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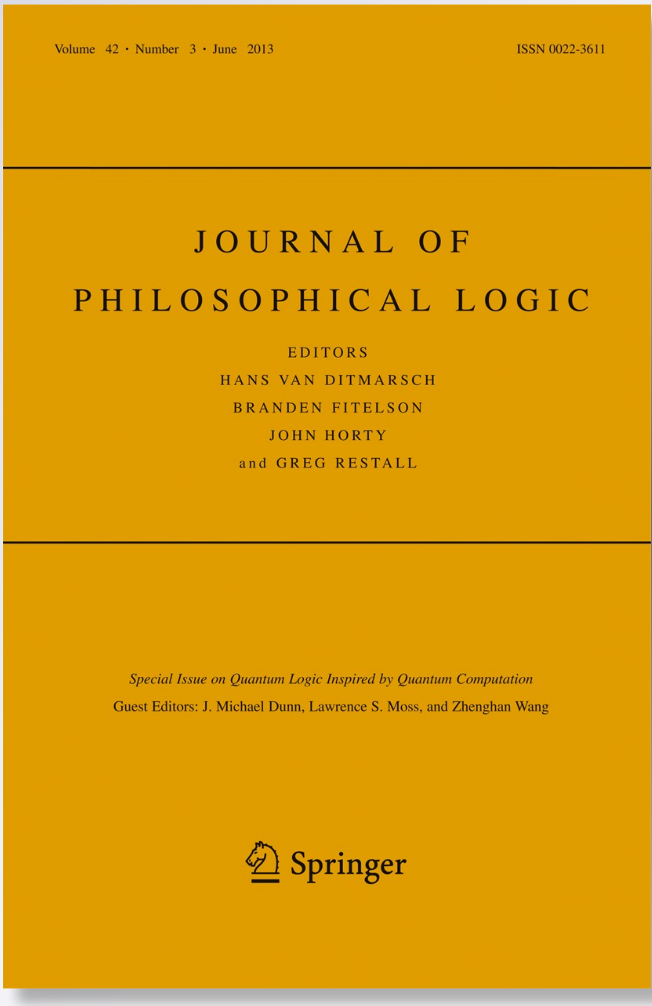

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Daggers, Kernels, Baer *-semigroups, and Orthomodularity

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Abstract We discuss issues related to constructing an orthomodular structure from an object in a category. In particular, we consider axiomatics related to Baer *-semigroups, partial semigroups, and various constructions involving dagger categories, kernels, and biproducts.

Keywords Baer *-semigroups · Orthomodular poset · Dagger kernel category · Biproduct

1 Introduction

There has been considerable recent interest in using categorical methods to address foundational issues in quantum mechanics [1, 2, 5, 29]. Here the objects of a suitable category are quantum systems and morphisms are processes. It is natural to connect this work to the older subject of quantum logic [23, 28] by constructing from each object in a category an orthomodular structure to represent the propositions of the quantum system represented by the object. This was considered in [19, 21], and likely will be of interest in future works as well. It is our purpose here to look in detail at the axiomatics related to this task.

Much, but not all, of what we consider is based on the simple and well known [17, 23] observation that the idempotents $E(R)$ of a ring R with unit form an orthomodular poset (abbreviated: OMP). Here $e \leq f$ if $ef = e = fe$ and $e' = 1 - e$. In this result, one can do with much less than a ring. The addition plays a very minor role, and one can make due with a multiplicative semigroup, or even partial semigroup, with suitable properties. This opens the door to creating an orthomodular structure

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from some collection of idempotent endomorphisms of an object A in a suitable category. Most, but not all, of what we say is based on this simple idea. A seemingly different approach is given in [18] where the direct product decompositions of a object A are shown to form an orthomodular structure.

If one seeks more than an OMP, such as an orthomodular lattice (abbreviated: OML) or a modular ortholattice (abbreviated: MOL), stronger properties are needed. Here the ideas are based on work of von Neumann on regular rings coordinatizing modular lattices [32], Baer *-rings giving certain OMLs, the Baer *-semigroups Foulis used to coordinatize general OMLs [9–13] and the various generalizations of Gudder and Schelp [16] and Gudder [15] to partial Baer *-semigroups and related structures to coordinatize OMPs. Our paper is largely a survey of these results from the 60's and 70's, adapted in places to the categorical setting at hand, with a number of small, but hopefully useful observations thrown in.

2 Baer *-semigroups and Orthomodularity

We begin with a few standard definitions. The reader should consult [23, 28] for background on OMLs and OMPs, and [20, 27] for general category theory. Specifics of the various types of rings below are found in [4, 26, 31].

Definition 1 A ring is called a (von Neumann) regular ring if for each x, there is some y with $xyx = x$, or equivalently, if each principal right ideal is generated by an idempotent; a Rickart ring if the right annihilator $\{y : xy = 0\}$ of each element x is the right ideal generated by an idempotent; and a Baer ring if the right annihilator of each subset is the right ideal generated by an idempotent.

