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Partial Orders on Fuzzy Truth Value Algebras

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The elements of the truth value algebra of type-2 fuzzy sets are the mappings of the unit interval into itself, with operations given by various convolutions of the pointwise operations. This algebra can be specialized and generalized in various interesting ways. First, we consider the more general case of all mappings of a bounded chain with an involution into a complete chain, and delimit some of the properties of the resulting algebra. These include two binary operations each of which give a partial order on the elements of that algebra. These partial orders and their intersection are the principal objects of interest. We specialize this situation in two cases: (1) all mappings of the unit interval into itself, the original version of the truth value algebra of type-2 fuzzy sets introduced by Zadeh, and (2) all mappings of a finite chain into another finite chain. Again, each of these two cases yields two partial orders on the elements of the resulting algebras, and in each case, our principal interest is in these partial orders and their intersection.

Keywords: Truth value algebra of type-2 fuzzy sets; partial order; bisemilattice; lattice.

1. Introduction

The truth value algebra of type-2 fuzzy sets was introduced by Zadeh in Ref. 1, and has been heavily investigated both as a mathematical object and for use in applications. Its elements are all the mappings of the unit interval into itself and its operations are convolutions of operations on the unit interval. Some of its basic properties are in Ref. 2 for example.

The basic construction of Zadeh is applicable in a very general setting. Suppose I is a complete lattice and J is an algebra in the sense of universal algebra. For an n-ary operation $\Gamma^J : J^n \to J$ of J define an n-ary operation Γ^{I^J} on the set I^J of all functions f from J to I by setting

$$\left(\Gamma^{I^J}(f_1, ..., f_n)\right)(x) = \bigvee\{f_1(x_1) \wedge \cdots \wedge f_n(x_n) : \Gamma^J(x_1, \ldots, x_n) = x\}$$

This operation Γ^{I^J} is called the **convolution** of Γ^J with respect to the meet and

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join of I. Convoluting all operations of J yields an algebra I^J that has the same type as the algebra J.

Here we stay close to Zadeh's path and consider the case where I is a complete chain and J is a bounded chain with involution. In both cases, we denote the bounds by 0 and 1, the sup and inf by $\vee$ and $\wedge$, and in J the involution by $'$.

Definition 1. Given I and J as described above, consider the set I^J of all functions from J into I furnished with the operations given below: the binary operations $\sqcup$ and $\sqcap$, the unary operation $'$, and the nullary operations $\bar{1}$ and $\bar{0}$.

$$(f \sqcup g)(x) = \bigvee_{y \vee z = x} (f(y) \wedge g(z))$$

$$(f \sqcap g)(x) = \bigvee_{y \wedge z = x} (f(y) \wedge g(z))$$

$$f'(x) = \bigvee_{y' = x} f(y) = \bigvee_{y = x'} f(y) = f(x')$$

$$\bar{0}(x) = \begin{cases} 1 \text{ if } x = 0 \\ 0 \text{ if } x \neq 0 \end{cases} \qquad \bar{1}(x) = \begin{cases} 0 \text{ if } x \neq 1 \\ 1 \text{ if } x = 1 \end{cases}$$

There are two other operations on the functions I^J, namely pointwise max and min of functions. We also denote these by $\vee$ and $\wedge$, respectively. One can express the operations $\sqcup$ and $\sqcap$ in terms of these pointwise operations and two auxiliary unary operations, making it rather easy to determine some equational properties of the algebra I^J. Details may be found in Ref. 2 where these results are established in the case where I and J are the unit interval. They remain valid when J is a bounded chain and I is a complete chain, and even extend to the situation where I is a complete lattice satisfying the infinite distributive law $x \wedge \bigvee y_i = \bigvee (x \wedge y_i)$.

Definition 2. For $f \in I^J$, let f^L and f^R be the elements defined by

$$f^L(x) = \bigvee_{y \leq x} f(y)$$

$$f^R(x) = \bigvee_{y \geq x} f(y)$$

The operations $\sqcup$ and $\sqcap$ on I^J can be expressed in terms of the pointwise max and min of functions, as follows.[2,4]

Theorem 1. *The following hold for all $f, g \in I^J$.*

(1) $f \sqcup g = (f \wedge g^L) \vee (f^L \wedge g) = (f \vee g) \wedge (f^L \wedge g^L)$
(2) $f \sqcap g = (f \wedge g^R) \vee (f^R \wedge g) = (f \vee g) \wedge (f^R \wedge g^R)$

It has been shown[2,4] that I^J satisfies the following equations.

Proposition 1. *Let $f,\ g,\ h \in I^J$.*

(1) $f \sqcup f = f;\ f \sqcap f = f$

(2) $f \sqcup g = g \sqcup f$; $f \sqcap g = g \sqcap f$
(3) $f \sqcup (g \sqcup h) = (f \sqcup g) \sqcup h$; $f \sqcap (g \sqcap h) = (f \sqcap g) \sqcap h$
(4) $f \sqcup (f \sqcap g) = f \sqcap (f \sqcup g)$
(5) $\bar{1} \sqcap f = f$; $\bar{0} \sqcup f = f$
(6) $f'' = f$
(7) $(f \sqcup g)' = f' \sqcap g'$; $(f \sqcap g)' = f' \sqcup g'$

A further property of these operations will be needed [2, Proposition 6].

Proposition 2. *For* $f, g \in I^J$

(1) $(f \sqcup g)^L = f^L \sqcup g^L$ *and* $(f \sqcup g)^R = f^R \sqcup g^R$.
(2) $(f \sqcap g)^L = f^L \sqcap g^L$ *and* $(f \sqcap g)^R = f^R \sqcap g^R$.

Since each of the operations $\sqcup$ and $\sqcap$ on I^J is idempotent, commutative and associative, they each induce partial orders as given by the following definition.

Definition 3. $f \sqsubseteq_{\sqcup} g$ if $f \sqcup g = g$, and $f \sqsubseteq_{\sqcap} g$ if $f \sqcap g = f$.

We often call $\sqsubseteq_{\sqcup}$ the **join order** and $\sqsubseteq_{\sqcap}$ the **meet order**.

Remark 1. It is easy to see that the operations $\sqcup$ and $\sqcap$ do not give the same partial orders. For example, $f \sqsubseteq_{\sqcap} \overline{1}$ since $f \sqcap \overline{1} = f$, but it is not true that $f \sqsubseteq_{\sqcup} \overline{1}$ since $f \sqcup \overline{1} = (f \vee \overline{1}) \wedge f^L \wedge \overline{1}^L$ has an entry $f^L(1)$ which may be less than 1. There are many easy ways to construct such examples.

The inequalities in Definition 3 may be expressed in terms of pointwise order of functions.[2]

Theorem 2. *For* f, g *in* I^J *we have*

(1) $f \sqsubseteq_{\sqcup} g$ *if and only if* $f \wedge g^L \leq g \leq f^L$
(2) $f \sqsubseteq_{\sqcap} g$ *if and only if* $f^R \wedge g \leq f \leq g^R$

The following property of these orders (,[2] Proposition 15) provides a link between the algebras I^J and the study of bisemilattices.

Theorem 3. *Each partial order* $\sqsubseteq_{\sqcup}$ *and* $\sqsubseteq_{\sqcap}$ *induces a semilattice. That is,* $f \sqcup g$ *is the supremum of the two elements* f *and* g*, under the partial order* $\sqsubseteq_{\sqcup}$*, and* $f \sqcap g$ *is the infimum of* f *and* g *under the partial order* $\sqsubseteq_{\sqcap}$*.*

Viewing a partial order on a set A as a subset of $A \times A$ that is reflexive, antisymmetric, and transitive, it is clear that the intersection of two partial orders on a set A is again a partial order on A.

Definition 4. Let $\sqsubseteq$ be the intersection of the join $\sqsubseteq_{\sqcup}$ and meet $\sqsubseteq_{\sqcap}$ orders. We call $\sqsubseteq$ the **double order**.

From Theorem 2 we get the following.

Theorem 4. *$f \sqsubseteq g$ if and only if $f \wedge g^L \leq g \leq f^L$ and $f^R \wedge g \leq f \leq g^R$.*

In this paper, we investigate the join, meet, and double orders in three cases: the general case where I is a complete chain and J is a bounded chain with involution, the case $I = J = [0, 1]$, and the case where I and J are finite chains.

The general case is considered in Sec. 2. We show that the set of convex normal functions is a subalgebra of I^J on which the join, meet, and double orders coincide. This subalgebra forms a distributive lattice under $\sqcup$ and $\sqcap$. We show there is a retraction Γ from I^J to this subalgebra of convex normal functions taking a function f to its convex hull $f^L \wedge f^R$, and that this retraction is order preserving with respect to the join, meet and double orders. We show also that several other collections of functions are subalgebras of I^J, such as the increasing functions, and the collection S_k of functions of given height k. Finally, we show that under the double order, the poset I^J is the sum of the disjoint posets S_k where k ranges over all possible values from I.

In Sec. 3 we consider the case where I and J are both the unit interval, the case encountered most directly in studies of type-2 fuzzy sets. Here we show that $[0, 1]^{[0,1]}$ is not a lattice under either the join or meet order. Results from the general case show it is not a lattice under the double order. The continuous functions form a subalgebra of $[0, 1]^{[0,1]}$, but again do not form a lattice under the join, meet or double order.

In Sec. 4 we give a detailed study of the case when I and J are finite chains m and n. In general, I^J is a bounded meet semilattice in the order $\sqsubseteq_\sqcap$ with finite meets given by $\sqcap$. Since m and n are finite, this meet semilattice m^n is a lattice with the join of two elements given by the meet of their upper bounds. Similarly m^n is a join semilattice in the order $\sqsubseteq_\sqcup$ with finite joins given by $\sqcup$, and meets given by the join of a set of lower bounds. We show that the set of convex functions is a sublattice of m^n under both the join order and the meet order, as is the set of normal functions. The intersection of these sets, the set of convex normal functions, is shown to be a distributive sublattice of m^n under the join order and under the meet order. This is the content of Subsecs. 4.1 and 4.2.

