm{Rel}(F_1X, Y) \approx \mathrm{Set}(X, G_1Y)$. ■

Definition 4.4 *The **forgetful functor*** $G_2 : \mathrm{FSet} \to \mathrm{Set}$ *takes an object* (X, A) *to* X *and a morphism* $R : (X, A) \to (Y, B)$ *to* $R : X \to Y$. *Similarly, there is a forgetful functor* $G_3 : \mathrm{FRel} \to \mathrm{Rel}$.

Definition 4.5 *Define* $F_2, H_2 : \mathrm{Set} \to \mathrm{FSet}$ *and* $F_3, H_3 : \mathrm{Rel} \to \mathrm{FRel}$ *for an object* X *and morphism* $R : X \to Y$ *by setting*

1. $F_2(X)$ *and* $F_3(X)$ *to be the object* $(X, 0)$.
2. $H_2(X)$ *and* $H_3(X)$ *to be the object* $(X, 1)$.
3. $F_2(R)$, $F_3(R)$, $H_2(R)$ *and* $H_3(R)$ *are* R.

Here, 0 *and* 1 *are the obvious constant functions, and in (3)* R *is of course considered with the appropriate domain and codomain.*

Theorem 4.6 *Each of the pairs* (F_2, G_2), (G_2, H_2), (F_3, G_3) *and* (G_3, H_3) *are adjoint situations.*

Proof. While a morphism $R : X \to Y$ in either Set or Rel will not lift to a morphism $R : (X, A) \to (Y, B)$ for any choice of functions A and B, the morphism R will lift if either A is the constant 0 or B is the constant 1, since the required inequality will then be trivial. So

$$\mathrm{FSet}((X,0),(Y,B)) \approx \mathrm{Set}(X,Y)$$
$$\mathrm{Set}(X,Y) \approx \mathrm{FSet}((X,A),(Y,1))$$

The first shows $\mathrm{FSet}(F_2(X),(Y,B)) \approx \mathrm{Set}(X,G_2(Y,B))$, and the second that $\mathrm{Set}(G_2(X,A),Y) \approx \mathrm{FSet}((X,A),H_2(Y))$. These lead to the adjoint situations (F_2,G_2) and (G_2,H_2). The arguments to show (F_3,G_3) and (G_3,H_3) are adjoint situations are essentially identical. ■

Definition 4.7 *Let* $F_4 : \mathrm{FSet} \to \mathrm{FRel}$ *be the inclusion functor and define* $G_4 : \mathrm{FRel} \to \mathrm{FSet}$ *for an object* (X,A) *and a morphism* $R : (X,A) \to (Y,B)$ *by setting*

1. $G_4(X,A) = (2^X, \inf A)$ *where* $\inf A(S) = \inf\{A(x) : x \in S\}$.
2. $G_4(R)$ *is the relational image function* $R[\,\cdot\,] : 2^X \to 2^Y$.

Theorem 4.8 *The pair* (F_4,G_4) *is an adjoint situation.*

Proof. Say $R : (X,A) \to (Y,B)$ in FRel. For $S \subseteq X$ and $y \in R[S]$ there is $x \in S$ with $x\,R\,y$. As $A(x) \le B(y)$, we have $(\inf A)(S) \le (\inf B)R[S]$. So $R[\,\cdot\,] : (2^X, \inf A) \to (2^Y, \inf B)$ is a morphism in FSet. It follows that G_4 is a functor. Also

$$\mathrm{FRel}((X,A),(Y,B)) \;\approx\; \mathrm{FSet}((X,A),(2^Y, \inf B)).$$

Indeed, if $R : (X,A) \to (Y,B)$, then $A(x) \le (\inf B)R[\{x\}]$, showing that $R[\{\cdot\}] : (X,A) \to (2^Y, \inf B)$. Conversely, for $f : (X,A) \to (2^Y, \inf B)$, define a relation $R : X \to Y$ by $x\,R\,y$ if $y \in f(x)$. Then $x\,R\,y$ implies $A(x) \le (\inf B)(f(x))$, hence $A(x) \le B(y)$. ■

One easily checks commutation properties of the above diagram. As adjoint situations compose to adjoint situations, it follows that the functors from Set to FRel sending a set X to $(X,0)$ and (X,A) to 2^X give an adjoint situation. We next consider two endofunctors on FRel that play an interesting role.

Definition 4.9 *Let* $Proj, Inj : \mathrm{FRel} \to \mathrm{FRel}$ *be given by*

1. $Proj = F_3 \circ G_3$. *and*
2. $Inj = H_3 \circ G_3$.

*We call these the **projective cover** and **injective hull functors**.*

Here the names are chosen because $Proj$ sends an object (X, A) to its projective cover $(X, 0)$ and Inj takes (X, A) to its injective hull $(X, 1)$. See Propositions 3.17 and 3.18. A simple calculation gives the following.

Proposition 4.10 $\ddagger \circ Proj = Inj \circ \ddagger$.

5 Monoidal Structure

Recall that a symmetric monoidal category $\mathcal{C}$ is a category equipped with a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, a distinguished object I, and natural isomorphisms $\alpha_{X,Y,Z} : X \otimes (Y \otimes Z) \to (X \otimes Y) \otimes Z$, $\sigma_{X,Y} : X \otimes Y \to Y \otimes X$, and $\lambda_X : X \to A \otimes I$, subject to certain coherence conditions [15, p. 157]. The bifunctor $\otimes$ is often called a tensor product, and I the tensor unit.

