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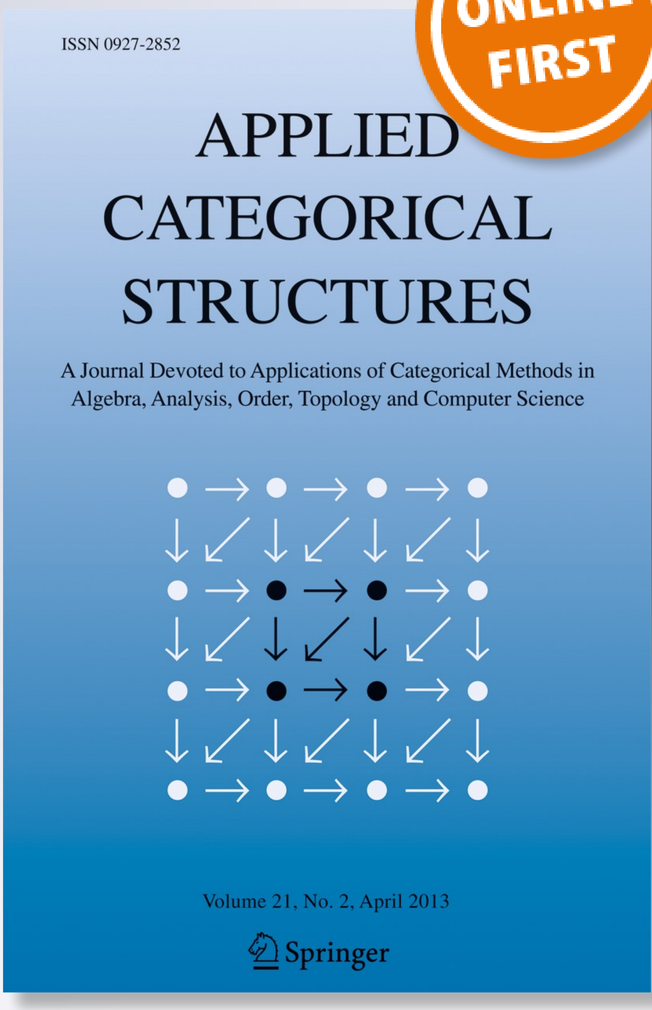

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Abstract It is well known that the category KHaus of compact Hausdorff spaces is dually equivalent to the category KRFrm of compact regular frames. By de Vries duality, KHaus is also dually equivalent to the category DeV of de Vries algebras, and so DeV is equivalent to KRFrm, where the latter equivalence can be described constructively through Booleanization. Our purpose here is to lift this circle of equivalences and dual equivalences to the setting of stably compact spaces.

The dual equivalence of KHaus and KRFrm has a well-known generalization to a dual equivalence of the categories StKSp of stably compact spaces and StKFrm of stably compact frames. Here we give a common generalization of de Vries algebras and stably compact frames we call proximity frames. For the category PrFrm of proximity frames we introduce the notion of regularization that extends that of Booleanization. This yields the category RPrFrm of regular proximity frames. We show there are equivalences and dual equivalences among PrFrm, its subcategories StKFrm and RPrFrm, and StKSp.

Restricting to the compact Hausdorff setting, the equivalences and dual equivalences among StKFrm, RPrFrm, and StKSp yield the known ones among KRFrm, DeV, and KHaus. The restriction of PrFrm to this setting provides a new category StrInc whose objects are frames with strong inclusions and whose morphisms and composition are generalizations of those in DeV. Both KRFrm and DeV are subcategories of StrInc that are equivalent to StrInc. For a compact Hausdorff space X, the category StrInc not only contains both the frame of open sets of X and the de Vries algebra of regular open sets of X, these two objects are isomorphic in StrInc, with the second being the regularization of the first. The restrictions of these categories

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are considered also in the setting of spectral spaces, Stone spaces, and extremally disconnected spaces.

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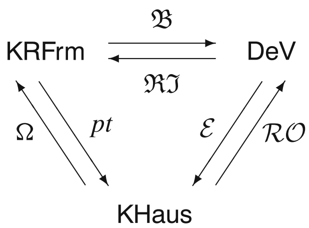
1 Introduction

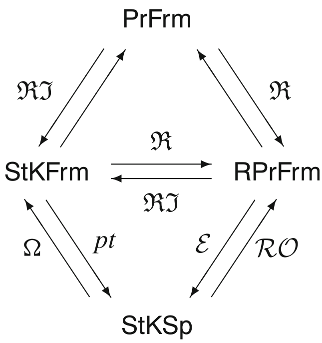
In extending Smirnov's characterization of compactifications of completely regular spaces [21] to the pointfree setting, de Vries introduced a category of structures we call DeV of de Vries algebras, and showed DeV is dually equivalent to the category KHaus of compact Hausdorff spaces. Also in the pointfree setting, Isbell [14] (see also [2, 15]) showed the category KRFrm of compact regular frames is dually equivalent to KHaus. This gives the situation in the diagram below. In [13, Theorem VI-7.4] Isbell's duality was extended to one between the categories StKSp of stably compact spaces and StKFrm of stably compact frames. Our purpose here is to lift the equivalences and dual equivalences involving DeV to the stably compact setting.



To briefly review, a de Vries algebra is a complete Boolean algebra equipped with a binary relation $\prec$, called a proximity, that satisfies certain conditions. Morphisms between de Vries algebras are certain order-preserving maps compatible with the proximities, but are not in general even lattice homomorphisms. Importantly, the composition $*$ of de Vries morphisms differs from ordinary function composition. The primary example of a de Vries algebra is the Boolean algebra of regular open sets of a compact Hausdorff space, with proximity given by $U \prec V$ if $\mathrm{cl}U \subseteq V$. This provides the regular open functor $\mathcal{RO}$ above. The maximal round filters, or ends, of a de Vries algebra form a compact Hausdorff space, much as in Stone duality, and this gives the end functor $\mathcal{E}$. The point pt and open set functor Ω are standard from pointfree topology. The functor $\mathfrak{RI}$ takes the compact regular frame of round ideals of a de Vries algebra, and $\mathfrak{B}$ is the usual Booleanization of a compact regular frame equipped with the proximity given by the restriction of the well inside relation on the frame [6].

Our first step in lifting DeV to the stably compact setting is to define the category PrFrm of proximity frames. Objects are frames with an additional relation $\prec$, called a proximity, that satisfies some quite general conditions. Examples of proximity frames include any frame with its partial ordering as its proximity, any stably compact frame with its way below relation as proximity, any de Vries algebra, and any frame with a strong inclusion in the sense of Banaschewski [1]. Morphisms in PrFrm are modeled after morphisms in DeV. They need not be frame homomorphisms, and their composition $*$ is not function composition. The round ideals of any proximity frame form a stably compact frame, providing a functor $\mathfrak{RI}$ from PrFrm to StKFrm. This, together with the inclusion functor, provide an equivalence. This counterintuitive situation, where any proximity frame is isomorphic to a stably compact frame, is caused by the fact that the composition $*$ is not function composition, allowing isomorphisms to be more general than structure-preserving isomorphisms.

While PrFrm is equivalent to StKFrm and contains DeV, it is not the generalization we seek. The equivalence between PrFrm and StKFrm does not restrict to the one between DeV and KRFrm, and we have lost touch with the Booleanization and regular open functors from the compact Hausdorff setting. Using a generalization of the Booleanization functor we call regularization, we will introduce a full subcategory RPrFrm of regular proximity frames, and establish the situation in the diagram below. Thus, it is RPrFrm that serves as our generalization of DeV to the stably compact setting.



For L a proximity frame and $a \in L$, set $k(a) = \bigwedge\{b \in L : a \prec b\}$, and define the nucleus $j(a) = \bigwedge\{(a \to k(b)) \to k(b) : b \in L\}$. The nucleus j then gives an endofunctor $\mathfrak{R}$ on PrFrm much as $\neg\neg$ gives the Booleanization functor $\mathfrak{B}$. Proximity frames where j is the identity are called regular, and the full subcategory RPrFrm of regular proximity frames serves as our analog of DeV for the stably compact setting.

The inclusions of StKFrm and RPrFrm into PrFrm are equivalences, and in PrFrm there are isomorphisms between L, its regularization $\mathfrak{R}(L)$, and its stably compact frame of round ideals $\mathfrak{RI}L$. The restrictions of the round ideal functor $\mathfrak{RI}$ and regularization functor $\mathfrak{R}$ then provide an equivalence between StKFrm and RPrFrm. The end functor restricts naturally to a functor $\mathcal{E}$ from RPrFrm to StKSp, and there is an analog of the regular open functor $\mathcal{RO}$ with $\mathcal{E}$ and $\mathcal{RO}$ providing a dual equivalence between StKSp and RPrFrm. Here $\mathcal{RO}$ applied to a stably compact space X takes the regular open sets of X in the sense of those that are the interior in the topology of X of their closures in the patch topology of X.

We have lifted our original diagram from the setting of compact Hausdorff spaces to stably compact spaces, but have gained an extra piece in the process, the category PrFrm of proximity frames. What becomes of this when we restrict back to the compact Hausdorff setting? It becomes the full subcategory StrInc of PrFrm whose objects are frames with strong inclusions in the sense of Banaschewski [1]. Both StKFrm and DeV are subcategories of StrInc, and the inclusion functors are equivalences. Further, in StrInc, there is an isomorphism between the frame of open sets and the frame of regular open sets of a compact Hausdorff space.

There is a fairly long history of considering relations akin to proximities on frames or on more general lattices. Here we mention several papers most closely related to our work, and direct the reader to [22] for a more thorough account of the background literature.

Perhaps most closely related to our work is Smyth's paper [22]. Smyth's purpose is to generalize Smirnov's characterization to the setting of T_0-spaces by introducing stable compactifications of a T_0-space X, quasi-proximities on X, and showing that the two are in bijective correspondence. In his treatment, there are a number of developments closely related to our work. Smyth's approximating auxiliary relations on a topology τ are essentially our proximity frames; he gives a key link between stably compact spaces and proximities; he develops properties of round ideals; his notion of strong covers leads to his proximal filters that parallel our prime round filters or ends; and his strong cover preservation amounts to condition 3.3.3 in our definition of proximity morphisms. Smyth's focus is different than ours, and he does not consider the matter of equivalences and dual equivalences between stably compact spaces and categories of proximity lattices. Also, the key notion of regularization, and its topological counterpart, does not appear in his work.