In Subsec. 4.3 we provide an efficient algorithm to compute directly from elements f and g in m^n, the meet of f and g in the join order. The argument above giving the existence of this meet as the join of the finite set of lower bounds requires searching through all m^n elements to be implemented. The algorithm in this subsection computes this meet directly from f and g and is of order n.

In Subsec. 4.4 we show that the functions S_k in m^n of some given height k form a sublattice of m^n under the double order. Frankly, we have no idea why this result should be true past the quite complex algorithm that efficiently yields joins and meets. We further show that the convolution of the (unique) involution of the finite chain m gives an involution on the lattice S_k. So this construction yields a rather interesting source of generally non-distributive involutive lattices. In Subsec. 4.5 we give a surprisingly simple algorithm to generate covers of an element f of m^n in the double order.

The paper concludes with Sec. 5. Here we state several open problems, and provide diagrams of several examples of the lattices m^n under the join, meet and double orders.

We hope these results contribute to a basic understanding of the algebra $[0,1]^{[0,1]}$ used as the truth value algebra for type-2 fuzzy sets. Also, as many practical applications involve only a limited range of values for the domain and range of functions, results on the algebras m^n are applicable here. Our study of the double order is motivated by our desire to place type-2 fuzzy sets in a categorical setting as in Ref. 3. Finally, the ordered structures arising in this paper seem to be of independent interest. Finite chains m^n under the double order produce an interesting and non-trivial family of finite involutive lattices, when there seems absolutely no rational reason for them to do so.

2. The General Case

Here we discuss properties of the algebras I^J in the case where I is a complete chain and J is a bounded chain with involution. So we consider $\sqcap, \sqcup, ', \overline{0}, \overline{1}$ to be basic operations of I^J as in Definition 1. We note that results not explicitly involving this involution are valid for any bounded chain J. We will consider several subalgebras of these algebras, or their reducts, and relate these to the join, meet, and double orders. Of course, these results are applicable to the more specialized situations we consider later.

Definition 5. For $f \in I^J$, the **height** of f is $\bigvee_{x\in J} f(x)$. The elements of maximal height are called **normal**.

The following is established in Ref. 2.

Proposition 3. *The collection of normal elements is a subalgebra of* I^J.

In the following, we use a definition of convex function common in fuzzy set theory (based on α-sets being convex). We note that this differs from the definition often used in analysis.

Definition 6. A function $f \in I^J$ is **convex** if for all $x \le y \le z$ in J, $f(y) \ge f(x) \wedge f(z)$. Equivalently, $f = f^L \wedge f^R$.

The fact that convexity of f is equivalent to the condition $f = f^L \wedge f^R$ may be found in Refs. 2 and 4. The proof of the following is found in Ref. 2.

Proposition 4. *The convex elements are a subalgebra of* I^J.

As an intersection of subalgebras is a subalgebra, the set of convex normal functions forms a subalgebra of I^J. This subalgebra is very well behaved, and perhaps forms a natural setting to consider Zadeh's algebra of truth values. We collect below several results about this subalgebra, established in Refs. 2 and 4.

Theorem 5. *The set of convex normal functions is a subalgebra of I^J. On this subalgebra, the join and meet orders $\sqsubseteq_\sqcup$ and $\sqsubseteq_\sqcap$ coincide. This subalgebra forms a bounded distributive lattice under the operations $\sqcap$ and $\sqcup$, and a DeMorgan algebra with its negation $'$.*

If I and J are both the unit interval, the lattice of convex normal functions is complete. It further has a natural quotient that is not only complete but completely distributive. These facts are discussed in Refs. 5 and 6. We next consider how the subalgebra of convex functions sits inside the poset I^J. The following result is substantially contained in Ref. 7.

Theorem 6. *There is a retraction Γ from the algebra I^J to its subalgebra of convex functions given by*

$$\Gamma f = f^L \wedge f^R$$

Further, $f \sqsubseteq_\sqcup \Gamma f \sqsubseteq_\sqcap f$ for each f.

Proof. It follows from Definition 6 that Γf is convex, and if f is convex, then $f = \Gamma f$. So Γ is idempotent and its image is the set of convex functions. Clearly Γ fixes the constants $\overline{0}$ and $\overline{1}$, and as $f'(x) = f(x')$, it is easily seen that Γ is compatible with the involution. To show Γ is a retraction, it remains to show that it preserves $\sqcup$ and $\sqcap$.

Note that

$$(f^L \wedge f^R)^L = f^L \text{ and } (f^L \wedge f^R)^R = f^R. \tag{1}$$

This follows since $f \leq f^L \wedge f^R \leq f^L$, so $f^L \leq (f^L \wedge f^R)^L \leq f^{LL} = f^L$, and symmetrically for the other statement.

Using this observation, Theorem 2, and Proposition 2 we have

$$\begin{aligned}
\Gamma(f \sqcup g) &= (f \sqcup g)^L \wedge (f \sqcup g)^R \\
&= (f^L \sqcup g^L) \wedge (f^R \sqcup g^R) \\
&= [(f^L \vee g^L) \wedge f^{LL} \wedge g^{LL}] \wedge [(f^R \vee g^R) \wedge f^{RL} \wedge g^{RL}] \\
&= f^L \wedge g^L \wedge (f^R \vee g^R)
\end{aligned}$$

$$\begin{aligned}
\Gamma f \sqcup \Gamma g &= (f^L \wedge f^R) \sqcup (g^L \wedge g^R) \\
&= [f^L \wedge f^R \wedge (g^L \wedge g^R)^L] \vee [(f^L \wedge f^R)^L \wedge g^L \wedge g^R] \\
&= (f^L \wedge f^R \wedge g^L) \vee (f^L \wedge g^L \wedge g^R) \\
&= f^L \wedge g^L \wedge (f^R \vee g^R)
\end{aligned}$$

This shows that Γ preserves $\sqcup$. The argument that it preserves $\sqcap$ is nearly identical. Further, Theorem 2 and equation (1) provide $f \sqcup \Gamma f = (f \wedge (f^L \wedge f^R)^L) \vee (f^L \wedge f^L \wedge f^R) = f \vee (f^L \wedge f^R) = f^L \wedge f^R = \Gamma f$. The argument that $f \sqcap \Gamma f = \Gamma f$ is similar. □

The result above shows that Γ is order preserving with respect to the join and meet orders, hence also with respect to the double order. We consider next a further property of convex elements with respect to these orders.

Proposition 5. *If f is convex and g has height at least that of f, then*

$$f \sqcap g \sqsubseteq f \sqsubseteq f \sqcup g$$

Proof. We show $f \sqsubseteq f \sqcup g$. The argument for $f \sqcap g \sqsubseteq f$ is similar. Since $f \sqcup (f \sqcup g) = (f \sqcup g)$, we have f is less than $f \sqcup g$ in the join order. It remains to show that f is less than $f \sqcup g$ in the meet order.

$$\begin{aligned}
f \sqcap (f \sqcup g) &= [f \vee (f \sqcup g)] \wedge f^R \wedge (f \sqcup g)^R \\
&= [f \vee ((f \vee g) \wedge f^L \wedge g^L)] \wedge f^R \wedge (f^R \sqcup g^R) \\
&= [f \vee ((f \vee g) \wedge f^L \wedge g^L)] \wedge f^R \wedge (f^R \vee g^R) \wedge f^{RL} \wedge g^{RL} \\
&= [f \vee ((f \vee g) \wedge f^L \wedge g^L)] \wedge f^R \\
&= (f \wedge f^R) \vee [((f \vee g) \wedge f^L \wedge g^L) \wedge f^R] \\
&= f \vee [(f \vee g) \wedge f^L \wedge g^L \wedge f^R] \\
&= f \vee [(f \vee g) \wedge f \wedge g^L] \\
&= f
\end{aligned}$$

This concludes the proof. □

We next turn our attention to several other subalgebras of (reducts of) I^J.

Proposition 6. *For I and J bounded chains with bounds $0, 1$, and I complete, set*

$$S = \{f \in I^J : f(0) = 1\}$$

Then S is a subalgebra of the algebra $(I^J, \sqcup, \sqcap)$. Further, S forms a lattice under the join order with joins given by $\sqcup$ and meets given by pointwise meet $\wedge$.

Proof. That S is a subalgebra of $(I^J, \sqcup, \sqcap)$ follows easily from Theorem 1. Suppose that f, g, h are in S. Then by Theorem 2 we have h is a lower bound of f, g under the join order if and only if

$$\begin{aligned}
h \wedge f^L \leq f \leq h^L \\
h \wedge g^L \leq g \leq h^L
\end{aligned}$$

Since f^L, g^L, h^L are the constant function 1, these conditions are equivalent to having $h \leq f, g$ in the pointwise order. Let k be the pointwise meet $f \wedge g$ and note that k belongs to S and is a lower bound of f, g under $\sqsubseteq_\sqcup$. If h is another such lower bound of f, g, then $h \leq k$, so

$$h \wedge k^L \leq k \leq h^L$$

Thus $h \sqsubseteq_\sqcup k$, showing k is the greatest lower bound of f, g under $\sqsubseteq_\sqcup$. □

Proposition 7. *For I a complete chain and J a bounded chain, set*

$$T = \{f \in I^J : f \text{ is monotone increasing}\}$$

Then T is a subalgebra of the algebra $(I^J, \sqcup, \sqcap)$. Further, T forms a lattice under the join order with joins given by $\sqcup$ and meets given by pointwise join $\vee$.