Proposition 5.1 *Each of* Set*,* Rel*,* FSet*, and* FRel *has finite products, so has a monoidal structure* $\otimes$ *given by products and called its Cartesian monoidal structure.*

1. *In* Set*,* $X \otimes Y$ *is ordinary Cartesian product* $X \times Y$*.*
2. *In* Rel*,* $X \otimes Y$ *is disjoint union* $X \sqcup Y$*.*
3. *In* FSet*,* $(X, A) \otimes (Y, B)$ *is* $(X \times Y, \min\{A, B\})$*.*
4. *In* FRel*,* $(X, A) \otimes (Y, B)$ *is* $(X \sqcup Y, A \sqcup B)$*.*

Note in Rel *and* FRel *this product tensor is the biproduct* $\oplus$ *discussed above.*

Proof. This is a reiteration of results on products in these categories given in a previous section, with the well known fact that products give a monoidal structure in any category with finite products. ■

The category Rel carries another monoidal structure. In Rel, set $X \otimes Y$ on objects to be the usual Cartesian product of sets (which is not the categorical product), and set $R \otimes S$ for morphisms to be usual product relation. With this monoidal structure, Rel behaves much like the category of finite dimensional vector spaces over a given field, and has found application in recent categorical treatments of quantum mechanics [1, 9]. Our main interest in monoidal structure here is to lift this situation to FRel.

Definition 5.2 *A **t-norm**, or triangular norm, is a function $T : \mathbb{I} \times \mathbb{I} \to \mathbb{I}$ that is order preserving in both coordinates and satisfies*

1. $T(x, y) = T(y, x)$.
2. $T(x, T(y, z)) = T(T(x, y), z)$.
3. $T(1, x) = x$.

*A **conorm** C is a function $C : \mathbb{I} \times \mathbb{I} \to \mathbb{I}$ satisfying the same conditions but with (3) replaced by $C(0, x) = x$.*

Common examples of t-norms include ordinary multiplication and the operation min, common examples of conorms include truncated addition and max. Such t-norms and conorms are used in fuzzy logic to play the role of the connectives "AND" and "OR" of classical 2-valued logic.

Proposition 5.3 *For any t-norm T on $\mathbb{I}$, there is a symmetric monoidal structure $\otimes_T$ on* FRel *defined as follows.*

1. $(X, A) \otimes_T (Y, B) = (X \times Y, T(A, B))$ *where* $T(A, B) = T \circ (A \times B)$.
2. $R \otimes_T S$ *is the ordinary product relation* $R \times S$.
3. *The tensor unit is* $I = (\{*\}, 1)$ *where* $\{*\}$ *is some one-element set.*

A corresponding result shows a conorm C yields a tensor $\otimes_C$, but with tensor unit $I = (\{*\}, 0)$.

Proof. Let $R : (X, A) \to (X', A')$ and $S : (Y, B) \to (Y', B')$ be morphisms. If (x, y) is $R \times S$ related to (x', y'), then $x\, R\, x'$ and $y\, S\, y'$. We then have that $A(x) \leq A'(x')$ and $B(y) \leq B'(y')$. Since the t-norm T is order preserving in each coordinate, we then have that $T(A, B)(x, y) \leq T(A', B')(x', y')$. So $R \times S$ is a morphism. That $\otimes_T$ is compatible with composition and identity morphisms follows from the corresponding results for the tensor on Rel defined through Cartesian product. So $\otimes_T$ is a bifunctor.

Results for the tensor for Rel provide a tensor unit $\{*\}$, and natural isomorphisms $\alpha_{X,Y,Z} : X \times (Y \times Z) \to (X \times Y) \times Z$, $\sigma_{X,Y} : X \times Y \to Y \times X$, and $\lambda_X : X \times I \to I \times X$ that satisfy the coherence conditions. To see these lift to FRel amounts to using the characterization of isomorphisms in FRel given in Proposition 3.14 as bijections preserving the additional mapping condition, and noting that $T(A, T(B, C)) = T(T(A, B), C)$, $T(A, B) = T(B, A)$, and $T(A,1) = A$. These of course follow from the the properties of t-norms given in Definition 5.2. The modification for conorms is trivial. ■

In Section 4 we provided a number of functors relating the categories Set, Rel, FSet, and FRel. The ways these functors interact with various monoidal structures are easily computed, and summarized below. We recall the monoidal tensor given by categorial product is called the Cartesian tensor as in Proposition 5.1.

Proposition 5.4 *The functors $F_2, G_2, H_2, F_3, G_3, H_3$ as well as G_1 and G_4 preserve Cartesian monoidal structure. The functors F_3, G_3, H_3 preserve monoidal structure when* Rel *has monoidal structure given by $X \times Y$ and* FRel *has that given by a t-norm or conorm. The functor F_1 preserves monoidal structure when* Set *has Cartesian monoidal structure and* Rel *has monoidal structure $X \times Y$, and F_4 preserves monoidal structure when the tensor on* FSet *is Cartesian and that on* FRel *is given by the t-norm* min.

We have put various types of structure on the category FRel. We now consider how these different types of structure relate to one another. For the following result we let $\neg : \mathbb{I} \to \mathbb{I}$ be the negation $\neg\, x = 1 - x$ and note that for a t-norm T there is a conorm $C = \neg\, T \neg$ given by $C(x,y) = \neg\, T(\neg\, x, \neg\, y)$. We call C the **complementary conorm** to T.