The paper [1] by Banaschewski aims to give a correspondence between the compactifications of a frame L and the strong inclusions on L. Thus, it extends Smirnov's characterization to the pointfree setting, much as does de Vries [10]. But unlike de Vries, who works with proximities on complete Boolean algebras, Banaschewski works with strong inclusions on frames. For the connection between the two, see [6] and Section 7 below. Our proximity frames are a direct generalization of what Banaschewski calls a strong inclusion on a frame; his strongly regular ideals amount to our round ideals; also his description of maximal strongly regular filters parallels our treatment of ends. As with Smyth's paper, Banaschewski's aim differs from ours, and does not treat equivalences and dual equivalences between categories of topological spaces and frames with proximities, nor does it touch on the notion of regularization. Of course, it is the Booleanization functor treated by Banaschewski and Pultr [4] that our regularization functor $\mathfrak{R}$ seeks to extend.

Frith [12] considers Banaschewski's frames with strong inclusions as the objects of a category he calls proximal frames, and that we denote here ProxFrm. These also are the objects of our category StrInc, but Frith's morphisms are a proper subclass of our morphisms, being frame homomorphisms preserving strong inclusion, and taken under regular function composition. Frith shows that ProxFrm is isomorphic to the coreflective subcategory of the category UniFrm of uniform frames consisting of totally bounded uniform frames. He also proves that there is a dual adjunction between UniFrm and the category UniSp of uniform spaces, which restricts to a dual equivalence between the full subcategory of UniFrm consisting of spatial uniform frames and the full subcategory of UniSp consisting of separated uniform spaces. We note that the composite UniSp $\to$ UniFrm $\to$ ProxFrm $\subseteq$ StrInc $\to$ KHaus is the functor taking a uniform space to its Samuel compactification [3], and that uniform frames can also be described in terms of Weil entourages [18].

In developing the regularization functor $\mathfrak{R}$, the paper [9] by Bruns and Lakser was one of our motivations. The regularization of the ideal frame of a distributive lattice D is the frame of distributive ideals of D that Bruns and Lakser use to create the injective hull of D in the category of meet semilattices. Roughly, this provides a version of regularization in the setting of coherent frames, a subcategory of the stably compact frames.

Finally, the paper of Jung and Sünderhauf [16] shares a similar aim with ours — that of giving a dual equivalence between the category of stably compact spaces and a category of lattices with proximities — but their techniques are quite different from ours. While we associate to a stably compact space X a proximity frame consisting of regular open sets of X (in a certain sense), they associate to X a strong proximity lattice whose elements are ordered pairs (U, K) consisting of an open set U and compact set K containing U; and while our proximity morphisms are functions under an altered notion of composition, theirs are relations under relational composition. So the techniques here differ substantially from those in [16]. Also, the notion of regularization given here has no counterpart in [16].

This paper is organized in the following way. Section 2 discusses preliminaries. Section 3 introduces the category PrFrm of proximity frames. Section 4 establishes equivalences and dual equivalences among StKSp, StKFrm, and PrFrm. Section 5 introduces regularization and the category RPrFrm of regular proximity frames. Section 6 extends the equivalences and dual equivalences of Section 4 to include RPrFrm. Section 7 restricts the results of Section 6 to the compact Hausdorff setting. Section 8 restricts the results of Section 6 to the spectral and Stone settings, and discusses links to the categories of distributive lattices and Boolean algebra. Section 9 restricts further to the setting of extremally disconnected spaces. Section 10 provides a summary of the results and a diagram illustrating the connections among the categories considered.

2 Preliminaries

In this preliminary section we recall a dual equivalence between the categories of compact Hausdorff spaces and compact regular frames, and its generalization to a dual equivalence between the categories of stably compact spaces and stably compact frames. These well-known facts can be found in [13, 15]. We also recall a dual equivalence between the categories of compact Hausdorff spaces and de Vries algebras. As a result, we obtain an equivalence between the categories of compact regular frames and de Vries algebras, which can be described constructively by means of Booleanization.

To begin, a *frame* is a complete lattice L satisfying the join infinite distributive law

$$a \wedge \bigvee S = \bigvee \{a \wedge s : s \in S\}.$$

For $a, b \in L$ we say a is *way below* b, and write $a \ll b$, if $b \leq \bigvee T$ implies there is a finite subset $S \subseteq T$ with $a \leq \bigvee S$. We say a is *compact* if $a \ll a$ and L is *compact* if 1 is compact in L. We say a is *well inside* b, and write $a \prec b$, if $\neg a \vee b = 1$ where $\neg a$ is the pseudocomplement of a.

Definition 2.1 For a frame L we say L is

(1) locally compact if $a = \bigvee \{x : x \ll a\}$ for each $a \in L$.
(2) regular if $a = \bigvee \{x : x \prec a\}$ for each $a \in L$.

(3) stable if $a \ll b, c$ implies $a \ll b \wedge c$ for all $a, b, c \in L$.
(4) stably compact if it is compact, locally compact, and stable.

A *frame homomorphism* is a map $f : L \to M$ between frames L and M that preserves finite meets (including 1) and infinite joins (including 0). We call a frame homomorphism f *proper* if $a \ll b$ implies $f(a) \ll f(b)$. Our usage of the term is motivated by [13, Definition VI-6.20, Lemma VI-6.21, and Theorem VI-7.4], where the notion of a proper continuous map is introduced and it is shown that such maps between stably compact spaces dually correspond to the frame homomorphisms between stably compact frames that preserve the way below relation. This should not be confused with the usage of the term a proper continuous map in [15, p. 104].

Let Frm be the category of frames and frame homomorphisms, KRFrm be the category of compact regular frames and frame homomorphisms, and StKFrm be the category of stably compact frames and proper frame homomorphisms. Clearly StKFrm is a proper subcategory of Frm. Also, since the well inside and way below relations coincide for compact regular frames, we have that KRFrm is a proper subcategory of StKFrm. In fact, since each frame homomorphism between compact regular frames is proper, we have that KRFrm is a full subcategory of StKFrm.

Next we recall the definition of a de Vries algebra. These algebras were first introduced by de Vries [10] under the name of complete compingent algebras. In [5] they were called de Vries algebras.

Definition 2.2 A de Vries algebra is a pair consisting of a Boolean frame (that is, a complete Boolean algebra) B together with a binary relation $\prec$ on B called a proximity that satisfies

(1) $1 \prec 1$.
(2) $a \prec b$ implies $a \leq b$.
(3) $a \leq b \prec c \leq d$ implies $a \prec d$.
(4) $a \prec b, c$ implies $a \prec b \wedge c$.
(5) $a \prec b$ implies $\neg b \prec \neg a$.
(6) $a \prec b$ implies there exists $c \in B$ such that $a \prec c \prec b$.
(7) $a \neq 0$ implies there exists $b \neq 0$ such that $b \prec a$.

A morphism between de Vries algebras A and B is a map $\varphi : A \to B$ that (1) preserves bounds, (2) preserves finite meets, and satisfies (3) $a \prec b$ implies $\neg\varphi(\neg a) \prec \varphi(b)$ and (4) $\varphi(a) = \bigvee\{\varphi(b) : b \prec a\}$. Note, each de Vries morphism satisfies $a \prec b$ implies $\varphi(a) \prec \varphi(b)$. The usual composition of two de Vries morphisms need not be a de Vries morphism. However, we have the following [10].

Theorem 2.3 *The de Vries algebras and de Vries morphisms form a category* DeV *where the composite $\psi \star \varphi$ of morphisms $\varphi : A \to B$ and $\psi : B \to C$ is given by*

$$(\psi \star \varphi)(a) = \bigvee\{\psi\varphi(b) : b \prec a\}.$$

A topological space X is *locally compact* if for each $x \in X$ and each open neighborhood U of x, there exist an open set V and a compact set K such that $x \in V \subseteq K \subseteq U$. A subset A of X is *irreducible* if $A \subseteq B \cup C$, with B, C closed, implies $A \subseteq B$ or $A \subseteq C$, and X is *sober* if each closed irreducible subset of X is the closure of a unique point of X. Clearly each sober space is T_0. A subset S of X is *saturated* if it is an intersection of open subsets of X. A detailed account of the following notion can be found in [13].

Definition 2.4 A space X is stably compact if it is compact, locally compact, sober, and the intersection of any two compact saturated sets is again compact.

With each stably compact space (X, τ) one associates two more topologies, the *co-compact topology* τ^k and the *patch topology* π. The co-compact topology is defined to have the compact saturated subsets of X as closed sets, and $\pi = \tau \vee \tau^k$ is defined to be the smallest topology containing τ and τ^k. It is well known that (X, π) is compact Hausdorff. A continuous map $f : X \to Y$ between stably compact spaces is called *proper* if the inverse image of each compact saturated set is compact, meaning that it is continuous with respect to both the τ and τ^k topologies. It follows that it is also continuous with respect to π. As we already pointed out, we follow the usage of the term from [13, Definition VI-6.20 and Lemma VI-6.21], and note that it differs from that in [15, p. 104]. The next proposition is well known.

Proposition 2.5 *The category* KHaus *of compact Hausdorff spaces and continuous maps is a full subcategory of* StKSp*, the stably compact spaces with proper continuous maps.*

For a topological space X, the open sets ΩX form a frame, and for a continuous map $f : X \to Y$, we let $\Omega f : \Omega Y \to \Omega X$ be the inverse image map $\Omega f = f^{-1}$. For a frame L, a frame homomorphism $p : L \to 2$ into the two-element frame is called a *point* of L. The collection $pt\,L$ of all points of L is topologized by $\{\varphi(a) : a \in L\}$, where $\varphi(a) = \{p : p(a) = 1\}$. For a frame homomorphism $f : L \to M$ we define $pt f : pt\,M \to pt\,L$ by setting $pt f(p) = p \circ f$. Then Ω, pt are contravariant functors giving an adjunction between Frm and the category Top of topological spaces. We further have the following key results (see [13, Theorem VI-7.4] and [15, Chapter III.1.10]).

Theorem 2.6 *The contravariant functors Ω, pt restrict to a dual equivalence between* StKSp *and* StKFrm*, as well as to a dual equivalence between* KHaus *and* KRFrm.