Proof. That T is a subalgebra again follows from Theorem 1. Suppose that f, g, h are monotone increasing. Then h is a lower bound of f and g under the join order if and only if

$$h \wedge f^L \leq f \leq h^L$$
$$h \wedge g^L \leq g \leq h^L$$

Since $f = f^L, g = g^L$ and $h = h^L$, these conditions are equivalent to having $f, g \leq h$ in the pointwise order. Let k be the pointwise join $f \vee g$ and note that k is monotone increasing and is a lower bound of f and g under $\sqsubseteq_\sqcup$. If h is another such lower bound of f and g, then $k \leq h$, so

$$h \wedge k^L \leq k \leq h^L$$

Thus $h \sqsubseteq_\sqcup k$, showing that k is the greatest lower bound of f and g under $\sqsubseteq_\sqcup$. □

To conclude this section, we make some basic observations about the double order in the general setting.

Proposition 8. *If two elements f and g in I^J are comparable in the double order $\sqsubseteq$, then they have the same height.*

Proof. By Theorem 4, if $f \sqsubseteq g$, then $g \leq f^L$ and $f \leq g^R$. Since $f^{LR} = f^{RL}$ is the height of f, it follows that f and g have the same height. □

Proposition 9. *For I a complete chain, J a bounded chain, and $k \in I$, set*

$$S_k = \{f \in I^J : f \text{ has height } k\}$$

Then S_k is a subalgebra of the algebra $(I^J, \sqcup, \sqcap)$. The least element of S_k is the function u given by $u(0) = k$ and $u(i) = 0$ otherwise. The greatest element is the function v given by $v(1) = k$ and $v(i) = 0$ otherwise.

Proof. That S_k is closed under the operations $\sqcup$ and $\sqcap$ follows from the facts that $(f \sqcup g)^L = f^L \sqcup g^L$, $(f \sqcup g)^R = f^R \sqcup g^R$, $(f \sqcap g)^L = f^L \sqcap g^L$, and $(f \sqcap g)^R = f^R \sqcap g^R$, established in Ref. 2 , and that the height of f is given by f^{LR}. That the indicated elements are bounds follows from a simple computation using Theorem 1. □

Corollary 1. *The partially ordered set $(I^J, \sqsubseteq)$ is the disjoint union of its incomparable bounded subposets $(S_k, \sqsubseteq)$, $k \in I$; that is*

$$(I^J, \sqsubseteq) = \bigoplus_{k \in I} \mathbf{S}_k$$

where $\bigoplus$ denotes disjoint union. In the special case $I = J = [0, 1]$, these subposets are each isomorphic to the subposet of normal functions.

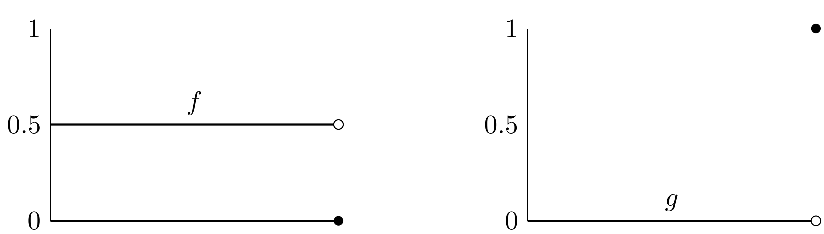
3. The Case When *I* and *J* are the Unit Interval

In this section, we consider the algebra $[0, 1]^{[0,1]}$. In the previous section, we mentioned several results on the subalgebra of this algebra consisting of convex normal functions[5,6] that took full advantage of the completeness and topological properties of the unit interval. Our purpose here is to provide several counterexamples showing poor behavior of this algebra when moving outside the setting of convex normal functions. These will contrast with results of the Sec. 4 where both I and J are finite chains.

Theorem 7. $[0, 1]^{[0,1]}$ *is not a lattice under the join or meet order.*

Proof. Let f and g be the elements of $[0, 1]^{[0,1]}$ given by

$$f(x) = \begin{cases} 0.5 \text{ if } x \in [0, 1) \\ \;0 \;\text{ if } \quad x = 1 \end{cases} \quad \text{and} \quad g(x) = \begin{cases} 0 \text{ if } x \in [0, 1) \\ 1 \text{ if } \quad x = 1 \end{cases}$$



By Theorem 2, a function h is a lower bound of f and g in the $\sqsubseteq_\sqcup$ order if and only if the following inequalities hold in the pointwise order.

$$h \wedge f^L \leq f \leq h^L \tag{2}$$

$$h \wedge g^L \leq g \leq h^L \tag{3}$$

Similarly, $h \sqsubseteq_\sqcup k$ if and only if

$$h \wedge k^L \leq k \leq h^L \tag{4}$$

Noting that $g = g^L$ and that $h(0) = h^L(0)$, the only conditions that the inequalities (2) and (3) impose on h are these:

$$h(1) = 0 \tag{5}$$

$$h^L(1) = 1 \tag{6}$$

$$h(0) \geq 0.5 \tag{7}$$

So any function that satisfies these conditions is a lower bound of f and g. To find a bigger lower bound, we must find a function k different from h that satisfies the conditions on h in (5), (6) and (7), and satisfies the conditions in (4).

To find such a k, we consider two cases. First, suppose it is not the case that $h = h^L$ on $[0, 1)$. Then let $k = h^L$ on $[0, 1)$ and let $k(1) = 0 = h(1)$. It is easy to check that (5), (6), and (7) hold for k and (4) holds. Now suppose that $h = h^L$ on $[0, 1)$, so h is monotone increasing on $[0, 1)$. If $h(0) > 0.5$, then let $k(0) = 0.5$, and $k = h$ elsewhere. If $h(0) = 0.5$, then there exists an x_0 such that $0 < x_0 < 1$ and $h(x_0) > 0.5$. In this case, let

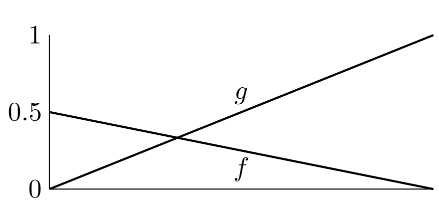
$$k(x) = \begin{cases} 0.5 & \text{if } x \in [0, x_0] \\ h(x) & \text{if } x \in (x_0, 1] \end{cases}$$

Using the fact that $h = h^L$ on $[0, 1)$, we see (5), (6), and (7) hold for k and that (4) holds. Here, a key point in establishing that (6) holds for k is that $(x_0, 1]$ is non-empty, which follows from basic properties of the unit interval.

Therefore the elements f and g have no greatest lower bound in the $\sqsubseteq_\sqcup$ order, and so the algebra is not a lattice under the join order. That it fails also to be a lattice under the meet order comes from the fact that negation $'$ is a dual isomorphism from this set under the join order to this set under the meet order. □

Theorem 8. *The continuous functions form a subalgebra of* $([0, 1]^{[0,1]}, \sqcup, \sqcap)$. *However, the continuous functions do not form a lattice under either the join or meet order.*

Proof. That the continuous functions are closed under $\sqcup$ and $\sqcap$ follows easily from Theorem 1. Let f and g be the elements of $[0, 1]^{[0,1]}$ given by setting $f(x) = 0.5 - 0.5x$ and $g(x) = x$.



If h is a lower bound of f, g in the join order, then by Theorem 2

$$h \wedge f^L \leq f \leq h^L \,,$$

$$h \wedge g^L \leq g \leq h^L \,.$$

On $(0,1]$, $h \leq f$ since $h \wedge f^L \leq f$ and $f < f^L$ on $(0,1]$, yet $g \leq h^L$. The latter implies that $h(0) = 1$. Such a continuous function does not exist. So the continuous functions are not a lattice in the join order, and by a similar argument, are not a lattice in the meet order either. □

Remark 2. In the theorem above, we have proved that two continuous functions do not necessarily have any continuous lower bound in the join order, let alone an infimum. However, the example above suggests the possibility that the subalgebra of upper semi-continuous functions might be a lattice in the join and meet orders. We do not know the answer to that question.

4. The Case Where I and J are Finite Chains

Here we consider lattice properties of the algebra I^J where I and J are finite chains. Other properties of these algebras have been considered in Refs. 8 and 9. We first settle on notation.

Definition 7. For a natural number n let $\mathbf{n}$ be the chain $\{1, 2, \ldots, n\}$ with the natural ordering. As with every finite chain, $\mathbf{n}$ has a unique involution $'$.

Let m and n be natural numbers. Taking the chains $\mathbf{m}$ and $\mathbf{n}$, the algebra of interest is $\mathbf{m}^{\mathbf{n}}$, the set of all functions f from $\{1, \ldots, n\}$ to $\{1, \ldots, m\}$ equipped with the operations from Definition 1. We will often represent such functions f as n-tuples or strings of elements in $\mathbf{m}$ of length n. To be more intuitive, we change the notation of the constants as follows.

$$\overline{1}\,(i) = \begin{cases} m \text{ if } i = 1 \\ 1 \ \text{ if } i \neq 1 \end{cases} \quad \text{and} \quad \overline{m}\,(i) = \begin{cases} m \text{ if } i = n \\ 1 \ \text{ if } i \neq n \end{cases}$$

Note that under the join order, the smallest element is $\overline{1}$ and the largest is $\overline{m}$.

4.1. *The join and meet orders in the finite case*

Recall from Theorem 3 that for any chains I and J the poset I^J under the join order is a join semilattice with join given by $\sqcup$, and I^J under the meet order is a meet semilattice with meet given by $\sqcap$. These semilattices are dual via the negation $'$. In contrast to Theorem 7 we have the following.