Proposition 5.5 *Consider the category* FRel *with involution ‡, biproduct $\oplus$, and tensor $\otimes_T$ from a t-norm T whose complementary conorm is C. Then for morphisms with appropriate domains and codomains we have*

1. $(Q \oplus R)^{\ddagger} \; = \; Q^{\ddagger} \oplus R^{\ddagger}$.
2. $Q \otimes_T (R \oplus S) \; = \; (Q \otimes_T R) \oplus (Q \otimes_T S)$.
3. $(Q \otimes_T R)^{\ddagger} \; = \; Q^{\ddagger} \otimes_C R^{\ddagger}$.

Proof. Suppose $Q : (X,A) \to (X',A')$ and $R : (Y,B) \to (Y',B')$ and $S : (Z,D) \to (Z',D')$. (1) Then $Q \oplus R$ is disjoint union $Q \sqcup R$ considered as a morphism from $(X \sqcup Y, A \sqcup B)$ to $(X' \sqcup Y', A' \sqcup B')$. Thus $(Q \oplus R)^{\ddagger}$ is the converse $(Q \sqcup R)^{\smile}$ considered as a morphism from $(X' \sqcup Y', 1-(A' \sqcup B'))$ to $(X \sqcup Y, 1-(A \sqcup B))$. This is the biproduct $Q^{\ddagger} \oplus R^{\ddagger}$ of the morphisms $Q^{\ddagger} : (X', 1-A') \to (X, 1-A)$ and $R^{\ddagger} : (Y', 1-B') \to (Y, 1-B)$.

(2) This is a similar computation using $Q \times (R \sqcup S) = (Q \times R) \sqcup (Q \times S)$ and $T(A, B \sqcup D) = T(A,B) \sqcup T(A,D)$, as well as several others of a similar nature involving $X \times (Y \sqcup Z)$, $X' \times (Y' \sqcup Z')$ and $T(A', B' \sqcup D')$.

(3) A similar computation noting $1 - T(A, B) = C(1 - A, 1 - B)$. ■

The above results involving tensors $\otimes_T$ from t-norms, $\otimes_C$ from conorms, and even our involution ‡, are all instances of a more general process of lifting structure from the unit interval $\mathbb{I}$ to FRel. Clearly $\otimes_T$ and $\otimes_C$ come from lifting t-norms and conorms, and ‡ comes from lifting negation $\neg$.

Definition 5.6 *We say an n-ary operation $f : \mathbb{I}^n \to \mathbb{I}$ is **monotone** if in each argument it either preserves or reverses order.*

Each monotone n-ary operation $f : \mathbb{I}^n \to \mathbb{I}$ can be considered an order preserving operation $f : \mathbb{I}^{\alpha_1} \times \cdots \times \mathbb{I}^{\alpha_n} \to \mathbb{I}$ where $\alpha_i \in \{+, -\}$ and $\mathbb{I}^+$ is considered as $\mathbb{I}$ with the usual order $\leq$, and $\mathbb{I}^-$ is the dual of $\mathbb{I}$, that is, $\mathbb{I}$ under the reverse order $\geq$. Using FRel^+ for FRel and FRel^- for the opposite category, we come to our key notion.

Proposition 5.7 *For a monotone $f : \mathbb{I}^{\alpha_1} \times \cdots \times \mathbb{I}^{\alpha_n} \to \mathbb{I}$, there is a functor $F(f) : \mathrm{FRel}^{\alpha_1} \times \cdots \times \mathrm{FRel}^{\alpha_n} \to \mathrm{FRel}$ defined on objects and morphisms by*

1. $F(f)((X_1, A_1), \ldots, (X_n, A_n)) \;=\; (X_1 \times \cdots \times X_n, f(A_1, \ldots, A_n))$.
2. $F(f)(R_1, \ldots, R_n) \;=\; R_1^{\alpha_1} \times \cdots \times R_n^{\alpha_n}$.

Here $R^+ = R$ and $R^- = R^\smile$ is the converse relation.

Proof. This is clearly well defined on objects. Suppose $(R_1, \ldots, R_n)$ is a morphism from $((X_1, A_1), \ldots, (X_n, A_n))$ to $((Y_1, B_1), \ldots, (Y_n, B_n))$ in the category $\mathrm{FRel}^{\alpha_1} \times \cdots \times \mathrm{FRel}^{\alpha_n}$. Then

$$\begin{array}{lll} \text{for } \alpha_i = + & \text{we have} & R_i : (X_i, A_i) \to (Y_i, B_i) \\ \text{for } \alpha_i = - & \text{we have} & R_i : (Y_i, B_i) \to (X_i, A_i) \end{array}$$

Suppose $(x_1, \ldots, x_n)(R_1^{\alpha_1} \times \cdots \times R_n^{\alpha_n})(y_1, \ldots, y_n)$. Then for $\alpha_i = +$ we have $x_i \, R_i \, y_i$, so $A(x_i) \leq B(y_i)$, and for $\alpha_i = -$ we have $y_i \, R_i \, x_i$, so $B(y_i) \leq A(x_i)$. Thus $f((A_1(x_1), \ldots, A_n(x_n)) \leq f(B_1(y_1), \ldots, B_n(y_n))$. So $F(f)(R_1, \ldots, R_n)$ is a morphism in FRel. Showing that $F(f)$ preserves identity morphisms and composition is routine. ■

Definition 5.8 *Suppose $f_1, \ldots, f_k$ are monotone with $f_i : \mathbb{I}^{m_i} \to \mathbb{I}$. We let*

$$(f_1, \ldots, f_k) : \mathbb{I}^{m_1 + \cdots + m_k} \to \mathbb{I}^k$$

be the obvious map.