A Baer ring is a Rickart ring, and a simple calculation shows a regular ring is a Rickart ring [4]. Regular rings were used by von Neumann [32] in his work on continuous geometries. The crucial point is the principal right ideals of a regular ring form a complemented modular lattice. This lattice is complete if, and only if, the ring is additionally a Baer ring. While the lattice of principal ideals of a regular ring is determined by its idempotents, this is not the same as the construction of an OMP from $E(R)$ described above. This lattice is constructed via the quasiorder on $E(R)$ given by $e \leq f$ if $fe = e$. The full utility of regular rings lies in a deep geometric coordinatization theorem of large classes of modular latices via regular rings [32].

Definition 2 A *-ring is a ring with an involution *, that is, a bijection of R with itself satisfying $(x + y)^* = x^* + y^*$, $(xy)^* = y^*x^*$, $x^{**} = x$, $0^* = 0$ and $1^* = 1$. The involution is proper if $x^*x = 0$ implies $x = 0$. A projection of a *-ring is an idempotent e that satisfies $e = e^*$.

Definition 3 A ring is a *-regular ring if it is a regular ring with a proper involution, or equivalently, a ring where every principal ideal is generated by a projection; a Rickart *-ring if the right annihilator of each element is generated by a projection; and a Baer *-ring if the right annihilator of each subset is generated by a projection.

Baer *-rings and *-regular rings are Rickart *-rings. If one considers the projections of a *-ring, having $ef = e$ is equivalent to having $ef = e = fe$ (as $ef = e$ implies $(ef)^* = e^*$ so $fe = e$). So the two kinds of orderings described above, one used for the idempotents of a regular ring, the other for the OMP constructed from the idempotents of a ring, coincide. For a Rickart *-ring, this ordering on the projections yields an OML; for a Baer *-ring, this yields a complete OML; for a *-regular ring, an MOL; and for a ring that is both Baer and *-regular, a complete MOL, and therefore a continuous geometry [4]. The meet in this lattice of projections is given by $e \wedge f = e(f'e)'$ where a' is the unique projection with $\{x : ax = 0\} = a'R$ [26, pg. 173]. Once again, we stress that in all these cases, the orthomodular structure obtained is a sub-OMP of the OMP $E(R)$ constructed from the idempotents of the ring as described in the introduction.

Definition 4 A *-semigroup is a semigroup $(S, \cdot)$ with an involution $*$ (so $(xy)^* = y^*x^*$ and $x^{**} = x$). A projection of a *-semigroup is an idempotent element e with $e = e^*$. A Baer *-semigroup is a *-semigroup with two-sided zero element 0 such that for each $a \in S$ there is a projection e with $\{x : ax = 0\} = eS$. This projection is unique and is written a'. The projections of the form a' for some $a \in S$ are called closed projections.

Baer *-semigroups were introduced by Foulis in [9]. The motivating example of a Baer *-semigroup is the multiplicative fragment of a Rickart *-ring. In this case, all projections are closed, and for a projection e we have $e' = 1 - e$. The apparent conflict in terminology (the name Baer *-semigroup rather than Rickart *-semigroup) arises from our keeping with ring terminology from Berberian that has since become standard. Baer *-semigroups are often called Foulis semigroups [23]. One half of the utility of Baer *-semigroups is provided by the following result of Foulis [9].

Theorem 5 *The closed projections of a Baer *-semigroup form an* OML *under the ordering $e \leq f$ if $ef = e = fe$ and orthocomplementation $'$. Meets in this lattice are given by $e \wedge f = e(f'e)'$.*

Of course, Rickart *-rings also provide OMLs, but not every OML can be obtained from the projections of a Rickart *-ring [14]. Foulis showed [9] that every OML can be obtained as the closed projections of a Baer *-semigroup. This is usually called coordinatizing an OML by a Baer *-semigroup. Foulis obtained this by constructing a Baer *-semigroup from the Galois connections on an OML. However, the coordinatization of an OML is not unique; for a MOL this coordinatization through Galois connections need not agree with the multiplicative fragment of the unique *-regular ring geometrically coordinatizing the MOL; and this coordinatization is not in any obvious way functorial [25]. Foulis also noted the following well-known result from the theory of Baer rings lifts to the Baer *-semigroup setting [9, 26].

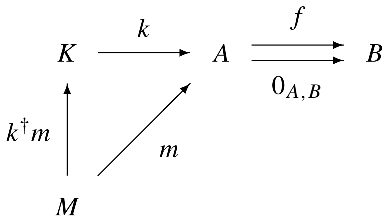
Theorem 6 *The closed projections of a Baer *-semigroup S form a complete* OML *iff for each $A \subseteq S$, there is a projection e with $\{x : ax = 0$ for all $a \in A\} = eS$.*

We turn our attention to matters related to categorical treatments of foundational issues in quantum mechanics. We briefly recall definitions from [1, 29, 30]. For general categorical issues, such as zero objects and kernels, see [20, 27].