Theorem 9. *For m and n natural numbers, $\mathbf{m}^{\mathbf{n}}$ is a lattice under the join order and also under the meet order.*

Proof. The supremum of f and g in the join order is $f \sqcup g$. The inf of f and g is the supremum of all elements below both. Such a supremum exists because $\mathbf{m}^{\mathbf{n}}$ is finite and there is at least one element below both, namely the element $\overline{1}$. Thus $\mathbf{m}^{\mathbf{n}}$ is a lattice in the join order. The result for the meet order follows from the dual isomorphism $'$. □

Figure 1 shows $\mathbf{2^3}$ under the join and meet orders. Each is a lattice, and the negation $'$ that takes the string xyz to its inverse zyx is a dual isomorphism between these lattices.

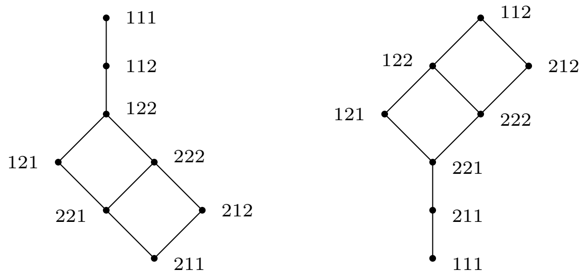


Fig. 1. 2^3 under the join order (left) and meet order (right).

Remark 3. The proof of Theorem 9 can be easily modified to show that any subalgebra of $(\mathbf{m^n}, \sqcup, \overline{1})$ is a lattice under the join order. However, such a subalgebra is not necessarily a sublattice of the lattice $\mathbf{m^n}$ under the join order. Joins in the two lattices will agree, but meets may differ.

4.2. *Normal and convex elements*

Two important subalgebras of $(\mathbf{m^n}, \sqcup)$, are the subalgebra of normal elements and the subalgebra of convex elements. As noted in Remark 3 they are both lattices under the join order. To get them to be sublattices of the lattice $\mathbf{m^n}$ under the join order, we need more.

Lemma 1. *In $\mathbf{m^n}$, if f is normal, then any element g below f in the join order is normal.*

Proof. Suppose that f is normal and $f \sqcup g = f$. We need that g is normal.

$$f = f \sqcup g = (f \vee g) \wedge f^L \wedge g^L$$

Since f assumes the value m, so does g^L, whence g is normal. □

Theorem 10. *The set of normal functions of $\mathbf{m^n}$ is sublattice of the lattice $\mathbf{m^n}$ under the join order, and a sublattice of the lattice $\mathbf{m^n}$ under the meet order.*

Proof. The least upper bound of two elements f and g in $(\mathbf{m^n}, \sqsubseteq_\sqcup)$ is $f \sqcup g$. So by Proposition 3 the normal elements are closed under supremums in this poset. The infimum of f and g in this poset is the supremum of all their lower bounds in the join order. If f and g are normal, Lemma 1 shows these lower bounds are normal, hence their supremum is again normal. This shows the normal elements are a sublattice of the lattice $\mathbf{m^n}$ under the join order, and the proof for the meet order follows via the dual isomorphism $'$. $\square$

A similar situation holds for the set of convex elements of $\mathbf{m^n}$, but is a bit more delicate. For example, it is not true that elements below convex elements are convex, as illustrated by Fig. 1 where the non-convex element 212 is below the convex element 222.

Theorem 11. *The set of convex functions of* $\mathbf{m^n}$ *is a sublattice of the lattice* $\mathbf{m^n}$ *under the join order, and a sublattice of the lattice* $\mathbf{m^n}$ *under the meet order.*

Proof. We give the proof for the join order. By Proposition 4, the convex elements are closed under $\sqcup$, and as $\sqcup$ provides the supremum in the join order, the convex elements are closed under supremums in $(\mathbf{m^n}, \sqsubseteq_\sqcup)$. The infimum of two elements in $(\mathbf{m^n}, \sqsubseteq_\sqcup)$ is the supremum of their common lower bounds in the join order. Suppose f and g are convex. Then for any lower bound h of f and g in the join order, Theorem 6 shows Γh is a convex element that lies above h. Also by Theorem 6, Γ preserves $\sqcup$, so is order preserving with respect to the join order. Theorem 6 gives Γ is idempotent, so Γh is a convex lower bound of f and g, and Γh lies above h. Thus the infimum of f and g is the supremum of convex elements that are common lower bounds, hence is convex. $\square$

Corollary 2. *The set of convex normal functions of* $\mathbf{m^n}$ *is a distributive sublattice of the lattice* $\mathbf{m^n}$ *under the join order. It is also a distributive sublattice of* $\mathbf{m^n}$ *under the meet order.*

Proof. That the convex normal functions are a sublattice follows directly from Theorems 10 and 11 since the intersection of sublattices is a sublattice. That this lattice is distributive follows from Theorem 5. $\square$

Inspection of Fig. 3 at the end of the paper shows that neither the lattice of convex functions, nor the lattice of normal functions, of $\mathbf{m^n}$ need be distributive.

4.3. *A description of meet in the join order*

For finite chains $\mathbf{m}$ and $\mathbf{n}$, we know $\mathbf{m^n}$ is a lattice under the join order where the join of f and g is given by $f \sqcup g$. The meet of two elements in this lattice, which we denote $f \odot g$, is described only as the join of all their common lower bounds. In this section we give an algorithm, polynomial in n, that computes $f \odot g$ directly from the functions f and g.

Remark 4. It would be desirable to have a simple description of $f \odot g$ as a term operation using $\text{L}, \text{R}, \wedge, \vee$ as was done with $\sqcup$ and $\sqcap$. This is not possible. Consider the elements $f = (2, 2, 1)$ and $g = (3, 3, 3)$ in $\mathbf{3^3}$ under the join order. Their meet

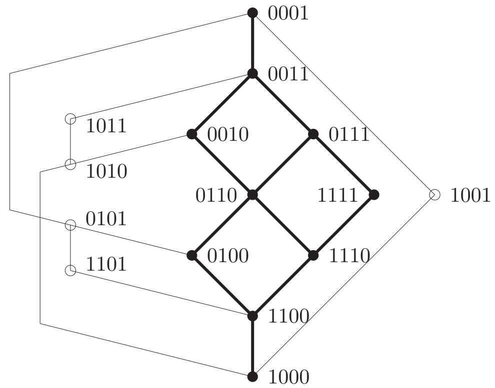


Fig. 2. The normal functions in $\mathbf{2^4}$ under the double order. The solid circles and darker lines indicate the convex normal functions.

$f \odot g$ is $(3, 3, 1)$ as can be seen by the results in this section, or from Fig. 2. However, the closure of $\{f, g\}$ under the operations L, R, $\wedge, \vee$ is $(2, 2, 1), (2, 2, 2)$ and $(3, 3, 3)$.

Throughout this subsection, let m and n be natural numbers and consider the chains $\mathbf{m} = \{1, \ldots, m\}$ and $\mathbf{n} = \{1, \ldots, n\}$.

Proposition 10. *Given f and g, there is a unique number l and for each $1 \leq i \leq l$ unique a_i and b_i in $\{1, \ldots, n\}$ such that*

(1) $a_1 = 1$,
(2) $a_i \leq b_i$ and $b_i < a_j$ for each $i < j \leq l$,
(3) $f = f^L$ and $g = g^L$ on each interval $[a_i, b_i]$,
(4) the intervals $[a_i, b_i]$ are maximal intervals having the property in (3).

We then define (for technical convenience) $a_{l+1} = n + 1$.

Proof. Let S be the set of all elements s in $\{1, \ldots, n\}$ with $f(s) = f^L(s)$ and $g(s) = g^L(s)$, and note that $1 \in S$. Find the family $\mathcal{F}$ of all intervals of $\{1, \ldots, n\}$ that are contained in S. Then the maximal members of $\mathcal{F}$ are pairwise disjoint, and their union is S. Suppose the number of these intervals is l. Then there are unique elements a_i, b_i for each $i \leq l$ with $a_i \leq b_i$ and $b_i < a_j$ for each $i < j$ so that these maximal intervals are exactly the $[a_i, b_i]$ for $i \leq l$. In effect, we have simply expressed S in the following manner

$$S = [a_1, b_1] \cup [a_2, b_2] \cup \cdots \cup [a_l, b_l]$$

We note that it may occur that $a_i = b_i$ for some i since some of these intervals may be singletons. Finally $a_1 = 1$ since $1 \in S$. $\square$

Remark 5. In Proposition 10, $\{1,\ldots,n\}$ is partitioned into disjoint intervals

$$\{1,\ldots,n\} = [a_1,b_1] \cup (b_1,a_2) \cup [a_2,b_2] \cup \cdots \cup [a_l,b_l] \cup (b_l,a_{l+1})$$

It may be that the last interval (b_l,a_{l+1}) is empty, depending on whether or not b_l equals n. If b_l does equal n, the interval (b_l,a_{l+1}) is empty, and if $b_l \neq n$ this interval is $(b_l,n]$. This is a somewhat artificial technical device that we use to avoid having to separate the cases where b_l does and does not equal n.

Definition 8. Given the partition a_i,b_i of f and g, we call the $[a_i,b_i]$ type-A intervals, and the (b_i,a_{i+1}) type-B intervals. Note that an element belongs to a type-A interval if and only if $f^L = f$ and $g^L = g$ at that element.

Before giving our construction of $f \odot g$, we require one further definition.