Similarly, there is a functor $(F(f_1),\ldots,F(f_k)) : \text{FRel}^{m_1+\cdots+m_k} \to \text{FRel}^k$ covariant in some arguments, contravariant in others, defined in a similar manner to $(f_1,\ldots,f_k)$. Further, using the obvious natural transformations for associativity of various cartesian products, we have the following.

Proposition 5.9 *Suppose $f_1,\ldots,f_k$ are monotone with $f_i : \mathbb{I}^{m_i} \to \mathbb{I}$, and $g : \mathbb{I}^k \to \mathbb{I}$ is monotone. Then the composite $g(f_1,\ldots,f_k)$ is monotone and there is a natural isomorphism of functors*

$$F(g(f_1,\ldots,f_k)) \simeq F(g) \circ (F(f_1),\ldots,F(f_k)).$$

Proof. To avoid cumbersome notation, suppose $f_1 : \mathbb{I}^2 \to \mathbb{I}$ is type $+,-$, that $f_2 : \mathbb{I} \to \mathbb{I}$ is type $+$ and $g : \mathbb{I}^2 \to \mathbb{I}$ is type $-,+$. Then $g(f_1,f_2)$ is of type $-,+,+$. Then $F(g(f_1,f_2))$ is a functor

$$F(g(f_1,f_2)) : \text{FRel}^{op} \times \text{FRel} \times \text{FRel} \to \text{FRel}.$$

This sends an object $((X,A),(Y,B),(Z,C))$ to $(X \times Y \times Z, h)$, where $h(x,y,z) = g(f_1(x,y), f_2(z))$, and it sends a morphism (R,S,T) to $R^{\smile} \times S \times T$.

We also have functors

$$\begin{aligned} (F(f_1),F(f_2)) &: \text{FRel} \times \text{FRel} \times \text{FRel} \to \text{FRel} \times \text{FRel} \\ F(g) &: \text{FRel} \times \text{FRel} \to \text{FRel} \end{aligned}$$

Here $(F(f_1),F(f_2))$ is covariant in its first argument, contravariant in its second, while $F(g)$ is contravariant in its first, covariant in its second. The composite $F(g) \circ (F(f_1),F(f_2))$ takes the object $((X,A),(Y,B),(Z,C))$ to $((X \times Y) \times Z, h')$ where $h'((x,y),z) = g(f_1(x,y), f_2(z))$. This composite takes a morphism (R,S,T) to $(R^{\smile} \times S) \times T$. The usual natural isomorphisms involved with associativity of cartesian products extend to give a natural isomorphism between these functors. ■

Corollary 5.10 *If f,g are mutually inverse isomorphisms of $\mathbb{I}$, then $F(f)$ and $F(g)$ are mutually inverse isomorphisms of* FRel.

Proof. This follows from Proposition 5.9 noting that the natural isomorphisms involved are actual equalities in this case. ■

We say t-norms T and T' are **equivalent**, and write $T \approx T'$, if there is an automorphism h of the ordered set $\mathbb{I}$ with $h(T(x,y)) = T'(h(x),h(y))$. Then, using Corollary 5.10, we immediately have the following.

Theorem 5.11 *If t-norms T, T' are equivalent via the isomorphism h of $\mathbb{I}$, then the functor $F(h)$ is an isomorphism between the monoidal categories* $(\text{FRel}, \otimes_T, I)$ *and* $(\text{FRel}, \otimes_{T'}, I)$. *Further, this functor $F(f)$ restricts to an isomorphism between* $(\text{FSet}, \otimes_T, I)$ *and* $(\text{FSet}, \otimes_{T'}, I)$.

6 Other Categories of Fuzzy Sets

Here we discuss some categories related to FRel and FSet. These include extensions to interval-valued fuzzy sets [5] and type-2 fuzzy sets [21]. These can be viewed as extensions of our earlier results obtained by replacing $\mathbb{I}$ by more general ordered structures. We begin with the following definition due to Goguen [7].

Definition 6.1 *For a poset V, let* $\text{Set}\,(V)$ *be the category whose objects are pairs (X, A) where $A : X \to V$ and whose morphisms from (X, A) to (Y, B) are functions $f : X \to Y$ with $A(x) \leq B(f(x))$.*

We make the obvious modification to incorporate relations as morphisms.

Definition 6.2 *For a poset V, let* $\text{Rel}\,(V)$ *be the category whose objects are pairs (X, A) where $A : X \to V$ and whose morphisms from (X, A) to (Y, B) are relations R from X to Y satisfying $x\,R\,y \Rightarrow A(x) \leq B(y)$.*

So FSet is $\text{Set}\,(\mathbb{I})$ and FRel is $\text{Rel}\,(\mathbb{I})$. Goguen paid particular attention to $\text{Set}\,(V)$ when V is a completely distributive lattice, giving an abstract axiomatization of such categories. Barr [2] also considered categories $\text{Set}\,(V)$ in the case where V is a kind of complete distributive lattice known as a frame, and noted such categories can be embedded into topoi. Here we consider instances of $\text{Rel}\,(V)$ arising from structures related to fuzzy sets. We begin with interval-valued fuzzy sets.

Definition 6.3 *Define $\mathbb{I}^{[2]} = \{(a, b) : 0 \leq a \leq b \leq 1\}$ and partially order this set by $(a, b) \leq (c, d)$ iff $a \leq c$ and $b \leq d$.*

For an account of the role of $\mathbb{I}^{[2]}$ in interval-valued fuzzy sets see [5]. We only mention that it is a completely distributive lattice with meets and joins given componentwise, and carries the further structure of a De Morgan algebra where $\neg\,(a, b) = (1 - b, 1 - a)$.