Definition 7 A dagger category is a category $\mathcal{C}$ equipped with a contravariant functor $\dagger : \mathcal{C} \to \mathcal{C}$ that is period two and the identity on objects. So for an object A, with identity $1_A : A \to A$ we have $A^\dagger = A$, and $(1_A)^\dagger = 1_A$, and for morphisms $f : A \to B$ and $g : B \to C$ we have $f^\dagger : B \to A$, $f^{\dagger\dagger} = f$, and $(gf)^\dagger = f^\dagger g^\dagger$.

Definition 8 A dagger category has zero morphisms if for each ordered pair of objects B, C, there is a distinguished morphism $0_{B,C} : B \to C$ with $(0_{B,C})^\dagger = 0_{C,B}$, so that for any $f : A \to B$ and $g : C \to D$ we have $0_{B,C} \circ f = 0_{A,C}$, and $g \circ 0_{B,C} = 0_{B,D}$.

Definition 9 For a morphism $f : A \to B$ in a dagger category with zero morphisms, we say $k : K \to A$ is a weak dagger kernel of f if $fk = 0_{K,B}$, and if $m : M \to A$ satisfies $fm = 0_{M,B}$ then $kk^\dagger m = m$.



A weak dagger kernel category is a dagger category with zero morphisms where every morphism has a weak dagger kernel.

One final definition is required.

Definition 10 A dagger kernel category is a dagger category with a zero object, hence zero morphisms, where each morphism f has a weak dagger kernel k that additionally satisfies $k^\dagger k = 1_K$.

In any category, the endomorphisms of a given object form a semigroup. In a dagger category, this is a *-semigroup, and in a dagger category with zero morphisms, a *-semigroup with two-sided zero. Kernels are clearly related to annihilators, and the idea, due to Crown [6], is to obtain a Baer *-semigroup from the endomorphisms of an object by requiring the existence of kernels that are well behaved with respect to the dagger. He considered what are called here dagger kernel categories, and showed that the endomorphisms of any object in such a category form a Baer *-semigroup. The following is a trivial, but somewhat useful, extension of Crown's result.

Theorem 11 *In a weak dagger kernel category, the endomorphisms of any object A form a Baer *-semigroup where $f' = kk^\dagger$ for any weak dagger kernel $k : K \to A$ of f. Thus the closed projections of this Baer *-semigroup form an* OML.

Proof The proof is the short proof of Crown [6]. One simply notes that only zero morphisms are required, not a zero object; and that for any weak dagger kernel k, one has $kk^\dagger k = k$, and the stronger condition that $k^\dagger k = 1_K$ is not needed. □

Remarks 12 One can make a case for the naturality of this extension of Crown's result. Crown [6] shows that if k, m are two weak dagger kernels of $f : A \to B$, then $kk^\dagger = mm^\dagger$, and this is the unique weak dagger kernel of f that is a self-adjoint idempotent of A. This applies also to dagger kernels, so even in dagger kernel categories, one naturally works with weak dagger kernels. There is also a conceptual advantage to considering weak dagger kernels. Considered as a one-element dagger category, a Baer *-semigroup is a weak dagger kernel category, but not a dagger kernel category. So weak dagger kernel categories include the Baer *-semigroups that motivate matters.

Remarks 13 Crown did more than described so far. He considered dagger categories where each ordered pair of objects A, B have a distinguished set of morphisms $D(A, B)$ subject to certain requirements regarding daggers and kernels. Choosing these to be all morphisms from A to B yields a dagger kernel category. In this setting he showed that the restricted endomorphisms $D(A, A)$ of an object form a structure called a partial Baer *-semigroup [16], and therefore produce an OMP. We discuss matters related to this in detail in the next section. Crown also showed that the category of all OMLs with morphisms being Galois connections forms a dagger kernel category, and that for each OML L we have L is isomorphic to the OML produced from the Baer *-semigroup of endomorphisms of the object L in this category.

Remarks 14 Heunen and Jacobs [21] rediscovered the fact that the dagger kernels of an object in a dagger kernel category form an OML, and Jacobs [22] later rediscovered Crown's observation that the category of OMLs and Galois connections is a dagger kernel category. However Heunen and Jacobs, and later Jacobs, add a considerable amount of new information, relating dagger kernel categories to categorical treatments of logic, and in filling out finer properties of the category of OMLs and Galois connections.

The link between dagger kernel categories and Baer *-semigroups opens the path to import information obtained in the Baer *-semigroup setting to the categorical setting. Foulis has considered completeness [9], modularity [10, 11], and dimension lattices [12]; and Maeda and Maeda [26] have considered O-symmetry. Additionally, Foulis' original work was with a more general notion of Baer *-semigroups with focal ideals replacing the zero [9], and this may have a useful extension to the categorical setting. One simple instance of translating a Baer *-semigroup result to the categorical setting is given below. Its proof is obvious from Theorem 6.