Definition 9. Define $\hat{f}$ by setting

$$\hat{f}(x) = \begin{cases} f(x) \text{ if } f(x) < f^L(x) \\ \ \ m \ \ \text{ if } f(x) = f^L(x) \end{cases}$$

Proposition 11. *The meet in the join order is given by*

$$(f \odot g)(x) = \begin{cases} (f \vee g)(x) & \text{if } x \in [a_i,b_i) \text{ for some } i \\ \sup\{(f \vee g)(y) : b_i \leq y < a_{i+1}\} & \text{if } x = b_i \text{ for some } i \\ (\hat{f} \wedge \hat{g})(x) & \text{otherwise} \end{cases}$$

Proof. Call the function defined above k. We will first show that k is a lower bound of f,g in the join order. For this, we must show

$$f^L \wedge k \leq f \leq k^L \tag{8}$$

$$g^L \wedge k \leq g \leq k^L \tag{9}$$

By symmetry, it is enough to show the statement involving f.

Claim 1. $f^L \wedge k \leq f$.

Proof of Claim. If $f^L(x) = f(x)$, clearly $f^L \wedge k \leq f$ at x. This includes the case where x belongs to an interval of type-A, so to some $[a_i,b_i]$. Suppose $f(x) < f^L(x)$. Then we are in the third case of the definition of $k(x)$ and have $\hat{f}(x) = f(x)$. Then $k(x) = (\hat{f} \wedge \hat{g})(x) \leq f(x)$.

Claim 2. $f \leq k^L$.

Proof of Claim. If x is in an interval of type-A, then the definition of $k(x)$ shows that $f(x) \leq k(x)$. In the second case, x is in the set over which the supremum is taken. So $f(x) \leq k^L(x)$. Suppose x belongs to an interval of type-B. Then there is a largest i with $b_i < x$. Then the definition of $k(b_i)$ gives $f(x) \leq k(b_i)$. Then, since $b_i < x$, we have $f(x) \leq k(b_i) \leq k^L(x)$.

We have shown k is a lower bound of f and g. Suppose h is another. Then

$$f^L \wedge h \leq f \leq h^L \tag{10}$$
$$g^L \wedge h \leq g \leq h^L \tag{11}$$

We must show $h \sqsubseteq_{\sqcup} k$, so we must show

$$k^L \wedge h \leq k \leq h^L \tag{12}$$

Claim 3. $k = k^L$ on $[a_1, b_1] \cup \cdots \cup [a_l, b_l]$.

Proof of Claim. Clearly f^L and g^L are increasing functions. On intervals of type-A we have that f and g agree with f^L and g^L. It follows that the restriction of k to $[a_1, b_1) \cup \cdots \cup [a_l, b_l)$ is increasing when we consider the natural order on this restricted domain since on this domain we have $k = f \vee g = f^L \vee g^L$. Further, the definition of $k(b_i)$ (the second case) gives $(f \vee g)(b_i) \leq k(b_i)$, hence k is increasing on each interval $[a_i, b_i]$. For $i < l$ we note $k(b_i) \leq k(a_{i+1})$. Indeed, if $b_i \leq y < a_{i+1}$ we have $(f \vee g)(y) \leq (f^L \vee g^L)(a_{i+1}) = k(a_{i+1})$. Thus we have k is increasing on $[a_1, b_1] \cup \cdots \cup [a_l, b_l]$. If y is an element in some interval of type-B, then there is a largest i with $b_i < y$, and the definition of $k(b_i)$ gives $(f \vee g)(y) \leq k(b_i)$. But $k(y) = (\hat{f} \wedge \hat{g})(y)$, and as y is in an interval of type-B, either $f(y) < f^L(y)$ or $g(y) < g^L(y)$, giving either $\hat{f}(y) = f(y)$ or $\hat{g}(y) = g(y)$, and in either case $k(y) = (\hat{f} \wedge \hat{g})(y) \leq (f \vee g)(y) \leq k(b_i)$.

Claim 4. $k^L \wedge h \leq k$.

Proof of Claim. This is obvious from the claim above if x belongs to an interval of type-A. Suppose x is in an interval of type-B. Then by definition of these intervals, we have either $f(x) < f^L(x)$ or $g(x) < g^L(x)$, or perhaps both. If $f(x) < f^L(x)$, then Equation (10) gives $f^L \wedge h \leq f$, hence $h(x) \leq f(x) = \hat{f}(x)$, and if $f(x) = f^L(x)$, then $\hat{f}(x) = m$, so surely $h(x) \leq \hat{f}(x)$. Similarly $h(x) \leq \hat{g}(x)$. So $h(x) \leq (\hat{f} \wedge \hat{g})(x) = k(x)$.

Claim 5. If x is in an interval of type-B, then $h(x) < h^L(x)$.

Proof of Claim. By definition of an interval of type-B, either $f(x) < f^L(x)$ or $g(x) < g^L(x)$. Without loss of generality, assume $f(x) < f^L(x)$, so there is $y < x$ with $f(x) < f(y)$. By Equation (10) we have $h(x) \leq f(x) < f(y)$ and $f(y) \leq h^L(y) \leq h^L(x)$.

Claim 6. $k \leq h^L$.

Proof of Claim. Equations (10) and (11) give $f \leq h^L$ and $g \leq h^L$, hence $f \vee g \leq h^L$. Then the definition of k gives $k \leq h^L$ on $[a_1, b_1) \cup \cdots \cup [a_l, b_l)$. Also, if x belongs to an interval of type-B, then by an argument we have used several times $k(x) = (\hat{f} \wedge \hat{g})(x) \leq (f \vee g)(x)$, so again we have $k \leq h^L$. It remains only to show $k(b_i) \leq h^L(b_i)$ for each $i \leq l$. Our definition has $k(b_i) = \sup\{(f \vee g)(y) : b_i \leq y <$

$a_{i+1}\}$. Our argument above shows that $(f \vee g)(b_i) \leq h^L(b_i)$, so we must only show that if $b_i < y < a_{i+1}$ then $(f \vee g)(y) \leq h^L(b_i)$. Claim 5 shows that $h^L(y) \leq h^L(b_i)$. If not, the first x in (b_i, a_{i+1}) with $h^L(x) > h^L(b_i)$ would have $h(x) = h^L(x)$. Since we know $(f \vee g)(y) \leq h^L(y)$, it follows that $(f \vee g)(y) \leq h^L(b_i)$.

This concludes the proof of the proposition. □

4.4. *The double order*

Corollary 1 states that for bounded chains I and J, the poset I^J under the double order is the disjoint sum $\bigoplus_{k\in I}$ of the subalgebras S_k of all functions of height k. It is our purpose in this subsection to show that for natural numbers m, n, each of these subalgebras S_k of $\mathbf{m}^{\mathbf{n}}$ for $k \leq m$ is a lattice under the double order.

Proposition 12. *For natural numbers m and n, and $k \in \mathbf{m}$, the subalgebra S_k of $\mathbf{m}^{\mathbf{n}}$ of functions of height k is isomorphic as an algebra, and as a poset under the double order, to the subalgebra of $\mathbf{k}^{\mathbf{n}}$ consisting of normal functions.*

Proof. This is immediate from Theorem 1 and Theorem 4. □

So it suffices to show that the normal functions S_m of $\mathbf{m}^{\mathbf{n}}$ form a lattice under the double order, and since this is a bounded poset with bounds $\overline{1}$ and $\overline{m}$, it suffices to show that any two elements of S_m have an infimum under the double order. Throughout, we assume f and g are normal functions in $\mathbf{m}^{\mathbf{n}}$. Immediate from Theorem 4 is the following.

Proposition 13. *$h \sqsubseteq f, g$ if and only if the following conditions hold.*

$$f^L \wedge h \;\leq\; f \;\leq\; h^L \tag{13}$$
$$g^L \wedge h \;\leq\; g \;\leq\; h^L \tag{14}$$
$$h^R \wedge f \;\leq\; h \;\leq\; f^R \tag{15}$$
$$h^R \wedge g \;\leq\; h \;\leq\; g^R \tag{16}$$

In the following definition, the meet in the chain $\mathbf{m}$ of the empty set is m.

Definition 10. With f and g given, define p_1, p_2, p and q as follows.

$$p_1(y) = \bigwedge\{f(x) : x \leq y,\ f^L(x) > f(x) < g(x)\}$$

$$p_2(y) = \bigwedge\{g(x) : x \leq y,\ g^L(x) > g(x) < f(x)\}$$

$$p(y) = p_1(y) \wedge p_2(y)$$

$$q(x) = \begin{cases} (f \vee g)(x) & \text{if } f = f^L \text{ and } g = g^L \\ f(x) & \text{if } f \neq f^L \text{ and } g = g^L \\ g(x) & \text{if } f = f^L \text{ and } g \neq g^L \\ (f \wedge g)(x) & \text{if } f \neq f^L \text{ and } g \neq g^L \end{cases}$$

We use $[\![f = f^L]\!]$ for $\{x : f(x) = f^L(x)\}$, with similar usages obvious.

Proposition 14. *If h is a lower bound of f and g in the double order, then*

(1) $h \leq p$.
(2) $h \leq q$ on $[\![f \neq f^L]\!] \cup [\![g \neq g^L]\!]$.

Proof. 1. We show $h(y) \leq p_2(y)$, and a similar argument shows that $h(y) \leq p_1(y)$, hence $h(y) \leq p(y)$. Suppose $x \leq y$ and $g^L(x) > g(x) < f(x)$. From this and (14), we have $h(x) \leq g(x)$. By (15) we have $h^R(x) \wedge f(x) \leq h(x)$, and as $h(x) \leq g(x) < f(x)$, we must have $h(x) = h^R(x)$. This implies $h(y) \leq h(x)$ and we had $h(x) \leq g(x)$, so $h(y) \leq g(x)$. So $h(y)$ lies under all the terms whose meet we take to form $p_2(y)$, hence $h(y) \leq p_2(y)$.