Definition 6.4 *Define the category* IFRel *of interval-valued fuzzy sets and relations to be* $\text{Rel}\,(\mathbb{I}^{[2]})$, *and let* IFSet *be* $\text{Set}\,(\mathbb{I}^{[2]})$.

A t-norm for interval-valued fuzzy sets is a function $T : \mathbb{I}^{[2]} \times \mathbb{I}^{[2]} \to \mathbb{I}^{[2]}$ satisfying conditions similar to Definition 5.2. Using these, we can again construct monoidal structure on IFRel and on IFSet. One can also use the De Morgan negation on $\mathbb{I}^{[2]}$ to construct an involution on IFRel much as above. However, just as with FRel, this involution does not restrict to an involution on IFSet.

Theorem 6.5 *For a t-norm T on $\mathbb{I}^{[2]}$ we can define a monoidal structure $\otimes_T$ on* IFRel *by setting*

1. $(X, A) \otimes_T (Y, B) = (X \times Y, T(A, B))$.
2. $R \otimes_T S = R \times S$.
3. $I = (\{*\}, 1)$ *is the tensor unit where* $1(*) = (1, 1)$.

Further IFRel *has an involution* $\ddagger$ *where* $(X, A)^{\ddagger} = (X, \neg A)$ *and* $R^{\ddagger} = R^{\smile}$.

We next consider matters for the type-2 fuzzy sets introduced by Zadeh. While the reader should consult [21] for a general background, we recall the key notion of the algebra of truth values for type-2 fuzzy sets.

Definition 6.6 *The* ***algebra of truth values for type-2 fuzzy sets*** *is*

$$\mathbf{M} = (\, [0,1]^{[0,1]}\,, \sqcup, \sqcap, ^{*}, \bar{0}, \bar{1})$$

where the operations are convolutions of the usual operations $\vee, \wedge, \neg, 0, 1$ on the unit interval:

$$\begin{aligned} (f \sqcup g)(x) &= \sup \{ f(y) \wedge g(z) : y \vee z = x \} \\ (f \sqcap g)(x) &= \sup \{ f(y) \wedge g(z) : y \wedge z = x \} \\ f^{*}(x) &= \sup \{ f(y) : \neg y = x \} \end{aligned}$$

The constants $\overline{0}(x)$ and $\overline{1}(x)$ are the characteristic functions of $\{0\}$ and $\{1\}$, respectively. The expression for f^* of course simplifies to $f^*(x) = f(\neg x)$.

This algebra has many interesting algebraic properties [21]. It satisfies all equations commonly used to define bounded lattices except the law of absorption $x \sqcap (x \sqcup y) = x = x \sqcup (x \sqcap y)$, and it does satisfy the version of this where the middle term in the three equalities is omitted. It forms a type of structure known as a **De Morgan bisemilattice**. While this structure is not a lattice, it can be treated in an order-theoretic way.

Definition 6.7 *Define relations* $\leq_\sqcup$, $\leq_\sqcap$ *and* $\leq_d$ *on* $\mathbf{M}$ *as follows:*

1. $f \leq_\sqcup g$ *iff* $f \sqcup g = g$.
2. $f \leq_\sqcap g$ *iff* $f \sqcap g = f$.
3. $f \leq_d g$ *iff* $f \leq_\sqcup g$ *and* $f \leq_\sqcap g$.

We call $\leq_\sqcup$ *the* ***join order****,* $\leq_\sqcap$ *the* ***meet order****, and* $\leq_d$ *the* ***double order****.*

Using basic properties of $\mathbf{M}$, it is easily seen that $\leq_\sqcup$ and $\leq_\sqcap$ are both partial orders on $\mathbf{M}$, and as $\leq_d$ is the intersection of these partial orders, it is a partial order on $\mathbf{M}$. Indeed, this holds for any bisemilattice. However, while a poset, $\mathbf{M}$ is not a lattice under any of these orders. We next use this poset $\mathbf{M}$ to define a category of type-2 fuzzy sets.

Definition 6.8 *Let* 2-FRel *be the category* $\mathrm{Rel}\,(\mathbf{M})$ *where* $\mathbf{M}$ *is a considered as a poset under its double order, and let* 2-FSet *be* $\mathrm{Set}\,(\mathbf{M})$*.*

We next consider the matter of additional structure on 2-FRel. We begin by using the operation $*$ on $\mathbf{M}$, given by $f^*(x) = f(\neg x)$, to define an involution on 2-FRel.

Proposition 6.9 *There is an involution* $\ddagger$ *on* 2-FRel *taking* (X, A) *to* (X, A^*) *and* R *to its converse* $R^{\smile}$*.*

Proof. The key points are that $*$ is a period two operation on $\mathbf{M}$ that is order inverting with respect to the double order. It is obvious that it is period two. In [21] it is shown that $f \leq_\sqcup g$ implies $g^* \leq_\sqcap f^*$, and $f \leq_\sqcap g$ implies $g^* \leq_\sqcup f^*$. From this it follows that $f \leq_d g$ implies $g^* \leq_d f^*$. ■

The matter of monoidal structure is somewhat problematic. We begin with the notion of a t-norm for type-2 fuzzy sets. The following is perhaps the most restrictive notion [21].

