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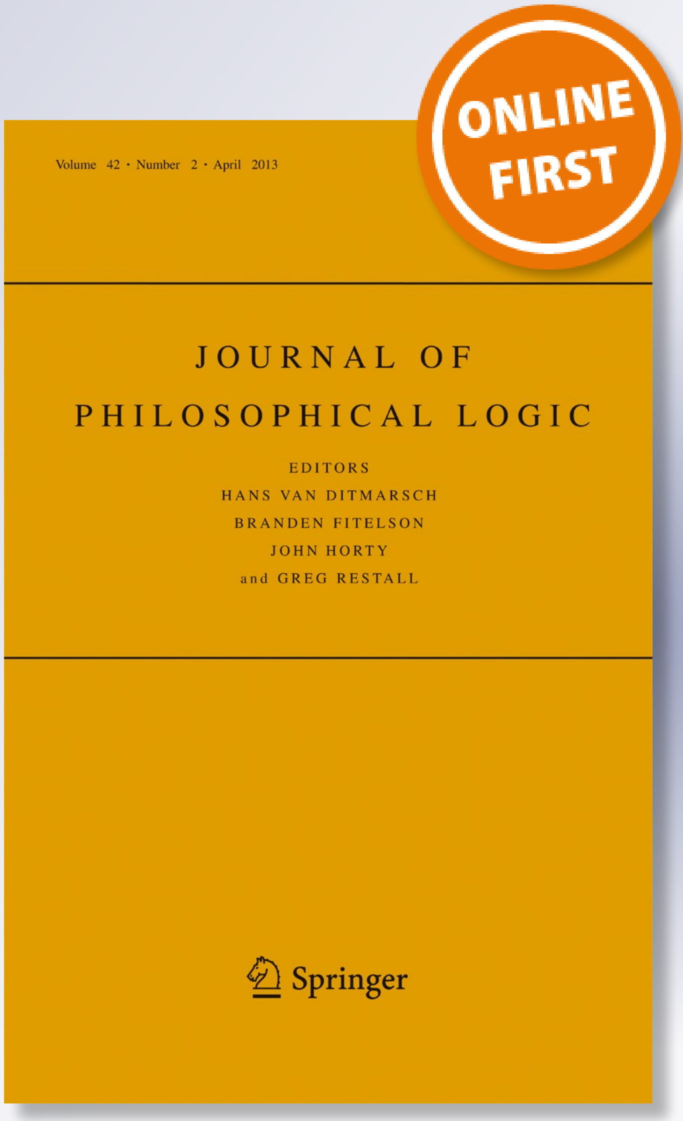

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Decidability of the Equational Theory of the Continuous Geometry $CG(\mathbb{F})$

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Abstract For $\mathbb{F}$ the field of real or complex numbers, let $CG(\mathbb{F})$ be the continuous geometry constructed by von Neumann as a limit of finite dimensional projective geometries over $\mathbb{F}$. Our purpose here is to show the equational theory of $CG(\mathbb{F})$ is decidable.

Keywords Continuous geometry · Equational theory · Decidability · Orthomodular lattice

Let $\mathbb{F}$ be the field of real or complex numbers. For $n \geq 1$, the subspaces of the n-dimensional inner product space $\mathbb{F}^n$ form a modular ortholattice $PG_{n-1}(\mathbb{F})$, or simply PG_{n-1}. This lattice has a normalized dimension function $d_n : PG_{n-1} \to [0, 1]$ that associates to a subspace A, its dimension divided by n. von Neumann [8] showed there is an embedding $PG_{n-1} \hookrightarrow PG_{2n-1}$ that preserves normalized dimensions. So the inductive limit of the chain $PG_1 \hookrightarrow PG_3 \hookrightarrow PG_7 \hookrightarrow \cdots$ yields a modular ortholattice $PG_\infty(\mathbb{F})$, or simply PG_∞. This ortholattice PG_∞ also has a dimension function, so is a metric lattice [1], and its metric space completion is a complete modular ortholattice $CG(\mathbb{F})$, or simply CG.