2. If $f(x) \neq f^L(x)$, then (13) gives $h(x) \leq f(x)$, and if $g(x) \neq g^L(x)$, (14) gives $h(x) \leq g(x)$. The statement follows from the definition of q in Definition 10. □

Proposition 15. *For $r = p \wedge q$ we have (using r in place of h),*

(1) r satisfies the first inequalities in (13) and (14).
(2) r satisfies the first inequalities in (15) and (16).
(3) r satisfies the second inequalities in (15) and (16).

Proof. 1. The first inequality in (13) is trivial when $f = f^L$. When $f \neq f^L$ the definition of q gives $q(x) \leq f(x)$, hence $r(x) \leq f(x)$. The argument for (14) is the same.

2. We show the first inequality in (15), that $r^R \wedge f \leq r$. The argument for (16) is similar. The definition of p shows that it is decreasing, so $p = p^R$. Since $(p \wedge q)^R \leq p^R, q^R$, we have $r^R = (p \wedge q)^R \leq p^R \wedge q^R = p \wedge q^R$. Suppose x is such that $f(x) \leq q(x)$. Then $(r^R \wedge f)(x) \leq (p \wedge q^R \wedge f)(x) \leq (p \wedge q)(x) = r(x)$. If x is such that $q(x) < f(x)$, then we must be in the third or fourth case of the definition of q, with $q(x) = g(x)$, and we must have $g^L(x) > g(x) < f(x)$. Then the definition of p gives $p(y) \leq g(x)$ for all $x \leq y$. Thus, since $r = p \wedge q$, we have $r(y) \leq g(x)$ for all $x \leq y$, hence $r^R(x) \leq g(x) = q(x)$. Clearly, since $r = p \wedge q$, we have $r^R \leq p^R = p$, so $r^R(x) \leq (p \wedge q)(x) = r(x)$. Thus $(r^R \wedge f)(x) \leq r(x)$.

3. We show the second inequality in (15), that $r \leq f^R$. The argument for (16) is similar. Suppose x is such that $f(x) \neq f^L(x)$. Then we are in the second or fourth case of the definition of q, so $q(x) \leq f(x)$. Then $r(x) \leq q(x) \leq f(x) \leq f^R(x)$. Suppose x is such that $f(x) = f^L(x)$. Then $f^R(x) = f^{LR}(x)$ is the maximum of f, which we have assumed is m. So $r(x) \leq f^R(x)$. □

As with any subset of $\{1, \ldots, n\}$, the set $[\![f = f^L]\!] \cap [\![g = g^L]\!]$ is comprised of disjoint closed intervals. We assume these intervals are $X_1, \ldots, X_l$ read left to right, and use $X_i = [a_i, b_i]$ for $i = 1, \ldots, l$.

Proposition 16. *Let $X_1, \ldots, X_l$ be the intervals described above.*

(1) On X_1 we have p is constantly m.
(2) If p is constantly m on X_i then $r = f \vee g$ on X_i.
(3) If p is not constantly m on X_i, then $r = p$ on X_i and is constant.

Proof. 1. The interval X_1 begins at 1 and continues until there is an element where $f \neq f^L$ or $g \neq g^L$. If y belongs to X_1 there is no $x \leq y$ with $f \neq f^L$ or $g \neq g^L$, so there is nothing to take the meet of when forming p. Therefore p is constantly m on this interval.

2. If p is constantly m on X_i, then $r = p \wedge q$ implies $r = q$ on X_i. On X_i this is case one of the definition of q, so $r = q = f \vee g$ on this interval.

3. Suppose y belongs to $[a_i, b_i]$ and $p(y) < m$. Then there is some $x \leq y$ with $f^L(x) > f(x) < g(x)$ or $g^L(x) > g(x) < f(x)$. In either case, the element x cannot be in one of our closed intervals, so $x < a_i$. So all points in $[a_i, b_i]$ have their value of p computed from the meet of the same sets of terms, so p is constant on X_i. Suppose it was $x \leq y$ with $f^L(x) > f(x) < g(x)$. Then $p(y) \leq f(x) \leq f^L(y) = f(y)$ and as $q(y) = (f \vee g)(y)$, we have $p(y) \leq q(y)$ for all $y \in X_i$. So $r = p$ on X_i. □

Definition 11. Let $X_1, \ldots, X_k$ be the intervals where p is constant m.

Proposition 17. *If $p(y) = m$, then $r(x) = (f \vee g)(x)$ for all $x \leq y$.*

Proof. Since $r = p \wedge q$, and the definition of q gives $q \leq f \vee g$ always, $r \leq f \vee g$ always. Since p is decreasing, if $p(y) = m$, then $p(x) = m$ for all $x \leq y$, so $r = q$ for all $x \leq y$. So it is enough to show that $q = f \vee g$ for all $x \leq y$, hence enough to show that $f \leq q$ and $g \leq q$ for all $x \leq y$. We show that $f(x) \leq q(x)$ for all $x \leq y$. This is obvious in the first two cases of the definition of q. In either of the last two cases $g^L(x) > g(x)$. Since $p(y) = m$, there are no terms in the meets used to define p, so we cannot have $g(x) < f(x)$, hence $f(x) \leq g(x)$. So in the last two cases of the definition of q we have $f(x) \leq q(x)$. □

Proposition 18. *The function r attains its largest value at b_k, where b_k is the right endpoint of the last interval X_k where p is constant m.*

Proof. Since $p(b_k) = m$, the previous result shows that $r = f \vee g$ to the left of b_k. Since $f = f^L$ and $g = g^L$ at b_k, it follows that $r(x) \leq r(b_k)$ for all $x \leq b_k$.

Consider the open interval (b_k, a_{k+1}) of all elements up to the next element where $f = f^L$ and $g = g^L$. Enumerate the elements of this interval as $y_1, \ldots, y_j$. At each of these y_i's we have either $f \neq f^L$ or $g \neq g^L$, or both.

Claim 7. For each $i \leq j$ either

(i) $(f \vee g)(y_i) \leq r(b_k)$, or
(ii) $p(y_i) \leq r(b_k)$.

Proof of Claim. Our proof is by induction on i. We note that if case (ii) ever applies, then since p is decreasing, case (ii) will apply for all further values of i and we are done.

For the base case $i = 1$ we must have either $f \neq f^L$ or $g \neq g^L$ at y_1. Say $f^L(y_i) > f(y_i)$. Then since $f^L(b_k) = f(b_k)$ we must have $f(y_1) < f(b_k) \leq r(b_k)$. If $g(y_1) \leq f(y_1)$, then case (i) applies at y_1. If $f(y_1) < g(y_1)$, then the definition of p gives $p(y_1) \leq f(y_1) \leq r(b_k)$, so case (ii) applies at y_1.

For the inductive case, suppose our claim holds up to y_i and consider y_{i+1}. We can assume case (ii) never applied before, so case (i) applied for all y_u where $u \leq i$. This means $(f \vee g)(y_u) \leq r(b_k)$ for all $u \leq i$, and since $r = f \vee g$ to the left of b_k, that $(f \vee g)(x) \leq r(b_k)$ for all $x \leq y_i$. Again, either $f \neq f^L$ or $g \neq g^L$ at y_{i+1}. Say $f(y_{i+1}) < f^L(y_{i+1})$. Then $f(y_{i+1}) < f(x)$ for some $x \leq y_i$, and it follows that $f(y_{i+1}) < r(b_k)$. If $g(y_{i+1}) \leq f(y_{i+1})$ we have case (i) at y_{i+1}. Otherwise we have $f^L(y_{i+1}) > f(y_{i+1}) < g(y_{i+1})$, and the definition of p gives $p(y_{i+1}) \leq f(y_{i+1}) \leq r(b_k)$. Thus case (ii) applies at y_{i+1}. This concludes the inductive proof of our claim.

The inductive proof has shown that $((f \vee g) \wedge p)(y_i) \leq r(b_k)$ for $i \leq j$. Since $r = p \wedge q$ and $q \leq f \vee g$, this shows that $r(y_i) \leq r(b_k)$ for $y_1, \ldots, y_j$. We already knew $r(x) \leq r(b_k)$ for all $x \leq b_k$, so $r(x) \leq r(b_k)$ for all $x \leq y_j$.

The element following y_j (if there is one) is a_{k+1}. By definition, b_k is the last element in the intervals $X_1, \ldots, X_l$ where p takes value m. So $p(a_{k+1}) < m$. Thus there is $x \leq a_{k+1}$ with $f^L(x) > f(x) < g(x)$ or $g^L(x) > g(x) < f(x)$. As $f = f^L$ and $g = g^L$ at a_{k+1} we must have $x < a_{k+1}$, and as $p = m$ at b_k, this x must be one of $y_1, \ldots, y_j$. From the choice of x, we have $p(x) \leq f(x)$ or $p(x) \leq g(x)$. So $p(x) \leq (f \vee g)(x)$, and Claim 7 then gives $p(x) \leq r(b_k)$. Since $r = p \wedge q$ and p is decreasing, we then have $r(z) \leq p(z) \leq p(x) \leq r(b_k)$ for all $z \geq x$. From the earlier comments, $r(z) \leq r(b_k)$ for all z. □

Remark 6. At this point r is *close* to being a greatest lower bound of f, g in the double order. It does not satisfy all inequalities in (13) through (16), we must modify r to get $f \leq r^L$ and $g \leq r^L$ so that it is a lower bound. To do so, we must make r larger, making sure we do not destroy the other inequalities we have. Proposition 14 says we cannot make r larger on $[\![f \neq f^L]\!] \cup [\![g \neq g^L]\!]$ and have a lower bound. So we can only increase r on the intervals $X_1, \ldots, X_l$, and we cannot increase it above p. Proposition 16 then says we can only increase r on some X_i where $p = m$, or in other words on some X_i with $i \leq k$.