Definition 6.10 *For a t-norm* T *on* $\mathbb{I}$*, define its convolution to be the binary operation* $\hat{T}$ *on* $\mathbf{M}$ *given by*

$$\hat{T}\,(f, g)\,(x) = \sup\,\{(f(y) \wedge g(z)) : T\,(y, z) = x\}$$

In [21, p. 39-41] it is shown that such $\hat{T}$ is commutative, associative, and has $\overline{1}$ as a unit. If $\hat{T}$ were order preserving in both coordinates with respect to the double order, then it would give a monoidal structure on 2-FRel, much as before. However, this seems not to be the case. There is a subalgebra of $\mathbf{M}$ where things are better behaved.

Definition 6.11 *For $f \in \mathbf{M}$ call f **normal** if its supremum is 1, and **convex** if it is never the case that $f(x) > f(y) < f(z)$ when $x < y < z$. Let $\mathbf{L}$ be the set of all convex normal functions in $\mathbf{M}$.*

In [21] it is shown that $\mathbf{L}$ is a subalgebra of $\mathbf{M} = (M, \sqcap, \sqcup, *, \overline{0}, \overline{1})$; on $\mathbf{L}$ the orders $\leq_\sqcap, \leq_\sqcup$ and $\leq_d$ agree and give lattice orders where meet is given by $\sqcap$ and join is given by $\sqcup$; and $\mathbf{L}$ is a distributive lattice with a De Morgan structure given by $*$.

Definition 6.12 *Let* CNRel *be the category* Rel $(\mathbf{L})$ *of convex normal sets and relations, and* CNSet *be* Set $(\mathbf{L})$.

Before considering structure on these categories, we require a lemma.

Lemma 6.13 *If T is a continuous t-norm on $\mathbb{I}$, then its convolution $\hat{T}$ restricts to an operation on $\mathbf{L}$ that is commutative, associative, has $\overline{1}$ as an identity, and is order preserving in both coordinates.*

Proof. We follow [21] and write $f \blacktriangle g$ for $\hat{T}(f, g)$. Proposition 61 of [21] shows $\hat{T}$ is commutative, associative, and has $\overline{1}$ as a unit on all of $\mathbf{M}$, hence these hold also on $\mathbf{L}$. Theorem 63 of [21] shows more than it states: for $p \in \mathbf{M}$, that p is convex iff $p \blacktriangle (q \sqcup r) = (p \blacktriangle q) \sqcup (p \blacktriangle r)$ for all $q, r \in \mathbf{M}$. Suppose f, g are convex and $q, r \in \mathbf{M}$. Then using this result and associativity,

$$\begin{aligned}
(f \blacktriangle g) \blacktriangle (q \sqcup r) &= f \blacktriangle (g \blacktriangle (q \sqcup r)) \\
&= f \blacktriangle ((g \blacktriangle q) \sqcup (g \blacktriangle r)) \\
&= (f \blacktriangle (g \blacktriangle q)) \sqcup (f \blacktriangle (g \blacktriangle r)) \\
&= ((f \blacktriangle g) \blacktriangle q) \sqcup ((f \blacktriangle g) \blacktriangle r)
\end{aligned}$$

proving $f \blacktriangle g$ is convex.

In [21] f^L and f^R are used for the pointwise least increasing and decreasing functions above f. So, [21, Proposition 30], f is normal iff $f = f^{LR}$. Proposition 62 of [21] gives $(f \blacktriangle g)^L = f^L \blacktriangle g^L$ and $(f \blacktriangle g)^R = f^R \blacktriangle g^R$. So if f, g are normal, then $(f \blacktriangle g)^{LR} = 1 \blacktriangle 1 = 1$ by Proposition 61. Hence f, g normal implies $f \blacktriangle g$ is normal. Thus $\hat{T}$ restricts to an operation on $\mathbf{L}$.

It remains to show $\hat{T}$ in order preserving in each argument. On $\mathbf{L}$ the orders $\leq_\sqcup, \leq_\sqcap, \leq_d$ agree, so $g \leq_d h$ is equivalent to $g \sqcup h = h$. Then for $f, g, h \in \mathbf{L}$ with $g \leq_d h$ we have by the convexity of f that $(f \blacktriangle g) \sqcup (f \blacktriangle h) = f \blacktriangle (g \sqcup h) = f \blacktriangle h$. Thus $g \leq_d h$ implies $f \blacktriangle g \leq_d f \blacktriangle h$. So $\hat{T}$ is order preserving in the second coordinate, and by commutativity in its first. ∎

Using this and the De Morgan negation $*$ on $\mathbf{L}$, we have the following.

Theorem 6.14 *For any continuous t-norm T on $\mathbb{I}$, there is a monoidal structure $\otimes_T$ on* CNRel *where*

1. $(X, A) \otimes (Y, B) = (X \times Y, \hat{T}(A, B))$
2. $R \otimes S = R \times S$.
3. $I = (\{*\}, 1)$ *is the tensor unit where* $1(*) = \overline{1}$.

Further, there is an involution $\ddagger$ on CNRel *where $(X, A)^\ddagger = (X, A^*)$ and $R^\ddagger = R^\smile$ is the converse.*

While $\mathbf{L}$ is complete and distributive, it is not completely distributive [10]. This can be remedied. Each convex normal function can be *straightened out* by taking its mirror image when it begins to decrease. Defining $f\theta g$ if the straightened out versions of f, g agree almost everywhere gives a congruence on $\mathbf{L}$ with the quotient $\mathbf{D} = \mathbf{L}/\theta$ being a completely distributive lattice with De Morgan negation. The reader should see [10, 11] for further details.