Theorem 15 *Suppose that $\mathcal{C}$ is a weak dagger kernel category and each family of morphisms F with common domain has a joint weak dagger kernel, meaning there is some k such that $fk = 0$ for each $f \in F$ and if m satisfies $fm = 0$ for each $f \in F$, then $kk^\dagger m = m$. Then the OMLs associated to objects of $\mathcal{C}$ are complete.*

We turn our attention to the role played by the dagger in the above results. In coordinatizing an MOL by a *-regular ring, the underlying lattice determines the ring, the orthocomplementation gives the involution. Different orthocomplementations on the lattice give different involutions on the ring. Reversing the process, the involution on a *-regular ring does not directly give the orthocomplementation on the lattice. Rather, it picks one element from each equivalence class of the quasiorder given by $e \leq f$ iff $fe = e$, namely the unique element in the class that is self-adjoint. This in turn provides a fragment of commutativity so that on these selected elements the order becomes $e \leq f$ iff $ef = e = fe$. The orthocomplement on these elements is then given simply by $e' = 1 - e$.

In the Rickart *-ring and Baer *-semigroup setting, the involution again selects one idempotent generator for each annihilator ideal, and for these elements, provides a fragment of commutativity. The purpose of the following result is to show that the involution can be entirely eliminated if we are willing to directly specify a choice of generators of annihilator ideals satisfying a certain fragment of commutativity.

Theorem 16 *Let S be a semigroup with two-sided* $0, 1$*, and a unary operation* $'$*, and set* $S' = \{a' : a \in S\}$*. Suppose*

1. *For each $a \in S$ that a' is idempotent and $a'S = \{x : ax = 0\}$.*
2. *For $e, f \in S'$ we have $ef = fe$ iff $e'f = fe'$ iff $ef = (ef)''$.*

Then S' is an OML *with $e \leq f$ if $ef = e = fe$, orthocomplement* $'$*, and $e \wedge f = e(f'e)'$.*

Proof The proof follows along the lines of [26, pg. 172-173] where similar conditions are used in showing closed projections of a Baer *-semigroup form an OML. □

The above result shows one can do without an involution in producing an OML if certain commutativity conditions are imposed on annihilators. Still, the commutativity conditions required are not entirely natural, and having them as a consequence of the presence of an involution that is matched with the annihilators is at least a wonderful convenience. In the following section, we see the situation is improved if one seeks to produce OMPs rather than OMLs.

3 Partial Semigroups

This section is based on results of Gudder and Schelp [16] where Foulis' notion of Baer *-semigroups is extended to partial Baer *-semigroups. Gudder and Schelp showed the closed projections of an orthomodular partial Baer *-semigroup form an OMP, and every OMP can be coordinatized this way. In fact, they showed more, that every OMP can be coordinatized by such a partial semigroup whose involution is trivial, and in doing so, essentially characterized the partial semigroups arising from commuting meets on OMPs. For our purposes, this result is quite useful, and as it is a bit hidden in their paper, we develop it below, in a slightly revised form.

Definition 17 Suppose S is a set with a partially defined binary multiplication, and write xDy to mean that the product xy is defined. We say S is a partial semigroup if it satisfies the following.

If xDy and yDz, then $xDyz$ iff $xyDz$, and when defined, $x(yz) = (xy)z$.

This condition is called weak associativity. We say a partial semigroup is commutative if xDy implies yDx and $xy = yx$, idempotent if xDx and $xx = x$ for all x, and that 0 is a zero element if $xD0$ and $x0 = 0$ for all x.

Definition 18 A commutative, idempotent, partial semigroup with zero is called an orthomodular partial semigroup (abbreviated: OMP) if for each $x \in S$ there is an $x' \in S$ satisfying the following conditions.

1. xDx' and $xx' = 0$.
2. xDy implies $x'Dy$.
3. If xDy and xDz, then $xy = xz$ and $x'y = x'z$ imply $y = z$.

We call the final condition in the above definition joint monicity. The condition in the above definition are reformulated below to match with [16]. We omit the proof, but remark that it follows from later results relating such semigroups to OMPs.

Proposition 19 *A commutative, idempotent partial semigroup with zero is an* OPS *iff for each x there is an x' satisfying the following conditions.*

1. $x'' = x$
2. *For each y, xDy and $xy = 0$ iff $x'Dy$ and $x'y = y$*
3. *For each y, xDy and $xy = x$ imply $x'Dy$.*

We have two main examples of OMPs. The first is the idempotents $E(R)$ of a ring R, with the partial multiplication being the multiplication of the ring restricted to those idempotents e, f that commute under the ring multiplication. In this case for each idempotent e we can consider the idempotent $e' = 1 - e$. That e and $1 - e$ are jointly monic follows in the ring setting from the fact that $e + (1 - e) = 1$. The second main example comes from taking an OMP and considering the partial multiplication of meet restricted to pairs of commuting elements. Here p' will be the orthocomplement of p. We return to this second example later.