This CG was von Neumann's first example of a continuous geometry. Our purpose is to show the equational theory of CG is decidable. The key tools are results of Herrmann and Roddy [4] on equations in modular ortholattices, and of Dunn, Hagge, Moss, and Wang [7] showing the first order theory of each PG_n is decidable.

This manuscript was prepared after the Quantum Logic Inspired by Quantum Computation Workshop in Bloomington IN in 2009. Since its preparation, the author has become aware of independent work by Christian Herrmann [3].

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Definition 1 Let *Eq(L)* be the set of equations $s \approx t$ holding in L.

Lemma 2 $Eq(CG) = Eq(PG_\infty)$.

Proof As PG_∞ is a subalgebra of *CG*, any equation holding in *CG* holds in PG_∞. Suppose $s(x_1, \ldots, x_n) \approx t(x_1, \ldots, x_n)$ holds in PG_∞. By [1], the operations of meet, join, and orthocomplementation on the metric lattice PG_∞ are uniformly continuous, and their continuous extensions to the metric completion *CG* are its meet, join, and orthocomplementation. The terms *s,t* give uniformly continuous *n*-ary operations on PG_∞ and *CG*, with the operations on *CG* extending those on PG_∞. As $s \approx t$ holds in PG_∞, the operations on *CG* agree on the dense subspace PG_∞^n of CG^n, and by continuity, agree on CG^n. So $s \approx t$ holds also in *CG*.

Lemma 3 $Eq(PG_1) \supset Eq(PG_2) \supset \cdots$ *is a strictly decreasing chain of sets whose intersection is* $Eq(PG_\infty)$.

Proof If $m \leq n$, PG_m is isomorphic to an interval of PG_n, so by [6] PG_m is a homomorphic image of a subalgebra of PG_n, giving $Eq(PG_m) \supseteq Eq(PG_n)$. If $m < n$, PG_n is of greater height than PG_m, so by Łoś' theorem is not a homomorphic image of a subalgebra of an ultrapower of PG_m. As PG_n is subdirectly irreducible, Jónsson's theorem [2] shows PG_n does not belong to the variety generated by PG_m, so $Eq(PG_m)$ strictly contains $Eq(PG_n)$. As PG_∞ is the union of the algebras PG_{2^n-1}, it follows that $Eq(PG_\infty)$ is the intersection of this decreasing chain.

We require two results from [4] that let us push a failure of $s \approx t$ in PG_∞ down to some PG_m where *m* is determined solely from the form of the equation.

Lemma 4 *Let* $s(x_1, \ldots, x_n)$ *and* $t(x_1, \ldots, x_n)$ *be ortholattice terms with variables among* $x_1, \ldots, x_n$, *and suppose s,t combined have a total of k occurrences of* $\wedge, \vee$. *Then there is a bounded lattice term* $r(z_1, \ldots, z_{2n})$ *with* $2n$ *variables and* $3k + 5$ *occurrences of* $\wedge, \vee$ *so that for any orthomodular lattice L the following are equivalent.*

1. $s(x_1, \ldots, x_n) \approx t(x_1, \ldots, x_n)$ *holds in L.*
2. $z_1 \leq z_2', \ldots, z_{2n-1} \leq z_{2n}' \Rightarrow r(z_1, \ldots, z_{2n}) \approx 0$ *holds in L.*

Proof The proof is that of [4, Lemma 3.1]. Set $u = (s \wedge (s' \vee t')) \vee (t \wedge (s' \vee t'))$, and note orthomodularity shows $s \approx t$ is equivalent to $u \approx 0$. Use DeMorgan's laws to rewrite *u* as a bounded lattice term in the variables $x_1, x_1', \ldots, x_n, x_n'$, and the result follows as bounded lattice terms are monotone in their arguments. As $s \approx t$ has a total of *k* occurrences of $\wedge, \vee$, then *u* has a a total of $3k + 5$ occurrences of $\wedge, \vee$, and hence so does *r*.