Theorem 6.15 *The category* $\mathrm{Rel}\,(\mathbf{D})$ *has an involution given by the De Morgan negation on $\mathbf{D}$, and the t-norm $T(x, y) = \min\{x, y\}$ on $\mathbb{I}$ gives a monoidal structure $\otimes_T$ on* $\mathrm{Rel}\,(\mathbf{D})$.

Proof. The convolution $\hat{T}$ of the t-norm min is the operation $\sqcap$ on $\mathbf{M}$, which is the meet of the lattice $\mathbf{L}$. In [11] it is shown that θ is a lattice congruence, so is compatible with the convolution $\hat{T}$. Thus, this operation on $\mathbf{D}$ is the meet of this lattice, so is commutative, associative, order preserving in each coordinate, and has an identity. Thus, it yields a tensor on $\mathrm{Rel}\,(\mathbf{D})$. ■

We suspect the above result holds for the convolution of any continuous t-norm. We record this below.

Problem 6.16 *Is the congruence θ on $\mathbf{L}$ compatible with the convolution $\hat{T}$ of each continuous t-norm on $\mathbb{I}$?*

We noted before that Goguen [7] axiomatized the categories arising as $\mathrm{Set}\,(V)$ for a completely distributive lattice V. We feel an answer to the following question(s) would be of interest.

Problem 6.17 *Axiomatize the categories* $\mathrm{Rel}\,(V)$ *where V is a completely distributive lattice, perhaps with De Morgan involution, or t-norm.*

7 Comparison with abstract categories of relations

The category FRel clearly has much in common with the category Rel. An abstraction of categories of relations has been given under the name of allegories [4]. FRel does not naturally form an allegory, but does have features in common with them. Here we discuss the relationship of FRel to allegories, and make some brief comments directed toward the relationship between FRel and other categories generalizing Rel [3, 1].

Definition 7.1 *An* ***allegory*** *is a 2-category $\mathcal{C}$ where homsets form posets that are meet semilattices, equipped with an involution $\dagger$ that is the identity on objects, that satisfies the* ***modular law***

$$(\beta\alpha) \wedge \gamma \;\;\leq\;\; \beta(\alpha \wedge (\beta^{\dagger}\gamma))$$

Here $\alpha : X \to Y$, $\beta : Y \to Z$ and $\gamma : X \to Z$.

The prime example of an allegory is Rel. Here homsets are complete Boolean algebras with meets given by intersection. The dagger on Rel is the identity on objects, and converse $R^{\dagger} = R^{\smile}$ on morphisms. This gives a unary operation $\dagger$ on each homset $\mathrm{Rel}\,(X, X)$ and makes these homsets into structures known as relation algebras [16].

Proposition 7.2 FRel *is a 2-category where homsets are complete Boolean algebras, and* FRel *has an involution $\ddagger$. However, $\ddagger$ is not the identity on objects, making a direct interpretation of the modular law meaningless.*

The modular law for allegories encodes properties of the converse of a relation. We feel we should be able to access this somehow using the involution $\ddagger$ on FRel as it does give converse relations, but with altered domains and codomains. One possibility is described below, essentially taking advantage of copies of Rel inside of FRel.

Theorem 7.3 FRel *is a 2-category where homsets are meet semilattices, equipped with an involution $\ddagger$ and an idempotent endofunctor $\mathbf{P}$ such that*

1. *$\mathbf{P}$ and $\ddagger$ preserve meets.*
2. *$\mathbf{P}\,\ddagger$ agrees with $\mathbf{P}$ on objects and $\mathbf{P}\,\ddagger\,\mathbf{P} = \mathbf{P}\,\ddagger$.*
3. *When $(\beta\alpha) \wedge \gamma$ is defined, $\mathbf{P}((\beta\alpha) \wedge \gamma) \;\leq\; \mathbf{P}\beta(\mathbf{P}\alpha \wedge (\mathbf{P}(\beta^{\ddagger})\mathbf{P}\gamma))$.*

Proof. Let $\mathbf{P}$ be the projective cover functor of Definition 4.9, so for an object (X, A) we have $\mathbf{P}(X, A) = (X, 0)$ and for $R : (X, A) \to (Y, B)$ we have $\mathbf{P}R$ is the same relation R but as a morphism from $(X, 0)$ to $(Y, 0)$. Then $\mathbf{P}$ is an endofunctor that is clearly idempotent.

(1) As meets are given by intersections, $\mathbf{P}(R \wedge S) = \mathbf{P}R \wedge \mathbf{P}S$ and $(R \wedge S)^\ddagger = (R \cap S)^\smile = R^\smile \cap S^\smile = R^\ddagger \wedge S^\ddagger$. (2) All of $\mathbf{P}$, $\mathbf{P}\,\ddagger$, $\mathbf{P}\,\ddagger\,\mathbf{P}$ take an object (X, A) to $(X, 0)$. Both $\mathbf{P}\,\ddagger\,\mathbf{P}$ and $\mathbf{P}\,\ddagger$ take a morphism $R : (X, A) \to (Y, B)$ to $R^\smile : (Y, 0) \to (X, 0)$. (3) The second condition ensures the domains and codomains of the morphisms are such that when the left side is defined, then so is the right side. As relations, ignoring domain and codomain, $\mathbf{P}R = R$ and $R^\ddagger = R^\smile$. So the identity becomes the usual modular law in Rel. ■

We note that the above result can be formulated also using the injective hull functor $\mathbf{I} = Inj$ of Definition 4.9. Here $\mathbf{I}$ takes an object (X, A) to $(X, 1)$, and a morphism R to the relation R with appropriately modified domain and codomain. So $\mathbf{I}$ provides access to a copy of Rel inside FRel, just as does $\mathbf{P}$. We next make a few comments on generalizing this situation.