For a compact Hausdorff space X, the collection $\mathcal{RO}X$ of regular open subsets of X forms a Boolean frame. Defining $U \prec V$ if $\mathrm{cl}\, U \subseteq V$ gives a de Vries algebra. For $f : X \to Y$ continuous, define $\mathcal{RO} f : \mathcal{RO}Y \to \mathcal{RO}X$ by setting $\mathcal{RO} f(U) = \mathrm{int}\,\mathrm{cl}\, f^{-1}[U]$. For a de Vries algebra B, call a filter F of B *round* if for each $a \in F$ there is some $b \in F$ with $b \prec a$, and call a maximal round filter an *end*. The set $\mathcal{E}B$ of ends of B is topologized by the basis $\{\varepsilon(a) : a \in B\}$, where $\varepsilon(a) = \{F : a \in F\}$. For a de Vries morphism $\varphi : A \to B$, let $\mathcal{E}\varphi : \mathcal{E}B \to \mathcal{E}A$ be given by $\mathcal{E}\varphi(F) = \{a : b \prec a \text{ for some } b \in \varphi^{-1}[F]\}$. We then have the following theorem of de Vries [10].

Theorem 2.7 *There is a dual equivalence between* KHaus *and* DeV *given by $\mathcal{RO}$ and $\mathcal{E}$.*

It follows from Theorems 2.6 and 2.7 that KRFrm is equivalent to DeV. This can be seen directly. For a frame L, the fixed points of the nucleus $\neg\neg$ on L

form a Boolean frame $\mathfrak{B}L$ called the Booleanization of L [4]. If L is compact and regular, the restriction of the well inside relation $\prec$ on L turns $\mathfrak{B}L$ into a de Vries algebra. If $f : L \to M$ is a frame homomorphism, let $\mathfrak{B}f : \mathfrak{B}L \to \mathfrak{B}M$ be given by $\mathfrak{B}f(a) = \neg\neg f(a)$ for each $a \in \mathfrak{B}L$. For a de Vries algebra B, the round ideals $\mathfrak{R}\mathfrak{I}B$ are a compact regular frame. For a de Vries morphism $\varphi : A \to B$, define $\mathfrak{R}\mathfrak{I}\varphi : \mathfrak{R}\mathfrak{I}A \to \mathfrak{R}\mathfrak{I}B$ by setting $\mathfrak{R}\mathfrak{I}\varphi(I) = \{b : b \prec a \text{ for some } a \in \varphi[I]\}$. We then have the following [6].

Theorem 2.8 *There is an equivalence between* DeV *and* KRFrm *given by* $\mathfrak{R}\mathfrak{I}$ *and* $\mathfrak{B}$.

Let A and B be arbitrary categories. We recall [17, p. 91] that functors $F : \mathsf{A} \to \mathsf{B}$ and $G : \mathsf{B} \to \mathsf{A}$ together with natural transformations $\eta : 1_{\mathsf{A}} \to G \circ F$ and $\varepsilon : F \circ G \to 1_{\mathsf{B}}$ give an *adjoint equivalence* provided $(F, G, \eta, \varepsilon)$ is an adjunction in which both η and ε are natural isomorphisms. The following well-known fact [17, Proposition IV.4.2] will be of use.

Lemma 2.9 *Suppose* A *is a full subcategory of* B *and* $i : \mathsf{A} \to \mathsf{B}$ *is the inclusion functor. If for each object* B *in* B *there is an object* FB *of* A *and mutually inverse isomorphisms* $\mu_B : B \to FB$ *and* $\nu_B : FB \to B$*, then*

(1) *F extends to a functor where $f : B \to B'$ is sent to $Ff = \mu_{B'} \circ f \circ \nu_B$.*
(2) *(F, i, μ, ν) is an adjoint equivalence.*

3 Proximity Frames

In this section we define the category PrFrm of proximity frames.

Definition 3.1 A proximity on a frame L is a binary relation $\prec$ on L satisfying

(1) $0 \prec 0$ and $1 \prec 1$.
(2) $a \prec b$ implies $a \leq b$.
(3) $a \leq b \prec c \leq d$ implies $a \prec d$.
(4) $a, b \prec c$ implies $a \vee b \prec c$.
(5) $a \prec b, c$ implies $a \prec b \wedge c$.
(6) $a \prec b$ implies there exists $c \in L$ with $a \prec c \prec b$.
(7) $a = \bigvee\{b \in L : b \prec a\}$.

If $\prec$ is a proximity on L, we call the pair $(L, \prec)$ a proximity frame, but refer to it as L.

Example 3.2 Some examples of proximity frames are the following. (1) Any de Vries algebra is a proximity frame. (2) The partial ordering of any frame is a proximity. (3) A strong inclusion $\lhd$ on a frame [1] is a proximity. (4) The way below relation on a stably compact frame is a proximity. (5) The well inside relation on any regular frame is a proximity. (6) The really inside relation on any completely regular frame [15, Chapter IV.1] is a proximity. One further example dispels some possible spots for confusion. Let B be the power set of the natural numbers ω. Then B is a Boolean frame. Define $\prec$ on B by setting $a \prec b$ iff a is finite and $a \subseteq b$ or $b = \omega$. This yields a proximity frame whose underlying frame is a Boolean frame, but it is not a de Vries algebra. This also shows that there are frames L such that strong inclusions on L are properly contained in proximities on L.

Definition 3.3 For proximity frames L and M, a map $\varphi : L \to M$ is a proximity morphism if it satisfies

(1) $\varphi(0) = 0$ and $\varphi(1) = 1$.
(2) $\varphi(a \wedge b) = \varphi(a) \wedge \varphi(b)$.
(3) $a_1 \prec b_1$ and $a_2 \prec b_2$ imply $\varphi(a_1 \vee a_2) \prec \varphi(b_1) \vee \varphi(b_2)$.
(4) $\varphi(a) = \bigvee\{\varphi(b) : b \prec a\}$.

Remark 3.4 It is easily seen from Definition 3.3.2 that a proximity morphism φ is order-preserving, and from Definition 3.3.3 that $a \prec b$ implies $\varphi(a) \prec \varphi(b)$. An easy inductive argument shows that Definition 3.3.3 applies to an arbitrary finite n, not just to $n = 2$. We note that a proximity morphism need not preserve even finite joins, but for maps φ that do preserve finite joins, 3.3.3 is equivalent to $a \prec b \Rightarrow \varphi(a) \prec \varphi(b)$. Finally, we remark that this key condition 3.3.3 amounts to Smyth's *preservation of strong covers* [22, p. 327].

Example 3.5 As mentioned above, proximity morphisms need not be frame homomorphisms, and this allows for some unexpected features. Primary among these is the fact that function composition of proximity morphisms need not be a proximity morphism. To see this, let $L = [0, 1]$ be the unit interval with $a \prec b$ iff $a < b$ or $a = b = 0$ or $a = b = 1$; and let 2 be the two-element frame with $\leq$ as its proximity. Define maps $\varphi : L \to L$ and $\psi : L \to 2$ by $\varphi(a) = \max\{2a, 1\}$, and $\psi(a) = 0$ if $a < 1$ and $\psi(1) = 1$. Then $\psi\varphi(\frac{1}{2})$ violates the fourth condition of the definition of a proximity morphism.

Proposition 3.6 *The proximity frames and proximity morphisms form a category* PrFrm *where the composite* $\psi \star \varphi$ *of morphisms* $\varphi : A \to B$ *and* $\psi : B \to C$ *is given by*

$$(\psi \star \varphi)(a) = \bigvee\{\psi\varphi(b) : b \prec a\}.$$

Further, for $\chi : C \to D$*, we have* $(\chi \star \psi \star \varphi)(a) = \bigvee\{\chi\psi\varphi(b) : b \prec a\}$.

Proof Clearly the composite $\psi \star \varphi$ preserves bounds. To see it preserves finite meets, observe $\{c : c \prec a \wedge b\} = \{d \wedge e : d \prec a, e \prec b\}$ and use Definition 3.3.2 for φ and ψ and the frame condition that infinite joins distribute over finite meets. For the third condition, suppose $a_1 \prec b_1$ and $a_2 \prec b_2$. Then there are $a_1 \prec x_1 \prec y_1 \prec b_1$ and $a_2 \prec x_2 \prec y_2 \prec b_2$. Therefore, $(\psi \star \varphi)(a_1 \vee a_2) \leq \psi\varphi(a_1 \vee a_2) \prec \psi(\varphi(x_1) \vee \varphi(x_2)) \prec \psi\varphi(y_1) \vee \psi\varphi(y_2) \leq (\psi \star \varphi)(b_1) \vee (\psi \star \varphi)(b_2)$. For the final condition, suppose $b \prec a$. Then there is some x with $b \prec x \prec a$. From the definition of $\star$ we have $\psi\varphi(b) \leq (\psi \star \varphi)(x)$, so $(\psi \star \varphi)(a) \leq \bigvee\{(\psi \star \varphi)(x) : x \prec a\}$, hence equality. So $\psi \star \varphi$ is a proximity morphism.

To show associativity, for any proximity morphisms α and β observe that (i) $(\alpha \star \beta)(a) \leq \alpha\beta(a)$ and (ii) $b \prec a$ implies $\alpha\beta(b) \leq (\alpha \star \beta)(a)$. Let $x = \bigvee\{(\chi \star \psi)\varphi(b) : b \prec a\}$, $y = \bigvee\{\chi(\psi \star \varphi)(b) : b \prec a\}$, and $z = \bigvee\{\chi\psi\varphi(c) : c \prec a\}$. Property (i) shows

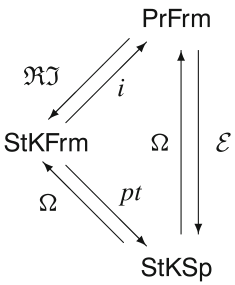
$x, y \leq z$. If $c \prec a$, then $c \prec b \prec a$ for some b, giving $\varphi(c) \prec \varphi(b)$, and by (ii) $\chi\psi\varphi(c) \leq (\chi \star \psi)\varphi(b)$. So $z \leq x$. Similarly, property (ii) gives $\psi\varphi(c) \leq (\psi \star \varphi)(b)$, so $\chi\psi\varphi(c) \leq \chi(\psi \star \varphi)(b)$, showing $z \leq y$. As $x = ((\chi \star \psi) \star \varphi)(a)$ and $y = (\chi \star (\psi \star \varphi))(a)$, associativity of $\star$ follows. Consequently, $(\chi \star \psi \star \varphi)(a) = \bigvee\{\chi\psi\varphi(b) : b \prec a\}$. Finally, for any proximity frame L, the identity map id_L on L is easily seen to be a proximity morphism. That $\varphi \star id_L = \varphi$ and $id_M \star \varphi = \varphi$ is trivial, implying that PrFrm forms a category. □

We conclude this section with an easily proved observation that will be of use. Here we use $\circ$ for usual set-theoretic composition of functions.