Definition 12. Let s equal r everywhere except possibly at b_k. Define $s(b_k)$ to be the least above $r(b_k)$ so that $f \leq s^L$ and $g \leq s^L$ to the right of b_k.

Proposition 19. *The function s is a lower bound of f and g in the double order.*

Proof. For (13) and (14), the first two inequalities in these equations hold because they held for r and we only modified r at an element where $f = f^L$ and $g = g^L$ where they trivially hold. For the second two inequalities of (13) and (14), note that Proposition 17 gives $r = f \vee g$ to the left of b_k, so $f \leq r \leq s \leq s^L$ and $g \leq r \leq s \leq s^L$ to the left of b_k, and $s(b_k)$ was specifically chosen to that $f \leq s^L$ and $g \leq s^L$ to the right of b_k.

For (15) and (16), consider the second inequalities. By Proposition 15 they held for r and we only modified r at an element where $f = f^L$ and $g = g^L$. So at this element, $f^R = f^{LR}$ and $g^R = g^{LR}$, which is the maximum m. So these inequalities hold also for s. For the first inequalities in (15) and (16), we must show $s^R \wedge f \leq s$ and $s^R \wedge g \leq s$. By Proposition 15, these were valid for r, and we only modified r at b_k, so they must hold at all $x > b_k$. By Proposition 18, r attains its maximum value at b_k, so s must attain its maximum value at b_k. This gives $s(b_k) = s^R(b_k)$, so these inequalities hold at b_k. By Proposition 17, we have $f, g \leq r$ to the left of b_k, so $f \leq s$ and $g \leq s$ to the left of b_k, giving our result. □

Proposition 20. *$s(b_k) = m$.*

Proof. We have seen that r attains its maximum at b_k. So if r is anywhere equal to m it is equal to m at b_k. We have $r \leq s$, and that s is comparable to f and g implies by Proposition 8 that s has the same height m as f and g. It must therefore be that $s(b_k) = m$. □

Remark 7. This result shows that we could define s to be constructed from r by changing the value of r at b_k to be m. However, the definition above will have its technical advantage later.

Proposition 21. *If h is a lower bound of f, g in the double order then*

(1) $s^L \wedge h \leq s$
(2) $s \leq h^L$
(3) $h^R \wedge s \leq h$
(4) $h \leq s^R$

That is, s is the greatest lower bound of f and g in the double order.

Proof. 1. Proposition 14 shows $h \leq p$ everywhere and $h \leq q$ on the set $[\![f \neq f^L]\!] \cup [\![g \neq g^L]\!]$. Proposition 16 says $r = p$ on $X_{k+1}, \ldots, X_l$. Thus $h \leq r$ except possibly on $X_1, \ldots, X_k$. Proposition 17 says $r = f \vee g$ to the left of b_k, so $r^L = f^L \vee g^L$ to the left of b_k, so $r = r^L$ on $X_1, \ldots, X_k$. As we only alter r at b_k to form s, strictly to the left of b_k we have $s^L = r^L = r = s$, and as we only increase r at b_k we have $s^L = s$ at b_k as well. Thus $s^L \wedge h \leq s$ to the left of b_k. As $h \leq r \leq s$ strictly to the right of b_k, the inequality holds there as well.

2. Since h is a lower bound of f, g, the second inequalities in (13) and (14) give $f \vee g \leq h^L$. Since $r = p \wedge q$, and the definition of q gives $q \leq f \vee g$ everywhere, we have $r \leq h^L$. Since $s = r$ everywhere except possibly at b_k, we have only to check that $s(b_k) \leq h^L(b_k)$. Now $s(b_k)$ was chosen to be the larger of $r(b_k)$ and the smallest value needed to get $f \vee g \leq s^L$ from b_k onward. Call this quantity t. The

argument from the first part has shown that $h \leq r$ strictly to the right of b_k, hence $h \leq s$ strictly to the right of b_k. So as h is a lower bound of f and g, there must be some x occurring at b_k or before that allows $f \vee g \leq h^L$ to the right of b_k. This means there is some $x \leq b_k$ with $h(x) \geq t$. But this then gives $s(b_k) \leq h^L(b_k)$.

3. Since h is a lower bound of f and g, the first equalities in (15) and (16) give $h^R \wedge f \leq h$ and $h^R \wedge g \leq h$. Thus $h^R \wedge (f \vee g) \leq h$. Since $r \leq f \vee g$ everywhere, it follows that $h^R \wedge r \leq h$. Since s agrees with r except possibly at b_k, we need only show $(h^R \wedge s)(b_k) \leq h(b_k)$. We will obtain this by showing $h^R(b_k) = h(b_k)$. Indeed, if $h^R(b_k) > h(b_k)$, then, since $h^R \wedge r \leq h$, we would have $r(b_k) \leq h(b_k)$. But $h \leq r$ strictly to the right of b_k and r attains its maximum at b_k. Thus $h(x) \leq r(b_k)$ for all $x > b_k$. Then since $r(b_k) \leq h(b_k)$ we must have $h^R(b_k) = h(b_k)$, a contradiction.

4. We have seen that $h \leq r$ except possibly on $X_1, \ldots, X_k$. As b_k is the right endpoint of X_k, all of $X_1, \ldots, X_k$ lies to the left of b_k. We have seen $s(b_k) = m$, so s^R is identically m everywhere to the left of b_k. □

Theorem 12. *The collection of normal functions in* $\mathbf{m^n}$ *forms an involutive lattice under the double order.*

Proof. That this poset is a lattice follows from Proposition 21. Surely $'$, which reverses the listing of a string, restricts to an operation on the normal functions. Proposition 1 (6) shows that $'$ is of period two. Since $f \sqsubseteq g$ if and only if $f \sqcup g = g$ and $f \sqcap g = f$, Proposition 1 (7) shows that $'$ is order inverting. □

Remark 8. It would be nice to have a simple term description of meet in the double order involving only L, R, $\wedge, \vee$. There is none. In $\mathbf{3^4}$ the functions $f = (3,3,1,2)$ and $g = (3,3,2,3)$ have meet $(3,3,1,1)$ in the double order, and this can not be expressed by applying these operations to f and g.

Proposition 22. *The convex normal functions are a sub-involutive lattice of the involutive lattice of normal functions of* $\mathbf{m^n}$ *under the double order. Further, these convex normal functions form a De Morgan algebra.*

Proof. Suppose f and g are convex normal functions. Note that Theorem 5 shows that the join order, meet order, and double order on the convex normal functions all agree, and that meet and join in the lattice of convex normal functions are given by $\sqcap$ and $\sqcup$. We show that if h is a lower bound of f and g in the double order, then $h \sqsubseteq f \sqcap g$. This shows that meet in the lattice of convex normal functions agrees with the meet of convex normal functions in the lattice of normal functions under the double order. A similar proof establishes the corresponding result for joins.

Assume h is a lower bound of f and g in the double order. This implies that (a) $f \sqcap h = h$, (b) $g \sqcap h = h$, (c) $f \sqcup h = f$, and (d) $g \sqcup h = g$. From the first two items and the associativity of $\sqcap$ we obtain $(f \sqcap g) \sqcap h = h$, hence h lies beneath $f \sqcap g$ in the meet order. For the join order, we use the first item to obtain $(f \sqcap g) \sqcup h = (f \sqcap g) \sqcup (f \sqcap h)$. By [2, Theorem 36] we may distribute to obtain

$(f \sqcap g) \sqcup (f \sqcap h) = f \sqcap (g \sqcup h)$. Using the fourth item this becomes $f \sqcap g$. Thus $(f \sqcap g) \sqcup h = f \sqcap g$, showing h lies beneath $f \sqcap g$ in the join order as well. $\square$

4.5. *An algorithm for covers in the double order*

In this subsection, for natural numbers m, n, we give a simple algorithm to compute covers in the lattice of normal functions of $\mathbf{m}^{\mathbf{n}}$ under the double order. Throughout, we assume f and g are elements of $\mathbf{m}^{\mathbf{n}}$. We shall consider these functions as n-tuples, with f given by $(x_1, \ldots, x_n)$.

Proposition 23. *Let g be constructed from f by one of the following rules:*

(1) For some x_i where $j < i \Rightarrow x_j < x_i$, change x_i to $x_i - 1$.
(2) For some x_i where $i < j \Rightarrow x_i \geq x_j$, change x_i to $x_i + 1$.

Then, if the resulting g is normal, it is a cover of f in the double order.

Remark 9. In this proposition, the requirement that g be normal means it takes values in $\{1, \ldots, m\}$ and achieves the value m. In particular, for the first rule to apply, we must have $x_i > 1$ and there must be some $k > i$ with $x_k = m$ so the resulting function attains value m. For the second rule to apply, we must have $x_i < m$. This means there must be $k < i$ with $x_k = m$.

Proof. By Theorem 4, $f \sqsubseteq g$ if and only if

$$f \wedge g^L \leq g \leq f^L \quad \text{and} \quad f^R \wedge g \leq f \leq g^R.$$

Claim 8. $f \sqsubseteq g$.

Proof of Claim. Say g is produced by the first rule. The first inequality above holds since $f = g$ except at i and $g^L(i) = g(i)$. The second inequality holds since $g \leq f \leq f^L$. The third inequality holds since $g \leq f$. For the fourth inequality, we have $f = g$ except at i, and $g^R(i) = m$ since $g(k) = m$ for some $k > i$ because g is normal. Suppose g is produced by the second rule. The first inequality holds since $f \leq g$. For the second inequality, $g = f$ except at i, and $f^L(i) = m$ since there is $k < i$ with $f(k) = m$. The third inequality holds since $g = f$ except at i and $f^R(i) = f(i)$. The fourth inequality holds since $f \leq g \leq g^R$.