Lemma 5 *Suppose L is a complemented, atomic, modular lattice, $r(x_1, \ldots, x_n)$ is a bounded lattice term whose variables are among $x_1, \ldots, x_n$ and having k occurrences of $\wedge, \vee$, a is an atom of L, and $b_1, \ldots, b_n \in L$ are such that $a \leq r(b_1, \ldots, b_n)$. Then there are $c_1, \ldots, c_n \in L$ of height at most 2^k with $c_i \leq b_i$, and $a \leq r(c_1, \ldots, c_n)$.*

Proof The inductive proof is that of [4, Lemma 2.1] with bookkeeping of the bounds on heights. If $k = 0$ then r is either the constant 1 or a variable x_j and the c_i can all be chosen to be either 0 or a. If $r = r_1 \wedge r_2$, the inductive hypothesis gives $c_{1i} \leq b_i$ of height at most 2^{k-1} with $a \leq r_1(c_{11}, \ldots, c_{1n})$ and $c_{2i} \leq b_i$ of height at most 2^{k-1} with $a \leq r_2(c_{21}, \ldots, c_{2n})$. Set $c_i = c_{1i} \vee c_{2i}$. If $r = r_1 \vee r_2$ things are trivial if $a \leq r_1(b_1, \ldots, b_n)$ or $a \leq r_2(b_1, \ldots, b_n)$. Otherwise, modularity is used to show there are atoms a_1, a_2 with $a_1 \leq r_1(b_1, \ldots, b_n)$, $a_2 \leq r_2(b_1, \ldots, b_n)$, and $a \leq a_1 \vee a_2$. The inductive hypothesis gives $c_{1i} \leq b_i$ of height at most 2^{k-1} with $a_1 \leq r_1(b_1, \ldots, b_n)$, and $c_{2i} \leq b_i$ of height at most 2^{k-1} with $a_2 \leq r_2(b_1, \ldots, b_n)$. Set $c_i = c_{1i} \vee c_{2i}$.

Proposition 6 *Suppose $s(x_1, \ldots, x_n)$ and $t(x_1, \ldots, x_n)$ are ortholattice terms whose variables are among $x_1, \ldots, x_n$ and that together they have k total occurrences of $\wedge, \vee$. Then the following are equivalent.*

1. $s \approx t$ *holds in* CG.
2. $s \approx t$ *holds in* PG_∞.
3. $s \approx t$ *holds in* PG_{m-1} *where* $m = 2n2^{3k+5}$.

Proof The equivalence of the first two statements is Lemma 1, and Lemma 2 shows the second implies the third.

To show the third imples the second, let $r(z_1, \ldots, z_{2n})$ be the bounded lattice term given in Lemma 4 with $2n$ variables and $3k + 5$ occurrences of $\wedge, \vee$. If $s \approx t$ does not hold in PG_∞, then by Lemma 3 it fails in PG_q for some q. So there are $b_1, \ldots, b_{2n}$ in PG_q with $b_1 \leq b_2', \ldots, b_{2n-1} \leq b_{2n}'$ and $r(b_1, \ldots, b_{2n}) \neq 0$. As PG_q is a complemented, atomic, modular lattice, there is an atom a with $a \leq r(b_1, \ldots, b_n)$. Lemma 5 provides $c_1, \ldots, c_{2n}$ in PG_q of height at most 2^{3k+5}, with $a \leq r(c_1, \ldots, c_{2n})$, and $c_i \leq b_i$. Let $c = c_1 \vee \cdots c_{2n}$, and note the height h of c is at most $m = 2n2^{3k+5}$.

With the orthocomplementation $x^- = x' \wedge c$, the interval [*0*,*c*] of PG_q forms an ortholattice isomorphic to PG_{h-1}. As $c_1 \leq c_2^-, \ldots, c_{2n-1} \leq c_{2n}^-$ and $r(c_1, \ldots, c_{2n}) \neq 0$, we have $s \approx t$ fails in the interval [*0*,*c*], so it fails in PG_{h-1}, and therefore also fails in PG_{m-1}.

With the result of Dunn, Hagge, Moss, and Wang [7] giving the decidability of the first order theory of PG_{m-1}, the equational theory of CG is decidable. We do not know if the full first order theory is decidable, or if the theory of quasi-identities (formulas of the form $s_1 \approx t_1 \,\&\cdots\&\, s_n \approx t_n \Rightarrow s \approx t$) in

CG is decidable. While perhaps not directly related to this problem, we include a small observation about quasi-identities in the setting of finite height orthomodular lattices.