Definition 7.4 *A **fuzzy allegory** is a 2-category $\mathcal{C}$ with involution $\ddagger$ and idempotent endofunctor $\mathbf{P}$ satisfying the conditions of Theorem 7.3.*

We note that an allegory is the same as a fuzzy allegory where the additional endofunctor $\mathbf{P}$ is the identity. We extend this further.

Proposition 7.5 *For a fuzzy allegory $\mathcal{C}$ with endofunctor $\mathbf{P}$ and involution $\ddagger$, the image category $\mathbf{P}(\mathcal{C})$ is an allegory under the dagger $\dagger = \mathbf{P}\,\ddagger$.*

Proof. As $\mathbf{P}$ is idempotent, the image $\mathbf{P}(\mathcal{C})$ is a full subcategory of $\mathcal{C}$, so homsets in this image are meet semilattices when given the same structure as the 2-category $\mathcal{C}$. For $\dagger = \mathbf{P}\,\ddagger$ we clearly have $\dagger$ is a contravariant functor from $\mathbf{P}(\mathcal{C})$ to itself. Note $\dagger\dagger = \mathbf{P}\,\ddagger\,\mathbf{P}\,\ddagger = \mathbf{P}\,\ddagger\ddagger$, and as $\ddagger$ is an involution $\dagger\dagger = \mathbf{P}$, hence is the identity on $\mathbf{P}(\mathcal{C})$. For an object x of $\mathbf{P}(\mathcal{C})$ we have $\dagger(x) = \mathbf{P}\,\ddagger\,\mathbf{P}x = \mathbf{P}\,\ddagger\,x$, and as $\mathbf{P}\,\ddagger = \mathbf{P}$ on objects, it follows that $\dagger(x) = \mathbf{P}x = x$. So $\dagger$ is the identity on objects.

Let α, β, γ be morphisms in $\mathbf{P}(\mathcal{C})$ with $\beta\alpha \wedge \gamma$ defined. As $\mathbf{P}$ restricts to the identity on $\mathbf{P}(\mathcal{C})$, 7.3 (3) gives $(\beta\alpha) \wedge \gamma \;\leq\; \beta(\alpha \wedge (\mathbf{P}(\beta^\ddagger)\gamma))$. Now $\mathbf{P}(\beta^\ddagger) = \mathbf{P}\,\ddagger\,\beta = \beta^\dagger$. This gives the modular law. ■

To conclude, we briefly mention some facts that help in considering the relationship of FRel to other generalizations of Rel considered by Carboni and Walters [3] and Abramsky and Coecke [1].

Definition 7.6 *For a set X, let Δ_X be the relation from X to $X \times X$, and ∇_X be the relation from $X \times X$ to X, given by*

$$\begin{aligned}\Delta_X &= \{(x,(x,x)) : x \in X\}\\ \nabla_X &= \{((x,x),x) : x \in X\}\end{aligned}$$

We remark that these morphisms in Rel yield a Frobenius structure that is key in Carboni and Walter's [3] treatment. We note the following.

Proposition 7.7 *For a t-norm T on $\mathbb{I}$, consider the following statements about the tensor $\otimes_T$ on* FRel*.*

1. $\Delta_X : (X, A) \to (X, A) \otimes_T (X, A)$ *is a morphism in* FRel.
2. $\nabla_X : (X, A) \otimes_T (X, A) \to (X, A)$ *is a morphism in* FRel.

Then (1) holds for all objects (X, A) iff T is the t-norm min*. However (2) holds for all objects (X, A) without any restriction on the t-norm.*

Proof. Having Δ_X be a morphism simply means $A(x) \leq T(A(x), A(x))$ for all $x \in X$. Having this hold for all objects (X, A) is equivalent to requiring $\alpha \leq T(\alpha, \alpha)$ for all $\alpha \in \mathbb{I}$. The one and only t-norm with this property is min. Having ∇_X be a morphism means $T(A(x), A(x)) \leq A(x)$ which follows as any t-norm satisfies $T(\alpha, \alpha) \leq \alpha$. ■

The matter of compact closure is even more problematic. Here, we use the Kelly-Laplaza formulation of compact closure (see [1] for further details).

Definition 7.8 *A **compact closed category** is a symmetric monoidal category in which each object A has a dual object A^*, a unit $\eta_A : I \to A^* \otimes A$ and a counit $\epsilon_A : A \otimes A^* \to I$ such that*

$$A \simeq A \otimes I \to A \otimes (A^* \otimes A) \simeq (A \otimes A^*) \otimes A \to I \otimes A \simeq A$$

evaluates to the identity, as does the dual expression for A^*.

Proposition 7.9 *For any t-norm T on $\mathbb{I}$, the symmetric monoidal category* $(\mathrm{FRel}, \otimes_T)$ *is not compact closed.*

Proof. Recall, the tensor unit for $\otimes_T$ is $(\{*\}, 1)$ where $1(*) = 1$. Then for any set X and any object (Y, B), the empty relation is the only morphism from I to $(Y, B) \otimes_T (X, 0)$ since $T(B(y), 0) = 0$. No matter what object is chosen for the dual of $(X, 0)$, the only candidate for the unit $\eta_{(X,0)}$ is the empty relation, so the expression in Definition 7.8 cannot evaluate to the identity. ■

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