Lemma 20 *If $\mathcal{S} = (S, D, \cdot)$ is an* OMP*, the zero element is unique, and for each $x \in S$ the element x' satisfying the above conditions is unique.*

Proof Uniqueness of the zero is a standard argument, uniqueness of x' is a simple argument using weak associativity and joint monicity to show that if u, v satisfy the conditions for x', then $u = uv = v$. □

We recall that two elements x, y of an OMP commute, written xCy, if there are pairwise orthogonal elements a, b, c with $a \vee b = x$ and $b \vee c = y$.

Definition 21 For $\mathcal{S} = (S, D, 0)$ is an OPS and $\mathcal{T} = (T, \leq, ', 0, 1)$ is an OMP, set

1. $\Psi\mathcal{S} = (S, \leq, ', 0, 1)$
2. $\Phi\mathcal{T} = (T, C, \cdot)$

In defining $\Psi\mathcal{S}$, $\leq$ is given by $x \leq y$ if xDy and $xy = x$; and in defining $\Psi\mathcal{T}$, C is the relation of commutativity, and $\cdot$ is meet restricted to commuting elements.

Theorem 22 *For $\mathcal{S}$ an* OPS, *$\Psi\mathcal{S}$ is an* OMP. *Further, $x, y \in S$ commute in $\Psi\mathcal{S}$ iff xDy, and in this case their meet is given by $x \wedge y = xy$.*

Proof That $\leq$ is a partial order follows from idempotence, commutativity, and weak associativity, and clearly the zero 0 is the least element.

Assume $x \leq y$. Then xDy and $xy = x$. As $xDx'Dy'$ and $xx'Dy'$, weak associativity gives $xDx'y'$. So $xDx'y'$ and xDy'. Then as $x(x'y') = xy'$ and $x'(x'y') = x'y'$, the joint monicity of x, x' gives $x'y' = y'$. So $'$ is order inverting. That $'$ is period two follows using joint monicity of x, x' to show $xx'' = x''$, and the joint monicity of x', x'' to show $xx'' = x$. A simple argument shows if xDy, then xy is the greatest lower bound of x, y. In particular, $x \wedge x' = 0$, and this shows $'$ is an orthocomplementation. If x, y are orthogonal, then $x'Dy'$, so x', y' have a meet, hence x, y have a join. To show orthomodularity, it is enough to show that $x \leq y$ and $x' \wedge y = 0$ imply $y \leq x$. In this case xDy and $x'y = 0$. Joint monicity of x, x' gives $xy = y$, hence $y \leq x$.

It remains to show xDy iff x, y commute in $\Psi\mathcal{S}$. We freely use known properties of commutativity in OMPs [3]. If xDy, then $x \wedge y$ exists, so to show x commutes with y it is enough to show $x \wedge (x' \vee y') = x \wedge y'$. This follows using weak associativity and the monicity of y, y' applied to $x(xy)'$ and xy'. Suppose x, y commute. Then x', y' commute, so there are pairwise orthogonal a, b, c with $a' \wedge b' = x$ and $b' \wedge c' = y$. So $c \leq x$, giving xDc, hence xDc'. As $x \leq b'$ we have xDb', and as $b \leq c'$ we have bDc', hence $b'Dc'$. So $xDb'Dc'$. Then as $xb' = x$ and xDc' we have $xb'Dc'$, so weak associativity gives $xDb'c'$, hence xDy. □

Theorem 23 *For $\mathcal{T}$ an* OMP, *$\Phi\mathcal{T}$ is an* OPS *where x' is the orthocomplement of x.*

Proof Commutativity, idempotence, and the zero property are obvious. In any OMP, if $xCyCz$ then $x \wedge yCz$ iff $xCy \wedge z$ [16], giving weak associativity. Surely xCx' and $x \wedge x' = 0$, and it is well known that xCy implies $x'Cy$. Finally, if xCy then $(x \wedge y) \vee (x' \wedge y) = y$, and this shows joint monicity of x, x'. □

We consider OPSs as a category whose morphisms are functions that satisfy (i) $xDy \Rightarrow f(x)Df(y)$ and $f(xy) = f(x)f(y)$, and (ii) $f(x') = f(x)'$; and we consider OMPs to be a category where morphisms are functions that satisfy (i) $xCy \Rightarrow f(x)Cf(x)$ and $f(x \wedge y) = f(x) \wedge f(y)$ and (ii) $f(x') = f(x)'$. The following is then immediate from the above results.

Theorem 24 *Ψ and Φ are mutually inverse isomorphisms between the categories of* OPSs *and* OMPs *with morphisms as described above.*

Recall that an OMP is regular if for any pairwise commuting set x, y, z we have x commutes with $y \wedge z$. This is equivalent to requiring that if any two of x, y, z belong to a Boolean subalgebra, then all three belong to a Boolean subalgebra [28]. Gudder [15] showed regular OMPs are essentially the same thing as the transitive partial Boolean rings of by Kochen and Specker. From the above the following is immediate.