Proposition 7 *Suppose L is an orthomodular lattice with an upper bound h on the lengths of its chains. If the equational theory of L is decidable, then the theory of quasi-identities in L is decidable.*

Proof This relies on results in [5]. There, the notion of a partial matrix in an orthomodular lattice is developed. A partial matrix is a doubly indexed finite sequence a_{ij} of elements of L. A partial matrix is admissible if it satisfies certain technical properties, and the size of a partial matrix is the sequence of lengths of its rows. The sizes are linearly ordered in the lexicographical ordering, and if L has height at most h, there is a finite upper bound on the possible sizes of admissible partial matrices in L. Begin with the largest possible size σ, as determined by h. Then [5, Lemma 6] provides a term t_σ so that $t_\sigma \approx 0$ holds in L if, and only if, L has no admissible partial matrix of this size. If L has none of the largest size, let σ' be the next largest size and produce $t_{\sigma'}$ to determine if L has an admissible partial matrix of this size. Continuing in this way, as L is guaranteed to have an admissible partial matrix, we can use the decidability of the equational theory of L to determine the maximal size of an admissible partial matrix in L, and we call this σ.

Then [5, Lemma 6] provides a term $t(x_1, \ldots, x_q)$, where q is the number of entries in a maximal size admissible partial matrix, such that $t(a_1, \ldots, a_q) = 1$ if $a_1, \ldots, a_q$ comprise the entries of an admissible partial matrix in L listed in the natural order, and $t(a_1, \ldots, a_q) = 0$ otherwise. Also, [5, Lemma 2], there is a term $p(x_1, \ldots, x_q, y)$ so that for any $a_1, \ldots, a_q$ in L, $p(a_1, \ldots, a_q, 0) = 0$ and if $a_1, \ldots, a_q$ naturally form an admissible partial matrix of size σ, then $p(a_1, \ldots, a_q, b) = 1$ for any $b \neq 0$.

Orthomodularity shows any equation $s \approx t$ is equivalent to one of the form $u \approx 0$. So any quasi-identity $s_1 \approx t_1 \,\&\cdots\&\, s_n \approx t_n \Rightarrow s \approx t$ is equivalent in orthomodular lattices to one of the form $u_1 \approx 0 \,\&\cdots\&\, u_n \approx 0 \Rightarrow v \approx 0$, and by taking $u = u_1 \vee \cdots \vee u_n$, to one of the form $u \approx 0 \Rightarrow v \approx 0$. We claim the following formulas are equivalent in L.

1. $u(y_1, \ldots, y_n) \approx 0 \Rightarrow v(y_1, \ldots, y_n) \approx 0$
2. $t(x_1, \ldots, x_q) \wedge p(x_1, \ldots, x_q, u(y_1, \ldots, y_n))' \wedge v(y_1, \ldots, y_n) \approx 0$

If the first fails, there are $b_1, \ldots, b_n$ with $u(b_1, \ldots, b_n) = 0$ and $v(b_1, \ldots, b_n) \neq 0$. Then the second fails when the admissible partial matrix $a_1, \ldots, a_q$ is substituted for $x_1, \ldots, x_q$ and $b_1, \ldots, b_n$ are substituted for $y_1, \ldots, y_n$ as the first and second terms in the second formula evaluate to 1 and the third is not 0. If the second formula fails when some $c_1, \ldots, c_q$ are substituted for $x_1, \ldots, x_q$ and some $b_1, \ldots, b_n$ are substituted for $y_1, \ldots, y_n$, then $t(c_1, \ldots, c_q) \neq 0$, so the properties of t give that $c_1, \ldots, c_q$ is an admissible partial matrix. Also $p(c_1, \ldots, c_q, u(b_1, \ldots, b_n)) \neq 1$, and as $c_1, \ldots, c_q$ is admissible, the properties of p give that $u(b_1, \ldots, b_n) = 0$. Finally, $v(b_1, \ldots, b_n) \neq 0$, and this gives a failure of the first formula.

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