Lemma 3.7 *If φ, ψ are proximity morphisms and ψ preserves joins, then $\psi \star \varphi = \psi \circ \varphi$.*

4 The Basic Equivalences and Dual Equivalences

In this section we establish the following equivalences and dual equivalences using the inclusion functor i.



Before beginning, we direct the reader to the definition of a stably compact frame, and the category StKFrm of stably compact frames and proper frame homomorphisms. The following result is immediate from the definition of $\ll$ and is noted already in Example 3.2.4.

Proposition 4.1 *If L is a stably compact frame, then $\ll$ is a proximity on L.*

With a slight abuse, we consider the objects of StKFrm to be objects of PrFrm.

Proposition 4.2 StKFrm *is a full subcategory of* PrFrm.

Proof A proper frame homomorphism between stably compact frames is easily seen to be a proximity morphism. Further, as frame homomorphisms preserve arbitrary joins, by Lemma 3.7 we have $f \star g = f \circ g$ for proper frame homomorphisms f, g. So StKFrm is a subcategory of PrFrm. For fullness, suppose $\varphi : L \to M$ is a proximity morphism between stably compact frames. By definition, φ preserves the bounds, finite meets, and satisfies $a \ll b$ implies $\varphi(a) \ll \varphi(b)$, so to show it is a proper frame homomorphism it remains to show φ preserves arbitrary joins.

Let $a = \bigvee S$. Clearly $\varphi(a) \geq \bigvee\{\varphi(s) : s \in S\}$ as φ is order-preserving. For the other inequality, as φ is a proximity morphism, $\varphi(a) = \bigvee\{\varphi(b) : b \ll a\}$. Suppose $b \ll a$. By interpolation, there is c with $b \ll c \ll a$. As $a = \bigvee S$, by definition of the way below relation, there are $s_1, \ldots, s_n \in S$ with $b \ll c \leq s_1 \vee \cdots \vee s_n$. Each s_i is the updirected join of the elements way below it, and as $b \ll s_1 \vee \cdots \vee s_n$, there are $t_1 \ll s_1, \ldots, t_n \ll s_n$ with $b \leq t_1 \vee \cdots \vee t_n$. So by the definition of a proximity morphism, we then have $\varphi(b) \leq \varphi(t_1 \vee \cdots \vee t_n) \ll \varphi(s_1) \vee \cdots \vee \varphi(s_n)$. Thus, $\varphi(b) \leq \bigvee\{\varphi(s) : s \in S\}$, and so $\varphi(a) \leq \bigvee\{\varphi(s) : s \in S\}$. □

We will show that each object of PrFrm is isomorphic in this category to some object from StKFrm. To do this, we will need to produce from each proximity frame a stably compact frame. The key will be the notion of a round ideal of a proximity frame, which generalizes the notion of a round ideal of a de Vries algebra and a strongly regular ideal of [1].

Definition 4.3 For a proximity frame L and $S \subseteq L$ define

(1) $\twoheaddownarrow S = \{a \in L : a \prec s \text{ for some } s \in S\}$
(2) $\twoheaduparrow S = \{a \in L : s \prec a \text{ for some } s \in S\}$.

For $a \in L$ we write $\twoheaddownarrow a$ and $\twoheaduparrow a$ for $\twoheaddownarrow\{a\}$ and $\twoheaduparrow\{a\}$, respectively.

Definition 4.4 Let L be a proximity frame.

(1) We say an ideal I of L is a round ideal if for each $a \in I$ there is $b \in I$ with $a \prec b$.
(2) We say a filter F of L is a round filter if for each $a \in F$ there is $b \in F$ with $b \prec a$.

The following is easily seen.

Lemma 4.5 *Suppose L is a proximity frame.*

(1) *For any ideal I of L, $\twoheaddownarrow I = \bigcup\{\twoheaddownarrow a : a \in I\}$ is the largest round ideal contained in I.*
(2) *For any filter F of L, $\twoheaduparrow F = \bigcup\{\twoheaduparrow a : a \in F\}$ is the largest round filter contained in F.*

In particular, for $a \in L$, we have $\twoheaddownarrow a$ is a round ideal and $\twoheaduparrow a$ is a round filter.

We recall that the collection $\mathfrak{I}L$ of ideals of a frame L is a frame, and the way below relation on $\mathfrak{I}L$ is given by $I \ll J$ iff I is contained in the principal ideal $\downarrow a$ for some $a \in J$. We give an analogous result for the collection $\mathfrak{RI}L$ of round ideals of a proximity frame L.

Proposition 4.6 *Let L be a proximity frame.*

(1) *$\mathfrak{RI}L$ is a subframe of $\mathfrak{I}L$.*
(2) *For round ideals I, J we have $I \ll J$ in $\mathfrak{RI}L$ iff $I \subseteq \twoheaddownarrow a$ for some $a \in J$.*
(3) *For $a, b \in L$, $\twoheaddownarrow a \ll \twoheaddownarrow b$ in $\mathfrak{RI}L$ iff $a \prec b$.*

Proof

(1) Let $I = \bigvee I_\alpha$ be the join in the ideal lattice of a family of round ideals. We show I is round. If $a \in I$, then $a = a_1 \vee \cdots \vee a_n$ for some $a_i \in I_{\alpha_i}$. As each I_α is round, there are $a_1 \prec b_1, \ldots, a_n \prec b_n$ with $b_i \in I_{\alpha_i}$. Therefore, $a_1 \vee \cdots \vee a_n \prec b_1 \vee \cdots \vee b_n \in I$. So I is round. Suppose I, J are round ideals. If $d \in I \cap J$ then $d \prec a$ for some $a \in I$ and $d \prec b$ for some $b \in J$, so $d \prec a \wedge b \in I \cap J$. Thus, $I \cap J$ is round.

(2) Suppose $I \ll J$. As J is round, $J = \bigcup\{\twoheaddownarrow a : a \in J\}$. As this is a directed join in $\mathfrak{R}\mathfrak{I}L$, the definition of the way below relation gives $I \leq \twoheaddownarrow a$ for some $a \in J$. Conversely, suppose $I \subseteq \twoheaddownarrow a$ for some $a \in J$ and that $J \subseteq \bigvee\{J_\alpha : \alpha \in \kappa\}$ for some directed family of round ideals J_α. As $\mathfrak{R}\mathfrak{I}L$ is a subframe of $\mathfrak{I}L$, this directed join is a union. So $J \subseteq \bigcup\{J_\alpha : \alpha \in \kappa\}$. Then as $a \in J$, we have $a \in J_\alpha$ for some $\alpha \in \kappa$, so $I \subseteq \twoheaddownarrow a \subseteq J_\alpha$. Thus, $I \ll J$.

(3) If $\twoheaddownarrow a \ll \twoheaddownarrow b$, then by (2), $\twoheaddownarrow a \subseteq \twoheaddownarrow c$ for some $c \prec b$. This gives $a = \bigvee \twoheaddownarrow a \leq \bigvee \twoheaddownarrow c = c$, so $a \leq c \prec b$, showing $a \prec b$. Conversely, if $a \prec b$, then $a \prec c \prec b$ for some c, so $\twoheaddownarrow a \subseteq \twoheaddownarrow c$ for some $c \in \twoheaddownarrow b$, and by (2), $\twoheaddownarrow a \ll \twoheaddownarrow b$.

□

Remark 4.7 The map $\twoheaddownarrow\cdot : \mathfrak{I}L \to \mathfrak{R}\mathfrak{I}L$ sending an ideal I to the round ideal $\twoheaddownarrow I$ is not a retraction. In the final example given in Example 3.2, the principal ideals I, J generated by the odds and evens have $\twoheaddownarrow(I \vee J) \neq \twoheaddownarrow I \vee \twoheaddownarrow J$.

Proposition 4.8 *For L a proximity frame, $\mathfrak{R}\mathfrak{I}L$ is a stably compact frame.*

Proof Suppose I is round. Then $I = \bigcup\{\twoheaddownarrow a : a \in I\}$, and by Proposition 4.6.2, $\twoheaddownarrow a \ll I$ for each $a \in I$, so $I = \bigvee\{J : J \ll I\}$. Therefore, $\mathfrak{R}\mathfrak{I}L$ is locally compact. As $1 \prec 1$, it follows from Proposition 4.6.3 that $\twoheaddownarrow 1 \ll \twoheaddownarrow 1$, so $\mathfrak{R}\mathfrak{I}L$ is compact. Suppose I, J, K are round ideals with $I \ll J, K$. Then $I \subseteq \twoheaddownarrow a, \twoheaddownarrow b$ for some $a \in J$ and $b \in K$. Then $I \subseteq \twoheaddownarrow a \cap \twoheaddownarrow b = \twoheaddownarrow(a \wedge b)$ and $a \wedge b \in J \cap K$. So $I \ll J \cap K$, showing $\mathfrak{R}\mathfrak{I}L$ is stable.

□

Proposition 4.6.1 has an analogue in [1, Lemma 2] and Proposition 4.8 has analogues in [1, Lemma 2] and [22, Theorem 1].

Proposition 4.9 *For a proximity frame L, the map $(\twoheaddownarrow\cdot)_L : L \to \mathfrak{R}\mathfrak{I}L$, sending a to $\twoheaddownarrow a$, and the map $(\bigvee\cdot)_L : \mathfrak{R}\mathfrak{I}L \to L$, sending I to $\bigvee I$, are mutually inverse proximity frame isomorphisms.*

Proof Clearly $\twoheaddownarrow\cdot$ preserves bounds and as $\twoheaddownarrow(a \wedge b) = \twoheaddownarrow a \cap \twoheaddownarrow b$, it preserves finite meets. Suppose $a_1 \prec b_1$ and $a_2 \prec b_2$. Then $a_1 \in \twoheaddownarrow b_1$ and $a_2 \in \twoheaddownarrow b_2$, so $a_1 \vee a_2 \in \twoheaddownarrow b_1 \vee \twoheaddownarrow b_2$, and so by Proposition 4.6.2, $\twoheaddownarrow(a_1 \vee a_2) \ll \twoheaddownarrow b_1 \vee \twoheaddownarrow b_2$. Finally,

$$\twoheaddownarrow a = \{b : b \prec a\} = \bigcup\{\twoheaddownarrow b : b \prec a\} = \bigvee\{\twoheaddownarrow b : b \prec a\}.$$

So $\twoheaddownarrow\cdot$ is a proximity morphism.