Corollary 25 *For an* OPS $\mathcal{S}$*, the following are equivalent.*

1. *For all* $x, y, z \in S$*, if* xDy*,* xDz *and* yDz*, then* $xDyz$*.*
2. *The* OMP $\Psi\mathcal{S}$ *is regular.*
3. *The* OMP $\Psi\mathcal{S}$ *naturally forms a transitive partial Boolean ring.*

Remarks 26 In [16] Gudder and Schelp developed the notion of orthomodular partial Baer *-semigroups. These are partial semigroups with involution and zero, not necessarily commutative or idempotent, where each element x has a projection x' satisfying conditions 2 and 3 of Proposition 19. They showed that the closed projections of an orthomodular partial Baer *-semigroup form an OMP. This result was in turn used by Crown [6] to attach OMPs to objects in various categories.

However, the closed projections of an orthomodular partial Baer *-semigroup also form an OPS if we simply forget the involution and restrict the partial multiplication to those elements where it is commutative. If our purpose is to construct an OMP from the endomorphisms of an object, showing some collection of idempotent endomorphisms forms an OPS is no more difficult, and a more general approach, than constructing an orthomodular partial Baer *-semigroup.

Axiomatics regarding OMPs are related structures have been extensively investigated. Usually, this is done in terms of an operation serving as an orthogonal join $\oplus$, or a difference operation $\ominus$. Clearly these can be of use in attaching orthomodular structures to objects in a category as well. We focused here on the axioms above as they seem most directly applicable to studying sets of idempotent endomorphisms.

4 Examples

In this section, we consider forming orthomodular structures from the objects in a category in the following situations: semiadditive categories, dagger semiadditive categories, categories with biproducts, categories with dagger biproducts, dagger kernel categories and variants, certain categories simply having finite products, and several others as well. In categories with combinations of these properties, we compare the orthomodular structures created via different techniques.

We recall some basics. A semiring is a structure $\mathcal{R} = (R, +, \cdot, 0, 1)$ that is a commutative monoid under $+$ with unit 0, a monoid under $\cdot$ with unit 1, where multiplication distributes over addition from both sides, and 0 is absorbing for multiplication. A category $\mathcal{C}$ is semiadditive [20] if it has a zero and each homset can be equipped with an addition $+$ making it a commutative monoid with unit 0 such that composition distributes over addition on both sides. A category $\mathcal{C}$ is said to have biproducts [20] if each finite family of objects has an object that is simultaneously a product and coproduct and whose injections μ_i and projections π_j satisfy $\pi_j\mu_i = \delta_{ij}$. It is known that a category with biproducts is semiadditve, and that the endomorphisms of an object in a semiadditive category form a semiring.

Definition 27 For an idempotent e in a semiring $\mathcal{R}$, we say an idempotent e' is a supplement of e if $e + e' = 1$, and $ee' = 0 = e'e$.

Proposition 28 *For a semiring $\mathcal{R}$, let S be all idempotents that have supplements, define D by setting eDf if $ef = fe$, and restrict the multiplication to D. Then $(S, D, \cdot)$ is an* OPS *whose associated* OMP *is regular.*

Proof Suppose eDf. Then $e'f = e'f(e+e') = e'fe + e'fe'$, and as $ef = fe$ and $ee' = 0$, we have $e'f = e'fe'$. Similarly $fe' = (e+e')fe' = e'fe'$, so $e'f = fe'$, giving $e'Df$. A similar calculation shows $e'Df$ implies $ef = efe = fe$, hence eDf. Then if eDf we see that ef and $ef' + e'$ are idempotent, $ef + ef' + e' = e(f + f') + e' = e + e' = 1$, and $ef(ef' + e') = 0 = (ef' + e')ef$. Thus $ef' + e'$ serves the role of $(ef)'$. So if eDf, we have ef indeed belongs to S. It is simple to see S is commutative, idempotent, has a zero, and weak associativity is a simple calculation. By construction, eDe' and $ee' = 0$. We have just shown eDf iff $e'Df$. Finally, if eDf, eDg, $ef = eg$ and $e'f = e'g$, then $f = (e+e')f = ef + e'f = eg + e'g = (e+e')g = g$. To see the associated OMP is regular, by Corollary 25, we must show if eDf, eDg and fDg, then $eDfg$, and this is trivial. □

Corollary 29 *Each object A in a semiadditive category $\mathcal{C}$ has an associated regular* OMP *constructed from its supplemented idempotent endomorphisms.*

A semiring with involution is semiring with a unary $*$ satisfying $(x+y)^* = x^* + y^*$, $(xy)^* = y^*x^*$, $x^{**} = x$, $0^* = 0$, and $1^* = 1$. We say e is self-adjoint if $e^* = e$. A dagger semiadditive category is a dagger category with semiadditive structure that satisfies $(f+g)^\dagger = f^\dagger + g^\dagger$. Clearly the endomorphisms of an object in a dagger semiadditive category form a semiring with involution. By Lemma 20, supplements in an orthomodular partial semigroup are unique, and this provides the following.