We turn now to the matter of g being a cover of f. Suppose g is built from f by one of the above rules, and that h is normal and satisfies $f \sqsubseteq h \sqsubseteq g$. We must show that h is equal to either f or g. By Theorem 4, the following inequalities hold. We have numbered these to allow easy reference.

$$\begin{array}{llll}
f \wedge g^L \ \leq_1 \ g \ \leq_2 \ f^L & \qquad & f^R \wedge g \ \leq_3 \ f \ \leq_4 \ g^R \\
f \wedge h^L \ \leq_5 \ h \ \leq_6 \ f^L & \qquad & f^R \wedge h \ \leq_7 \ f \ \leq_8 \ h^R \\
h \wedge g^L \ \leq_9 \ g \ \leq_{10} \ h^L & \qquad & h^R \wedge g \ \leq_{11} \ h \ \leq_{12} \ g^R
\end{array}$$

Suppose g is built by rule 1. In this case we have $g \leq f$ and $f^R = g^R$. So $g \leq_{10} h^L$ implies that $g = g \wedge h^L \leq f \wedge h^L \leq_5 h$, and $h \leq_{12} g^R = f^R$ implies that $h = f^R \wedge h \leq_7 f$. So $g \leq h \leq f$. Since f covers g in the pointwise order, it follows that h is equal to either f or g. Suppose g is built by rule 2. In this case we have $f \leq g$ and $f^L = g^L$. So $f \leq_8 h^R$ implies that $f = f \wedge h^R \leq g \wedge h^R \leq_{11} h$, and $h \leq_6 f^L = g^L$ implies that $h = h \wedge g^L \leq_9 g$. In this case g covers f in the pointwise order, so h is equal to either f or g. $\square$

Having shown that each g produced from f by one of the above rules produces a cover of f in the double order, we turn to the matter of showing there are no other covers.

Proposition 24. *If g covers f in the double order and $g(i) < f(i)$ at some i, then g is built from f by rule 1.*

Proof. Let i be largest with $g(i) < f(i)$, and define h to agree with g everywhere except at i, and set $h(i) = g(i) + 1$. We consider the 12 inequalities listed in the above proof that amount to the conditions $f \sqsubseteq h \sqsubseteq g$. These are labelled $\leq_1$ through $\leq_{12}$. We note that our assumption that g covered f provides $\leq_1$ through $\leq_4$. We will establish the others.

Since $g \leq h$ we have $\leq_{10}$ and $\leq_{11}$ are true. Also, since $h = g$ except at i and $h(i) \leq f(i) \leq_4 g^R(i)$, we have $h \leq g^R$, hence $\leq_{12}$ is always true. At i we have $g(i) < f(i)$ and since $f \wedge g^L \leq_1 g$ at i, we must have $g^L(i) = g(i)$. It follows that $h \wedge g^L \leq g$ at i, and since $h = g$ everywhere but i, we have $h \wedge g^L \leq_9 g$ everywhere. Thus $\leq_9$ holds.

Since $g \leq_2 f^L$ and $g = h$ except at i, we have $h \leq f^L$ everywhere but i. But $h(i) \leq f(i)$. So $h \leq f^L$ everywhere, showing that $\leq_6$ holds. Since $g \leq h$ we have $g^R \leq h^R$. Since $f \leq_4 g^R$ it follows that $f \leq h^R$, so $\leq_8$ holds. Also, $f^R \wedge g \leq_3 f$, and since $g = h$ everywhere but i, we have $f^R \wedge h \leq f$ except at i. At i we have $h(i) \leq f(i)$ so $f^R \wedge h \leq f$ everywhere, so $\leq_7$ holds.

It remains to consider $\leq_5$. We know $g^L(i) = g(i)$, and since $h = g$ except $h(i)$ is one bigger than $g(i)$, we must have $h^L = g^L$ to the left of i and $h^L = h$ at i. Since $f \wedge g^L \leq_1 g$ and $g \leq h$, we must have $f \wedge h^L \leq g \leq h$ to the left of i and since $h^L = h$ at i we have $f \wedge h^L \leq h$ at i.

From the preceding paragraph, a failure of $\leq_5$ means there is a $j > i$ where it fails. At this j we must have $h(j) < f(j)$, and since $g = h$ except at i, we must have that $g(j) = h(j) < f(j)$. This contradicts our choice of i as the largest with $g(i) < f(i)$. So $\leq_5$ holds as well.

We have shown $f \sqsubseteq h \sqsubseteq g$. As g covers h in the double order, and $h \neq g$, we must have $h = f$. Viewing the construction of h from a different perspective, we see that g is constructed from $h = f$ by decreasing the value at i by one. Above we have shown $g^L(i) = g(i)$. Then since $f = h$ and $h = g$ except that $h(i)$ is one bigger than $g(i)$, it follows that $j < i$ implies that$f(j) < f(i)$. Thus rule 1 may be applied to f at i, and the result is g. So g is constructed from f by rule 1. □

To consider the other case for covers, we use the involution $'$ on the lattice of normal functions. We recall that f' is defined by setting $f'(j) = f(j')$ where in this context $'$ is the negation on the chain $\mathbf{n}$ given by $j' = n + 1 - j$.

Proposition 25. *If g covers f in the double order and $g(k) > f(k)$ at some k, then g is built from f by rule 2.*

Proof. The assumptions on f and g yield that f' covers g' and $f' < g'$ at some element. Then Proposition 24 gives that f' is built from g' by applying rule 1 to g' at some i. This means

$$j < i \Rightarrow g'(j) < g'(i)$$

Further, $f'(i) = g'(i) - 1$ and f' agrees with g' otherwise. Let $u = i'$. Note $f(u) = f(i') = f'(i) = g'(i) - 1 = g(u) - 1$. Also, $u < v$ implies $v' < u' = i$, hence $g'(v') < g'(i)$, and therefore giving $g(v) < g(u)$. So

$$u < v \Rightarrow f(v) = g(v) < g(u) = f(u) + 1.$$

So $u < v$ implies that $f(v) \leq f(u)$. So we may apply rule 2 to f at u, with the result being g. So g is obtained from f by an application of rule 2. □

The considerations above give the desired algorithm.

Theorem 13. *In the lattice of normal functions of $\mathbf{m^n}$ under the double order, f is covered by g if and only if g is obtained from f by an application of one of the rules in Proposition 23.*

5. Some Open Problems and Examples

In this section we list some open problems and provide several figures that illustrate lattices obtained from $\mathbf{m^n}$ under the join order or double order for some small values of m and n.

Problem 1. For natural numbers m and n, give a algorithmic description of covers in the lattice $\mathbf{m^n}$ under the join order similar to that given in Theorem 13 for the double order.

Problem 2. Give necessary and sufficient conditions on chains I and J for I^J to be a chain under the join order.

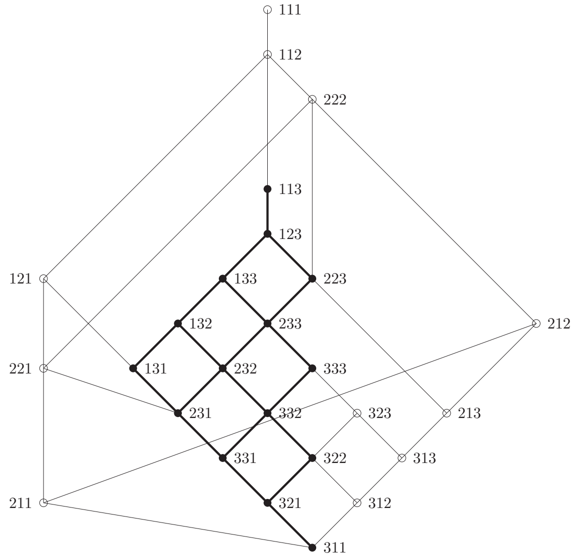


Fig. 3. The lattice $\mathbf{3^3}$ under the join order. The solid circles and darker lines indicate the convex normal functions.

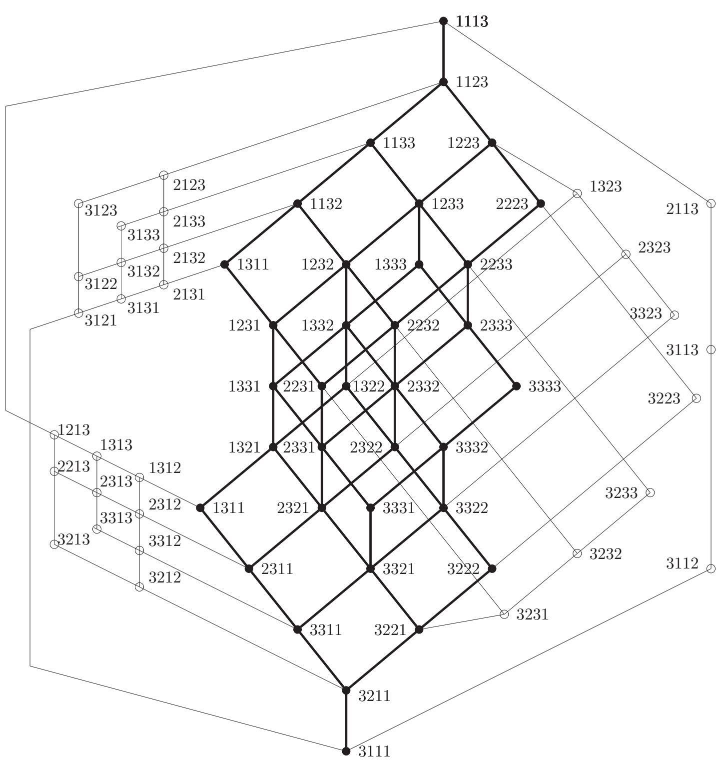


Fig. 4. The normal functions in $\mathbf{3^4}$ under the double order. The solid circles and darker lines indicate the convex normal functions.

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