As $\bigvee \twoheaddownarrow 0 = 0$ and $\bigvee \twoheaddownarrow 1 = 1$, $\bigvee \cdot$ preserves bounds, and basic properties of frames show $\bigvee \cdot$ preserves finite meets. Suppose $I_1 \ll J_1$ and $I_2 \ll J_2$. By Proposition 4.6.2, $I_1 \subseteq \twoheaddownarrow a_1$ and $I_2 \subseteq \twoheaddownarrow a_2$ for some $a_1 \in J_1, a_2 \in J_2$. As J_1, J_2 are round, there exist $b_1 \in J_1, b_2 \in J_2$ with $a_1 \prec b_1, a_2 \prec b_2$. Then $\bigvee(I_1 \vee I_2) \leq a_1 \vee a_2 \prec b_1 \vee b_2 \leq \bigvee J_1 \vee \bigvee J_2$. Finally, for a round ideal I we have $\bigvee I = \bigvee\{\bigvee \twoheaddownarrow a : a \in I\}$, so it follows from Proposition 4.6.2 that $\bigvee I = \bigvee\{\bigvee J : J \ll I\}$. Therefore, $\bigvee \cdot$ is a proximity morphism.

For a round ideal I we have $((\twoheaddownarrow \cdot) \star (\bigvee \cdot))(I) = \bigvee\{\twoheaddownarrow \bigvee J : J \ll I\}$. By Proposition 4.6.2, this equals $\bigvee\{\twoheaddownarrow \bigvee \twoheaddownarrow a : a \in I\}$, which is $\bigvee\{\twoheaddownarrow a : a \in I\}$, hence equal to I. For $a \in L$ we have $((\bigvee \cdot) \star (\twoheaddownarrow \cdot))(a) = \bigvee\{\bigvee \twoheaddownarrow b : b \prec a\} = \bigvee\{b : b \prec a\} = a$. □

Remark 4.10 The isomorphisms produced in the above result are isomorphisms in the category of proximity frames, but are in general not bijections. For instance, any frame L is a proximity frame with its partial ordering $\leq$ as its proximity. In this case all ideals of L are round, so $\mathfrak{RI}L$ is simply the ideal lattice $\mathfrak{I}L$, and surely the above maps between L and $\mathfrak{I}L$ are not bijections. This somewhat counter-intuitive behavior occurs as composition in our category is given by $\star$ rather than by usual composition of functions.

As StKFrm is a full subcategory of PrFrm, the above result, in the context of Lemma 2.9, leads us to consider the functor $\mathfrak{RI} : \mathsf{PrFrm} \to \mathsf{StKFrm}$ taking a proximity frame L to its round ideals $\mathfrak{RI}L$, and a proximity morphism $\varphi : L \to M$ to $\mathfrak{RI}\varphi = (\twoheaddownarrow \cdot)_M \star \varphi \star (\bigvee \cdot)_L$. Then for $i : \mathsf{StKFrm} \to \mathsf{PrFrm}$ the inclusion functor, Propositions 4.2, 4.9 and Lemma 2.9 give the following.

Theorem 4.11 *$(\mathfrak{RI}, i, \bigvee \cdot, \twoheaddownarrow \cdot)$ is an adjoint equivalence between* PrFrm *and* StKFrm.

The functor $\mathfrak{RI}$ will play a key role, and its description can be simplified.

Proposition 4.12 *For φ a proximity morphism, $\mathfrak{RI}\varphi = \twoheaddownarrow \varphi[\cdot]$.*

Proof By Proposition 3.6 we have $((\twoheaddownarrow \cdot)_M \star \varphi \star (\bigvee \cdot)_L)(I) = \bigvee\{\twoheaddownarrow \varphi \bigvee J : J \ll I\}$. By Proposition 4.6.2, this is equal to $\bigvee\{\twoheaddownarrow \varphi \bigvee \twoheaddownarrow a : a \in I\}$, and hence to $\bigvee\{\twoheaddownarrow \varphi(a) : a \in I\}$. □

We turn our attention to the dual equivalence between PrFrm and StKSp. Recall (see Proposition 4.6.1) that the collection $\mathfrak{RI}L$ of round ideals of a proximity frame forms a subframe of the frame $\mathfrak{I}L$ of all ideals of L. The same proof shows that the collection $\mathfrak{RF}L$ of round filters, partially ordered again by set inclusion, forms a subframe of the frame $\mathfrak{F}L$ of all filters of L. We recall that an element p of a distributive lattice is *meet-prime* if $a \wedge b \leq p$ implies either $a \leq p$ or $b \leq p$.

Definition 4.13 Let L be a proximity frame.

(1) We call a round ideal of L prime if it is a meet-prime element of $\mathfrak{RI}L$.

(2) We call a round filter of L prime if it is a meet-prime element of $\mathfrak{R}\mathfrak{F}L$.
(3) Let $\mathcal{P}L$ be the set of all prime round ideals of L.
(4) Let $\mathcal{E}L$ be the set of all prime round filters of L.

Following common terminology from the studies of compactifications and de Vries algebras, we call prime round filters of a proximity frame *ends*, so $\mathcal{E}L$ is the set of ends of L.

Lemma 4.14 *For a proximity frame L, there is a bijection α_L from the points of $\mathfrak{R}\mathfrak{I}L$ to the set $\mathcal{E}L$ of ends of L given by $\alpha_L(p) = F_p$ where $F_p = \{a \in L : p(\downarrow a) = 1\}$.*

Proof Clearly F_p is an upset, and as $\downarrow a \cap \downarrow b = \downarrow(a \wedge b)$ it is a filter of L. Suppose $a \in F_p$. As $\downarrow a = \bigvee\{\downarrow b : b \prec a\}$ and p preserves arbitrary joins, there is $b \prec a$ with $p(\downarrow b) = 1$. So F_p is a round filter. Suppose G, H are round filters with neither contained in F_p. Then there are $a \in G - F_p$ and $b \in H - F_p$. Therefore, there are $x \in G$ with $x \prec a$ and $y \in H$ with $y \prec b$, and we have $\downarrow(x \vee y) \subseteq \downarrow a \vee \downarrow b$ since $x \vee y$ belongs to the join of the ideals $\downarrow a$ and $\downarrow b$. As $p(\downarrow a) = 0$ and $p(\downarrow b) = 0$, we have $p(\downarrow(x \vee y)) = 0$, giving $x \vee y$ belongs to $G \cap H$ but not to F_p. So F_p is a prime round filter.

Given a prime round filter F, define $p_F : \mathfrak{R}\mathfrak{I}L \to 2$ by setting

$$p_F(I) = \begin{cases} 0 \text{ if } I \cap F = \emptyset \\ 1 \text{ if } I \cap F \neq \emptyset \end{cases}$$

Clearly p_F preserves the bounds. Suppose I, J are round ideals with $p_F(I) = 1$ and $p_F(J) = 1$. Then there are $a, b \in F$ with $a \in I$ and $b \in J$. Then $a \wedge b \in I \cap J$, showing $p_F(I \cap J) = 1$. It follows that p_F preserves finite meets.

Claim $p_F(\downarrow a_1 \vee \cdots \vee \downarrow a_n) = p_F(\downarrow a_1) \vee \cdots \vee p_F(\downarrow a_n)$ for each $a_1, \ldots, a_n \in L$.

Proof of Claim Say $p_F(\downarrow a_1 \vee \cdots \vee \downarrow a_n) = 1$. Then there is $a \in F$ with $a \in \downarrow a_1 \vee \cdots \vee \downarrow a_n$. So there are $b_1, \ldots, b_n$ with $b_i \prec a_i$ for $i \leq n$ with $a = b_1 \vee \cdots \vee b_n$. Then $\uparrow b_1 \cap \cdots \cap \uparrow b_n = \uparrow a \subseteq F$. As F is meet-prime in the lattice of round filters, there is $i \leq n$ with $\uparrow b_i \subseteq F$. Use interpolation to find x_i with $b_i \prec x_i \prec a_i$. Then $x_i \in \downarrow a_i \cap F$, so $p_F(\downarrow a_i) = 1$.

To see p_F preserves joins, suppose I_α $(\alpha \in \kappa)$ is a family of round ideals with $p_F(\bigvee_\kappa I_\alpha) = 1$. Then there is $a \in F$ with $a \in \bigvee_\kappa I_\alpha$. So there are $b_1, \ldots, b_n$ with $b_i \in I_{\alpha_i}$ and $a = b_1 \vee \cdots \vee b_n$. As the I_α are round, there are $x_i \in I_{\alpha_i}$ with $b_i \prec x_i$ for $i \leq n$. Then $a \in \downarrow x_1 \vee \cdots \vee \downarrow x_n$, showing $p_F(\downarrow x_1 \vee \cdots \vee \downarrow x_n) = 1$. By the above claim $p_F(\downarrow x_i) = 1$ for some $i \leq n$, hence $\bigvee_\kappa p_F(I_\alpha) = 1$. This shows p_F preserves joins, so is a point of $\mathfrak{R}\mathfrak{I}L$. It is routine to see $F_{p_F} = F$ and $p_{F_p} = p$, providing the bijective correspondence. □

We next use this bijection to lift the topology from the points to the prime round filters. We recall that for a proximity frame L, the open sets of the space $pt\,\mathfrak{R}\mathfrak{I}L$ are the sets of the form $\varphi(I)$ for a round ideal I, where $\varphi(I) = \{p \in pt\,\mathfrak{R}\mathfrak{I}L : p(I) = 1\}$.

Proposition 4.15 *For a proximity frame L and $a \in L$, set*

$$\mathfrak{p}(a) = \{F \in \mathcal{E}L : a \in F\}.$$

Then $\alpha_L[\varphi(\Downarrow a)] = \mathfrak{p}(a)$ for each $a \in L$. Therefore, the collection of sets $\{\mathfrak{p}(a) : a \in L\}$ is a basis for a topology on $\mathcal{E}L$, and with this topology α_L is a homeomorphism.

Proof Note $\varphi(\Downarrow(a \wedge b)) = \varphi(\Downarrow a) \cap \varphi(\Downarrow b)$ and $\varphi(I) = \bigcup\{\varphi(\Downarrow a) : a \in I\}$ for any round ideal I. So $\{\varphi(\Downarrow a) : a \in L\}$ is a basis for the topology on $pt\,\mathfrak{R}\mathfrak{I}L$. Noting that $p(\Downarrow a) = 1$ iff $a \in F_p$ gives $\alpha_L[\varphi(\Downarrow a)] = \mathfrak{p}(a)$. The rest is immediate. □

We next use the bijections α_L to deal with morphisms.