Proposition 30 *In a semiring with involution, a supplement of a self-adjoint idempotent is self-adjoint. So the self-adjoint supplemented idempotents form an* OPS, *hence a regular* OMP.

Corollary 31 *In a dagger semiadditive category, each object has an associated regular* OMP *built from its self-adjoint supplemented idempotent endomorphisms.*

A category $\mathcal{C}$ with biproducts has a semiadditive structure defined from the biproducts [20]. So each object A in a category with biproducts has an associated regular OMP consisting of its supplemented idempotent endomorphisms. These supplemented idempotent endomorphisms are related to biproduct decompositions. If $A_i \xrightarrow{\mu_i} A \xrightarrow{\pi_j} A_j$ for $i, j = 1, 2$ is a biproduct diagram, then $\mu_1\pi_1$ is an idempotent endomorphism of A whose supplement is $\mu_2\pi_2$. If idempotents in $\mathcal{C}$ split, then every supplemented idempotent endomorphism of A is of this form [19, Thm 5.7]. This provides the following.

Proposition 32 *In a category with biproducts, each object has an associated regular* OMP *constructed through its supplemented idempotent endomorphisms. If idempotents split, these supplemented idempotents correspond to binary biproduct decompositions of the object.*

Even in the case where idempotents do not split, one can build an orthomodular structure known as an orthoalgebra (abbreviated: OA) from the biproduct decompositions of an object in a category with biproducts. This is a special case of a more general construction described below, and we leave further comment until then. We remark also that the above results have analogs in the dagger category setting. A dagger biproduct category is a dagger category with a biproduct that satisfies $\pi_i^\dagger = \mu_i$. We say self-adjoint idempotents strongly split in a dagger category if each such e is written as $e = f^\dagger f$ where f is an epimorphism.

Proposition 33 *In a dagger biproduct category, each object has an associated regular* OMP *constructed through its idempotents that are self-adjoint and supplemented. If self-adjoint idempotents strongly split, these self-adjoint supplemented idempotents correspond to binary dagger biproduct decompositions of the object.*

Suppose e, e' are self-adjoint supplementary idempotent endomorphisms of an object A in a dagger semiadditive category. We claim e' is a weak dagger kernel of e and conversely. Indeed, $ee' = 0$, and if $m : M \to A$ has $em = 0$, then $m = (e + e')m = e'm = e'(e')^\dagger m$. Then the definitions of the partial orderings and orthocomplementations of the structures involved yields the following.

Proposition 34 *In a dagger semiadditive weak dagger kernel category, the* OMP *constructed from the self-adjoint idempotent endomorphisms having self-adjoint supplements is a sub-*OMP *of the* OML *of weak dagger kernels of the object.*

We turn our attention to some examples related to dagger kernels.

Remarks 35 The category $FDHilb$ of finite dimensional real Hilbert spaces with morphisms being linear maps is a dagger category with dagger being the usual adjoint of a map. This is a dagger kernel category with dagger biproducts. For an object A, the OML of dagger kernels of A is the familiar OML of closed subspaces of A. This agrees with the OMP of self-adjoint supplemented idempotent endomorphisms of A (all idempotents are supplemented here), as well as with the OMP of dagger biproduct decompositions of A.

Remarks 36 For a field K let Mat_K be the category whose objects are natural numbers and whose morphisms are matrices over K. Taking dagger as usual transpose of a matrix, this yields a dagger biproduct category [19]. This category has a dagger and kernels, but does not have dagger kernels. Still, one can consider the orthomodular structures of self-adjoint supplemented idempotents, and of biproduct decompositions, and this is done in [19].

Remarks 37 The category Rel of relations has sets as its objects with relations being morphisms. This is a dagger category with dagger being the converse of a relation. Intuitively, this is similar to taking matrices over $\{0, 1\}$ with suitable addition and multiplication. This is a dagger kernel category with dagger biproducts. For a set A, the OML of dagger kernels of A is isomorphic to the power set of A [21]. This coincides with the OMP of self-adjoint supplemented idempotents of A, and with that of dagger biproduct decompositions [19]. The category $PInj$ of partial injections is a subcategory of Rel, and similar comments hold there. This example has a long history. The connection between relations on a set A and Baer *-semigroups was already present in [11]; Crown [6] considered the category of partial injections; Abramsky and Coecke [1] used Rel as a motivating example; and both Rel and $PInj$ are key examples in the work of Heunen and Jacobs [21].

Remarks 38 Crown [6] considered the category whose objects are OMLs and whose morphisms are Galois connections $f = (f_*, f^*)$. He showed this is a dagger kernel category, and the OML of dagger kernels of an object L is isomorphic to L. In [22], Jacobs showed this category has dagger biproducts constructed through Cartesian products of the objects with suitable morphisms. The induced semiadditive structure $+$ is described using pointwise meet of functions as $f + g = (f_* \wedge g_*, f^* \wedge g^*)$.