Lemma 4.16 *For $\varphi : L \to M$ a proximity morphism, $\alpha_L \circ (pt\,\mathfrak{R}\mathfrak{I}\,\varphi) \circ \alpha_M^{-1} = \Uparrow\varphi^{-1}[\cdot]$.*

Proof Let $G \in \mathcal{E}M$. Then $G = G_q$ for some point q of $\mathfrak{R}\mathfrak{I}M$. Applying $\alpha_L \circ (pt\,\mathfrak{R}\mathfrak{I}\,\varphi) \circ \alpha_M^{-1}$ to G_q gives the end $F_{q\circ f}$, where $f = \mathfrak{R}\mathfrak{I}\,\varphi$. By Proposition 4.12, $F_{q\circ f} = \{a \in L : q(\Downarrow\varphi[\Downarrow a]) = 1\}$. Noting $\Downarrow\varphi[\Downarrow a] = \bigvee\{\Downarrow\varphi(c) : c \prec a\}$ and using the fact that q is a point then gives that $F_{q\circ f} = \{a \in L : q(\Downarrow\varphi(c)) = 1 \text{ for some } c \prec a\}$, so $F_{q\circ f} = \Uparrow\varphi^{-1}[G_q]$. □

The following is now immediate.

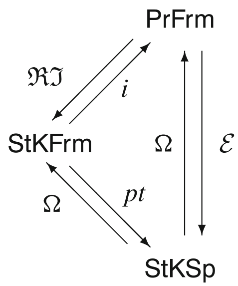
Theorem 4.17 *There is a contravariant functor $\mathcal{E}$: PrFrm → StKSp taking a proximity frame L to its space $\mathcal{E}L$ of ends and a proximity morphism $\varphi : L \to M$ to $\Uparrow\varphi^{-1}[\cdot]$. Further, this functor is naturally equivalent to the composite $pt \circ \mathfrak{R}\mathfrak{I}$ via α, and therefore $\mathcal{E}$ and Ω provide a dual equivalence.*

We summarize our results below.

Theorem 4.18 *There is a circle of equivalences and dual equivalences among* StKFrm, StKSp, *and* PrFrm *given by pt, Ω, $\mathcal{E}$, $\mathfrak{R}\mathfrak{I}$, and i where*

(1) *pt takes L to its space of points and f to $(\cdot) \circ f$.*
(2) *Ω takes X to its frame of opens and f to $f^{-1}[\cdot]$.*
(3) *$\mathcal{E}$ takes L to its space of ends and φ to $\Uparrow\varphi^{-1}[\cdot]$.*
(4) *$\mathfrak{R}\mathfrak{I}$ takes L to its frame of round ideals and φ to $\Downarrow\varphi[\cdot]$.*

(5) *i is the inclusion functor from* StKFrm *to* PrFrm.



We conclude this section with a few further observations.

Remark 4.19 For a stably compact space X, we consider its open sets ΩX as a proximity frame whose proximity is the way below relation $\ll$. It follows from [13, Proposition I-1.4] that for $U, V \in \Omega X$ we have $U \ll V$ iff there exists a compact saturated K such that $U \subseteq K \subseteq V$. Note this gives $U \ll V$ iff $\mathrm{cl}_\pi U \subseteq V$. Then each open set V is the union of ones of the form $\mathrm{int}\,\mathrm{cl}_\pi(U)$ where $U \ll V$. So open sets of the form $\mathrm{int}\,\mathrm{cl}_\pi(U)$ give a basis for the topology on X. These sets will play a prominent role later in the paper.

Remark 4.20 We have seen there is a bijection between points p of $\mathfrak{RI}L$ and prime round filters F of L. By basic principles there is a bijection between points and the meet-prime elements of $\mathfrak{RI}L$, and these are the prime round ideals I of L. These are given by

$$p \rightsquigarrow F_p \text{ where } F_p = \{a : p(\downarrow\!\!\!\!\!\downarrow a) = 1\}$$

$$p \rightsquigarrow I_p \text{ where } I_p = \bigvee\{\downarrow\!\!\!\!\!\downarrow a : p(\downarrow\!\!\!\!\!\downarrow a) = 0\}.$$

So there is a bijection between the prime round ideals and the prime round filters of L. These correspondences can be found using the inverse of $p \rightsquigarrow F_p$ given in the proof of Lemma 4.14 and the well-known correspondence between points and meet-prime elements. We obtain

$$F \rightsquigarrow I_F \text{ where } I_F = \{a : \uparrow\!\!\!\!\!\uparrow a \not\subseteq F\}$$

$$I \rightsquigarrow F_I \text{ where } F_I = \{a : \downarrow\!\!\!\!\!\downarrow a \not\subseteq I\}.$$

Noting that $\varphi(\downarrow\!\!\!\!\!\downarrow a) = \{p : p(\downarrow\!\!\!\!\!\downarrow a) = 1\}$ corresponds to $\chi(a) = \{I : \downarrow\!\!\!\!\!\downarrow a \not\subseteq I\}$ we obtain $\{\chi(a) : a \in L\}$ is a basis for a topology on the prime round ideals making this space homeomorphic to the space of ends of L.

Remark 4.21 As the stably compact frames are a full subcategory of the proximity frames, the points of $\mathfrak{RI}L$ are exactly the proximity morphisms into the two-element proximity frame 2. As L is isomorphic to its frame of round ideals, the points of $\mathfrak{RI}L$ are in bijective correspondence with the proximity morphisms from L to 2, which we call the *proximity points* of L. We could then topologize the set $pptL$ of proximity points to obtain another space homeomorphic to the space of prime round filters, and extend this to yet another functor providing a duality. The details are left to the reader.

Remark 4.22 We have shown the frame $\mathfrak{RI}L$ of round ideals of L is isomorphic to the frame of open sets of the stably compact space $\mathcal{E}L$. The round filters of L also play a natural role. It is not a difficult exercise to show that there are mutually inverse frame isomorphisms $\Gamma : \mathfrak{RF}L \to \mathfrak{RF}(\mathfrak{RI}L)$ and $\Delta : \mathfrak{RF}(\mathfrak{RI}L) \to \mathfrak{RF}L$ given by

$$\Gamma(F) = \{I : I \cap F \neq \emptyset\}$$

$$\Delta(\mathcal{F}) = \{\bigvee I : I \in \mathcal{F}\}.$$

It then follows from the Hofmann–Mislove Theorem [13, Theorem II-1.20] that the frame of round filters of L is isomorphic to the frame of closed sets of the co-compact topology of $\mathcal{E}L$ (equivalently, of compact saturated sets of $\mathcal{E}L$) ordered by reverse inclusion.

5 Regular Proximity Frames

We show each proximity frame carries a special nucleus j; use this nucleus to define regular proximity frames; and show j gives a functor $\mathfrak{R}$, much as $\neg\neg$ gives the Booleanization functor $\mathfrak{B}$, providing an equivalence between PrFrm and its full subcategory RPrFrm of regular proximity frames. We begin with a few basics [15, Chapter II.2].

Definition 5.1 A nucleus on a frame L is a map $j : L \to L$ that satisfies

(1) $j(a \wedge b) = j(a) \wedge j(b)$.
(2) $a \leq j(a)$.
(3) $jj(a) \leq j(a)$.

If in addition $j(0) = 0$, then j is called a dense nucleus.

Let L_j denote the fixed points of a nucleus j. It is well known that L_j is a frame, although not necessarily a subframe of L, and that $j : L \to L_j$ is an onto frame homomorphism. It is also well known that for any element c of a frame L, the map $w_c : L \to L$ defined by $w_c(a) = (a \to c) \to c$ is a nucleus, where $\to$ is the usual Heyting implication of a frame, and that the pointwise meet of nuclei is a nucleus.

Definition 5.2 Let L be a proximity frame. Define maps k and j from L to L by

(1) $k(a) = \bigwedge \Uparrow a$.
(2) $j(a) = \bigwedge\{w_{k(b)}(a) : b \in L\}$.

We let L_k be the fixed points of k and L_j be the fixed points of j.

Proposition 5.3 *For a proximity frame L and $a \in L$,*

(1) $a \leq j(a) \leq k(a)$.

(2) $\Uparrow a = \Uparrow j(a) = \Uparrow k(a)$.
(3) *k is a closure operator on L with $k(0) = 0$.*
(4) *j is a dense nucleus on L.*

Proof (1) As j is the pointwise meet of nuclei, it is a nucleus, so $a \leq j(a)$. As $a \leq \bigwedge \Uparrow a = k(a)$, the definition of j gives $j(a) \leq w_{k(a)}(a) = (a \to k(a)) \to k(a) = k(a)$. (2) As $a \leq j(a) \leq k(a)$ we have $\Uparrow k(a) \subseteq \Uparrow j(a) \subseteq \Uparrow a$. If $b \in \Uparrow a$, then $a \prec b$, so there is c with $a \prec c \prec b$. Then $k(a) \leq c \prec b$, showing $k(a) \prec b$, so $b \in \Uparrow k(a)$. (3) Surely k is order-preserving, $a \leq k(a)$, and by the second part $kk(a) = \bigwedge \Uparrow k(a) = \bigwedge \Uparrow a = k(a)$. So k is a closure operator, and $0 \prec 0$ gives $k(0) = 0$. (4) As $j(0) \leq k(0) = 0$, the nucleus j is dense. □

Remark 5.4 The above shows the fixed points of k are fixed by j, so $L_k \subseteq L_j$. Both L_k and L_j are complete lattices with meets agreeing with those in L, and joins given by applying either k or j to the join taken in L. In general neither L_k nor L_j is a sublattice of L. The lattice L_k need not be distributive, but L_j forms a frame.

We come to our key definition.

Definition 5.5 For a proximity frame L, call $a \in L$ regular if it is a fixed point of j. We say the proximity frame L is regular if each element of L is regular. Let RPrFrm be the full subcategory of PrFrm whose objects are the regular proximity frames.

Example 5.6 We give an example of a proximity frame that is not regular. Let $L = \omega + 2$. For $a, b \in L$ with $a \neq \omega$, set $a \prec b$ iff $a \leq b$, and set $\omega \prec \omega + 1$. It is easy to verify that L is a proximity frame such that $a \prec a$ for each $a \in L - \{\omega\}$, so ω is the only element of L that is not reflexive. Note that $k(a) = a$ for each $a \in L - \{\omega\}$ and $k(\omega) = \omega + 1$. Therefore, $w_{k(a)}(\omega) = \omega + 1$ for each $a \in L$, so $j(\omega) = \omega + 1$. Thus, ω is not a regular element of L, and so L is not a regular frame. Other examples will be given in Sections 6 and 7.

We now give the key results of this section.

Proposition 5.7 *For a proximity frame L, the restriction $\prec_j$ of $\prec$ to L_j is a proximity.*

Proof The first three conditions of Definition 3.1 are clear. (4) If $a, b, c \in L_j$ and $a, b \prec_j c$, then $a, b \prec c$, so $a \vee b \prec c$, and by Proposition 5.3.2 we have $j(a \vee b) \prec c$. Then as $j(a \vee b)$ is the join of a, b in L_j, the result follows. (5) This follows as meets in L_j agree with those in L. (6) Suppose $a, b \in L_j$ with $a \prec_j b$. Then $a \prec b$ so there is $c \in L$ with $a \prec c \prec b$. Then by Proposition 5.3.2, $a \prec j(c) \prec b$. (7) Let $a \in L_j$. Then a is the join in L of $\{b \in L : b \prec a\}$. But for any $b \prec a$ we have $j(b) \in L_j$ and $j(b) \prec a$. So a is the join in L of $\{b \in L_j : b \prec a\}$, hence a is the join in L_j of $\{b \in L_j : b \prec_j a\}$. So $\prec_j$ is a proximity on L_j. □

In the following we have need to consider several proximity frames in the same argument. We shall often use k_L, $\to_L$, j_L, $\bigwedge_L$, $\bigvee_L$ to indicate the operations belonging to the proximity frame L. In particular, j_{L_j} is the nucleus in the derived frame L_j.

Lemma 5.8 *For a proximity frame L and $a, b \in L_j$,*

(1) $k_{L_j}(a) = k_L(a)$.
(2) $a \to_{L_j} b = a \to_L b$.
(3) $j_{L_j}(a) = j_L(a)$.

Proof (1) Note first that meets in L_j agree with those in L. We have $k_L(a) = \bigwedge\{c \in L : a \prec c\}$. But if $a \prec c$, there is some $d \in L$ with $a \prec d \prec c$, so by Proposition 5.3.2 we have $a \prec j_L(d) \prec c$. It follows that $\bigwedge\{c \in L : a \prec c\} = \bigwedge\{d \in L_j : a \prec_j d\}$, so $k_L(a) = k_{L_j}(a)$. (2) This is well known; see, e.g., [7, Proposition 7]. (3) By definition, $j_L(a) = \bigwedge\{(a \to_L k_L(c)) \to_L k_L(c) : c \in L\}$. For any $c \in L$ we have $k_L(c) \in L_j$. It then follows from part (1) that $\{k_L(c) : c \in L\}$ equals $\{k_{L_j}(b) : b \in L_j\}$. So $j_L(a) = \bigwedge\{(a \to_{L_j} k_{L_j}(b)) \to_{L_j} k_{L_j}(b) : b \in L_j\}$, hence is equal to $j_{L_j}(a)$. □

The following is now trivial.

Proposition 5.9 *For a proximity frame L, the proximity frame L_j is regular.*

Lemma 5.10 *For a proximity frame L, consider the map $j : L \to L_j$ and let $i : L_j \to L$ be the identical embedding. Then*

(1) *j is both an onto frame homomorphism and a proximity morphism.*
(2) *i is a one-one proximity morphism.*
(3) $j \star i = j \circ i = 1_{L_j}$
(4) $i \star j = 1_L$.

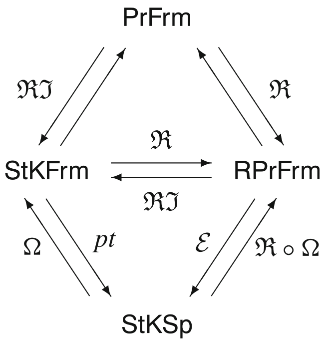
Proof (1) As j is a nucleus, $j : L \to L_j$ is an onto frame homomorphism. As j is a dense nucleus, it preserves $0, 1$, so the first condition in the definition of a proximity morphism is satisfied. As $j : L \to L_j$ preserves finite joins and meets the second and third conditions follow. For the fourth condition, as $j : L \to L_j$ preserves joins, $j(a) = j(\bigvee_L\{b : b \prec a\}) = \bigvee_{L_j}\{j(b) : b \prec a\}$. (2) Surely i preserves $0, 1$, and as meets in L_j agree with those in L it preserves finite meets as well. If $a_1 \prec b_1$ and $a_2 \prec b_2$, then $a_1 \vee a_2 \prec b_1 \vee b_2$, so by Proposition 5.3.2 $j(a_1 \vee a_2) \prec b_1 \vee b_2$. As the join $a_1 \vee_{L_j} a_2$ of a_1, a_2 in L_j is $j(a_1 \vee a_2)$, we have $i(a_1 \vee_{L_j} a_2) \prec i(b_1) \vee i(b_2) \leq i(b_1) \vee_{L_j} i(b_2)$. Finally, for $a \in L_j$, Proposition 5.3.2 shows a is the join in L of $\{b \in L_j : b \prec_j a\}$. (3) As the map $j : L \to L_j$ preserves arbitrary joins, by Lemma 3.7 $j \star i$ is given by ordinary function composition and clearly $j \circ i = 1_{L_j}$. (4) By definition, $(i \star j)(a) = \bigvee_L\{i\, j(b) : b \prec a\}$, and again, by Proposition 5.3.2, we have $a = \bigvee_L\{j(b) : b \prec a\}$. □

Lemmas 2.9 and 5.10 provide the following.

Theorem 5.11 *There is a functor $\mathfrak{R} : \mathsf{PrFrm} \to \mathsf{RPrFrm}$, called the regularization functor, that takes a proximity frame L to its frame L_j of regular elements, and a proximity morphism $\varphi : L \to M$ to $\mathfrak{R}(\varphi) = j_M \star \varphi \star i_L$. Further, this functor and the inclusion functor give an adjoint equivalence between PrFrm and RPrFrm.*

6 Equivalences and Dual Equivalences via Regularization

The equivalences and dual equivalences of Section 4, with the equivalence of the previous section, provide the circle of equivalences and dual equivalences shown below. Here we consider further aspects of these, and in particular, show $\mathfrak{R} \circ \Omega$ has a topological description in terms of sets that are regular open in a certain sense.



For a stably compact space X with topology τ we use int for interior with respect to τ, cl_k for closure with respect to the co-compact topology τ^k and cl_π for closure with respect to the patch topology π. For a stably compact frame L, we recall the points $pt\,L$ are a stably compact space with $\{\varphi(a) : a \in L\}$ as a topology.

Proposition 6.1 *For L a stably compact frame with $X = pt\,L$ and $a \in L$,*

(1) $\varphi(k(a)) = \mathrm{int}\,\mathrm{cl}_k\,\varphi(a)$.
(2) $\varphi(j(a)) = \mathrm{int}\,\mathrm{cl}_\pi\,\varphi(a)$.

Proof

(1) The round filters of L are exactly the Scott open filters of L in the sense of [13, p. 139]. So the Hofmann–Mislove Theorem [13, Theorem II-1.20] gives a frame isomorphism from the frame of round filters of L to the frame of closed subsets of τ^k partially ordered by reverse set inclusion. Here a round filter F is taken to $\bigcap\{\varphi(a) : a \in F\}$, which we denote $\varphi(F)$. Clearly $\varphi(b) \subseteq \varphi(F)$ iff $b \leq \bigwedge F$. It follows that $\mathrm{cl}_k\,\varphi(a)$ is given by the largest round filter contained in $\uparrow a$, so by $\twoheaduparrow a$. Then as $k(a) = \bigwedge \twoheaduparrow a$, we have $\varphi(k(a)) = \mathrm{int}\,\mathrm{cl}_k \varphi(a)$.

(2) As every round filter is the join of ones of the form $\twoheaduparrow c$, the $\varphi(\twoheaduparrow c)$ where $c \in L$ are a basis for the closed sets of τ^k. Since $\pi = \tau \vee \tau^k$, the sets $\varphi(\twoheaduparrow c) \cup -\varphi(d)$ where $c, d \in L$ are a basis for the closed sets of π. So for $b \in L$, having $\varphi(b)$ contained in $\mathrm{cl}_\pi\,\varphi(a)$ is equivalent to $\varphi(b) \subseteq \varphi(\twoheaduparrow c) \cup -\varphi(d)$ for each $c, d \in L$ with $\varphi(a) \subseteq \varphi(\twoheaduparrow c) \cup -\varphi(d)$. This is equivalent to $b \wedge d \leq k(c)$ for each $c, d \in L$ with $a \wedge d \leq k(c)$. For any $c \in L$, $d = a \to k(c)$ is the largest element whose meet with a is beneath $k(c)$. So $\varphi(b) \subseteq \mathrm{cl}_\pi\,\varphi(a)$ is equivalent to $b \wedge (a \to k(c)) \leq$

$k(c)$ for each $c \in L$, hence to $b \leq (a \to k(c)) \to k(c)$ for each $c \in L$. As $j(a) = \bigwedge\{(a \to k(c)) \to k(c) : c \in L\}$ it follows that $\varphi(j(a)) = \operatorname{int}\operatorname{cl}_\pi \varphi(a)$. □

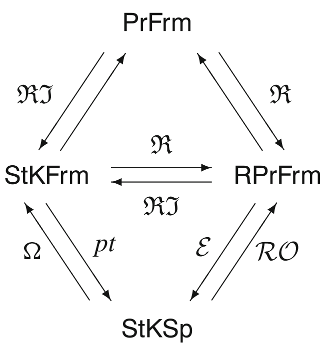
The above result shows that for a stably compact space X, the nucleus j on ΩX is given by $j(U) = \operatorname{int}\operatorname{cl}_\pi U$. So the fixed points of j are those U that are regular open in the sense that $U = \operatorname{int}\operatorname{cl}_\pi U$. For these reasons, we often call $\mathfrak{R}$ the *regularization* functor, and the fixed points of j *regular* elements. From the discussion in Remark 4.19 the following is now immediate.

Theorem 6.2 *Using $\mathcal{RO}$ for $\mathfrak{R} \circ \Omega$ we have $\mathcal{RO}$ takes a stably compact space X to its frame of regular open sets $\mathcal{RO}X$, in the above sense, with proximity $U \prec V$ if $\operatorname{cl}_\pi U \subseteq V$; and for $f : X \to Y$ a proper continuous map, $\mathcal{RO}f : \mathcal{RO}Y \to \mathcal{RO}X$ takes a regular open set U to $\operatorname{int}\operatorname{cl}_\pi f^{-1}[U]$.*

The functor $\mathfrak{R}$ from StKFrm to RPrFrm is the restriction of the functor $\mathfrak{R}$ given in the previous section. For $\varphi : L \to M$ a morphism between stably compact frames, we then have $\mathfrak{R}(\varphi) : L_j \to M_j$ is given by $\mathfrak{R}(\varphi) = j_M \star \varphi \star i_L$. But in this setting both j_M and φ preserve arbitrary joins, so reduces to $j_M \circ \varphi \circ i_L$, hence to $j_M \circ \varphi$. We now have the following.