Using [22, Lemma 6] and general comments above, the the self-adjoint idempotent endomorphisms of L that are weak dagger kernels are the $f_a = (\varphi_a', \varphi_a')$ where φ_a is the Sasaki projection for some $a \in L$. If such f_a is supplemented, it is supplemented by $f_{a'}$, meaning $f_a + f_{a'} = 1$. A calculation shows this is equivalent to a belonging to the center of L, and this in turn is equivalent to f_a coming from a biproduct decomposition. So the OML of dagger kernels of L is isomorphic to the OML L; while the OMP of self-adjoint idempotent endomorphisms having self-adjoint supplements, and the OA of biproduct decompositions, are isomorphic to the center of L.

Remarks 39 As mentioned in Remark 13, Crown's results on dagger kernel categories were more general, showing that OMPs can be attached to objects in a category where certain sets of morphisms have dagger kernels. His proof consisted of showing that a suitable set of endomorphisms of an object form an orthomodular partial Baer *-semigroup in the sense of Gudder and Schelp [16]. This extension is a valuable one, but it is likely just as easy, and more general still, to show some set of idempotent endomorphisms forms an OPS as described in Definition 18. The equivalent form of OPSs given in Proposition 19 is perhaps more convenient for this task when working with kernels.

Remarks 40 Crown [6] extended results on $FDHilb$ to any category $VB(X)$ of real or complex vector bundles over a space X. This is a dagger category with the dagger being the lifting of the usual adjoint, even a strongly compact closed category [1], but it is not a dagger kernel category as not all morphisms have kernels. Crown used the strict morphisms (ones with locally constant rank) to attach an OMP to each vector bundle A using the technique described in the previous remark. Specifically, the OMP consists of all self-adjoint idempotent endomorphisms of A that are kernels of strict morphisms having domain A. But every idempotent endomorphism p of A is strict [24, pg. 27], and if p is self-adjoint, then it is the weak dagger kernel of $1 - p$. So the OMP Crown constructs is simply the self-adjoint idempotents of the endomorphism ring of A. Further, the category $VB(X)$ has dagger biproducts, and as idempotents split, these correspond to biproduct decompositions. So the OMP constructed by Crown agrees with the one constructed through biproduct decompositions.

We have discussed several methods for constructing orthomodular structures from an object in a category. So far, all are based on using some set of idempotent endomorphisms, and fragments of ring-like properties. There is another method that is of a different nature, and based on direct product decompositions. We sketch the outline, for details see [17–19].

Definition 41 A binary product decomposition of an object A is an isomorphism $A \xrightarrow{\alpha} A_1 \times A_2$. Two such decompositions are equivalent if there are isomorphisms between their factors making the obvious diagram commute. The collection of all equivalence classes binary decompositions of A is written $\mathcal{D}A$.

Define a unary operation on $\mathcal{D}A$ by letting $[A \to A_1 \times A_2]^\perp$ be the equivalence class $[A \to A_2 \times A_1]$. We aim also for a partial binary operation $\oplus$ defined for all pairs occurring as $[A \to A_1 \times (A_2 \times A_3)]$ and $[A \to A_2 \times (A_1 \times A_3)]$ for a ternary decomposition $A \to A_1 \times A_2 \times A_3$, and whose result is $[A \to (A_1 \times A_2) \times A_3]$. Below we give sufficient conditions for these operations to be well-defined and to yield a well-behaved structure. Slightly more general conditions are given in [19].

Proposition 42 *If a category has finite products, projections are epimorphisms, and the natural diagrams $A \times B \times C$, $A \times B$, $B \times C$, B are pushouts for all objects A, B, C, then $\mathcal{D}A$ is an* OA *for each object A.*

The categories of non-empty sets, groups, topological spaces, etc. fall in the scope of the above result. Other categories also yield OAs via their decompositions. For example, in any category with biproducts, each product $A_1 \times A_2$ with projections π_1, π_2 has unique injections μ_1, μ_2 making it a biproduct. So each direct product decomposition $A \overset{\alpha}{\to} A_1 \times A_2$ gives an idempotent endomorphism $\alpha^{-1}\mu_1\pi_1\alpha$ that is supplemented by $\alpha^{-1}\mu_2\pi_2\alpha$. Calculations using the matrix calculus for biproducts show two decompositions yield the same idempotent endomorphism iff they are equivalent in the sense of Definition 41, and that $\mathcal{D}A$ is a sub-OA of the OMP built in Proposition 32 from the supplemented idempotent endomorphisms. If idempotents split, these agree.

5 Conclusions

Attaching orthomodular structures to objects in a category is a useful task in categorical treatments of the foundations of quantum mechanics. It not only fleshes out the theory to include propositions of a single system, but points to finer properties required of the tensor so that it interacts properly with these structures. The existing quantum logic literature may be of considerable use, and we hope to have helped point the path to this work.

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