Theorem 6.3 *There is a circle of equivalences and dual equivalences among* StKSp, StKFrm, PrFrm, *and* RPrFrm *given by pt, Ω, $\mathfrak{RI}$, $\mathfrak{R}$, $\mathcal{RO}$, and $\mathcal{E}$, where*

(1) *pt takes L to its space of points and f to $(\cdot) \circ f$.*
(2) *Ω takes X to its frame of opens and f to $f^{-1}[\cdot]$.*
(3) *$\mathfrak{RI}$ takes L to its frame of round ideals and φ to $\twoheaddownarrow\varphi[\cdot]$.*
(4) *$\mathfrak{R}$:* PrFrm → RPrFrm *takes L to its fixed points L_j and f to $j * f * i$.*
(5) *$\mathfrak{R}$:* StKFrm → RPrFrm *takes L to its fixed points L_j and f to $j \circ f$.*
(6) *$\mathcal{RO}$ takes X to $\{U : U = \operatorname{int}\operatorname{cl}_\pi U\}$ and f to $\operatorname{int}\operatorname{cl}_\pi f^{-1}[\cdot]$.*
(7) *$\mathcal{E}$ takes L to its space of ends and φ to $\twoheaduparrow\varphi^{-1}[\cdot]$.*



The equivalences and dual equivalences described above have a counter-intuitive aspect, due to the fact that isomorphisms in the category PrFrm are not what one would expect. For instance, each proximity frame is isomorphic to its frame of round ideals and to its regularization. Also, for each stably compact space, the proximity frames of its open and regular open sets are isomorphic. Of course, these isomorphisms are not bijections, and this occurs because composition is given by $\star$ rather than function composition. While composition is still given by $\star$ in RPrFrm, we will show isomorphisms are well behaved.

Lemma 6.4 *For a proximity frame L, in* $\mathfrak{RI}L$ *we have*

(1) $k(I) = \twoheaddownarrow k(\bigvee I)$.

(2) $j(I) = \twoheaddownarrow j(\bigvee I)$.

Proof

(1) By definition, $k(I) = \bigwedge\{J : I \ll J\}$. If $I \ll J$, it follows from Proposition 4.6.2 and roundness that $I \ll \twoheaddownarrow b$ for some $\twoheaddownarrow b \subseteq J$. Proposition 4.6 also shows $I \ll \twoheaddownarrow b$ iff $\bigvee I \prec b$. For any $S \subseteq L$, we have $\bigwedge\{\twoheaddownarrow s : s \in S\} = \twoheaddownarrow \bigwedge S$, giving $k(I) = \bigwedge\{\twoheaddownarrow b : I \ll \twoheaddownarrow b\} = \twoheaddownarrow k(\bigvee I)$.

(2) Let I be a round ideal and $a \in L$. If $b \in I \cap \twoheaddownarrow(\bigvee I \to a)$, then $b \le \bigvee I \wedge (\bigvee I \to a) \le a$, so by roundness $I \cap \twoheaddownarrow(\bigvee I \to a) \subseteq \twoheaddownarrow a$. Suppose J is round and $I \cap J \subseteq \twoheaddownarrow a$. If $b \in J$, then $b \wedge c \le a$ for all $c \in I$, hence $b \wedge \bigvee I \le a$, showing $b \le \bigvee I \to a$. Again by roundness, $J \subseteq \twoheaddownarrow(\bigvee I \to a)$. This shows $I \to \twoheaddownarrow a = \twoheaddownarrow(\bigvee I \to a)$. We then have

$$\begin{aligned} j(I) &= \bigwedge\{(I \to k(J)) \to k(J) : J \in \mathfrak{RI}L\} \\ &= \bigwedge\{(I \to \twoheaddownarrow ka) \to \twoheaddownarrow ka : a \in L\} \\ &= \bigwedge\{\twoheaddownarrow(\bigvee I \to ka) \to \twoheaddownarrow ka : a \in L\} \\ &= \bigwedge\{\twoheaddownarrow((\bigvee I \to ka) \to ka) : a \in L\} \end{aligned}$$

Using again the fact that for any $S \subseteq L$ we have $\bigwedge\{\twoheaddownarrow s : s \in S\} = \twoheaddownarrow \bigwedge S$, the result follows.

□

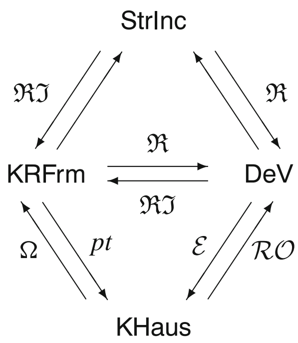
So in any regular proximity frame L, the fixed points in $\mathfrak{RI}L$ of j are exactly the $\twoheaddownarrow a$ where $a \in L$. So for L regular, there are structure-preserving bijections between L and $(\mathfrak{RI}L)_j$ sending a to $\twoheaddownarrow a$ and conversely. As $j : \mathfrak{RI}L \to (\mathfrak{RI}L)_j$ and $\bigvee \cdot : \mathfrak{RI}L \to L$ preserve joins, $j \star (\twoheaddownarrow\cdot) = j \circ (\twoheaddownarrow\cdot)$ and $(\bigvee \cdot) \star i = (\bigvee \cdot) \circ i$, so these are the isomorphisms giving the described structure-preserving bijections. We extend this with the following.

Proposition 6.5 *Isomorphisms in* RPrFrm *are structure-preserving bijections.*

Proof Let $\varphi : L \to M$ be an isomorphism between regular proximity frames. The previous result shows that the fixed points under j in $\mathfrak{RI}L$ are the $\twoheaddownarrow a$ where $a \in L$, and the fixed points under j in $\mathfrak{R}\mathfrak{I}M$ are the $\Downarrow b$ where $b \in M$. By general considerations of functors, $\mathfrak{R}\mathfrak{I}\varphi$ is an isomorphism between $\mathfrak{R}\mathfrak{I}L$ and $\mathfrak{R}\mathfrak{I}M$, and by Proposition 4.8 we have $\mathfrak{R}\mathfrak{I}L$ and $\mathfrak{R}\mathfrak{I}M$ are stably compact frames. By Proposition 4.2 morphisms between stably compact frames are proper frame homomorphisms under ordinary composition, so $\mathfrak{R}\mathfrak{I}\varphi$ is a structure-preserving bijection. Therefore, $\mathfrak{R}\mathfrak{I}\varphi(\Downarrow a)$ is a fixed point of j in $\mathfrak{R}\mathfrak{I}M$. So by the description of $\mathfrak{R}\mathfrak{I}\varphi$ in Theorem 6.3 and Lemma 6.4.2, we have $\mathfrak{R}\mathfrak{I}\varphi(\Downarrow a) = \Downarrow\varphi[\Downarrow a] = j(\Downarrow\varphi[\Downarrow a]) = \Downarrow j(\bigvee\varphi[\Downarrow a]) = \Downarrow j\varphi(a) = \Downarrow\varphi(a)$. It follows that φ equals the usual composite of functions $(\bigvee\cdot)_M \circ (\mathfrak{R}\mathfrak{I}\varphi) \circ (\Downarrow\cdot)$. As we noticed in the discussion prior to this result, for regular proximity frames the maps $\bigvee\cdot$ and $\Downarrow\cdot$ are structure-preserving bijections, so as φ is the usual composite of structure-preserving bijections, it is a structure-preserving bijection. □

7 Restricting to the Compact Hausdorff Setting

Having established a circle of equivalences and dual equivalences among the categories StKSp, StKFrm, PrFrm, and RPrFrm for the stably compact setting, we show these restrict to, and expand upon, those we began with in the compact Hausdorff setting. We first define a full subcategory StrInc of PrFrm we call the category of *frames with strong inclusions* that will serve as a counterpart for PrFrm in the compact Hausdorff setting. We then show KHaus, KRFrm, StrInc, and DeV are full subcategories of StKSp, StKFrm, PrFrm, and RPrFrm respectively, and that the functors providing equivalences and dual equivalences in the stably compact setting restrict to the compact Hausdorff setting, and that these restrictions include the original equivalences and dual equivalences we considered in the compact Hausdorff setting. The situation is shown below.



Definition 7.1 A strong inclusion on a frame L is a proximity $\prec$ on L that satisfies

(1) $\prec$ is a subset of the well inside relation on L.
(2) $a \prec b$ implies $\neg b \prec \neg a$.

We let StrInc be the full subcategory of PrFrm whose objects are frames with strong inclusions.

Remark 7.2 Strong inclusions were introduced by Banaschewski [1] in his pointfree treatment of compactifications. He did not consider frames with strong inclusions as a category. Frith [12] did consider frames with strong inclusions as a category, but with different morphisms than we use. Frith's morphisms were frame homomorphisms compatible with the strong inclusions. Frith called his category that of *proximal frames*, which we denote ProxFrm, and showed it was isomorphic to the full subcategory of the category UniFrm of uniform frames consisting of totally bounded uniform frames. Clearly ProxFrm is a non-full subcategory of StrInc. Fletcher and Hunsaker [11] also considered Frith's category ProxFrm, but under the name proximity frames. We have kept the name proximity frame for our more general notion, preferring to use the term strong inclusion for the more restrictive notion.

Lemma 7.3 *Suppose $\prec$ is a strong inclusion on a frame L and $a, b \in L$.*

(1) $a \prec b \Rightarrow \neg\neg a \leq b$.
(2) $a \prec \neg b \Leftrightarrow b \prec \neg a$.

Proof (1) If $a \prec b$, then by Definition 7.1.1 a is well inside b, and this implies $\neg\neg a \leq b$. (2) If $a \prec \neg b$, then by Definition 7.1.2 $\neg\neg b \prec \neg a$, hence $b \prec \neg a$. The converse follows by symmetry. □

By definition, StrInc is a full subcategory of PrFrm. It is well known that KHaus is a full subcategory of StKSp and KRFrm is a full subcategory of StKFrm, but it is easy to see these directly as well. From basic topology, a compact Hausdorff space is locally compact, irreducible sets are singletons, each subset is saturated, and compact sets are closed sets, so it is stably compact. Further, the topology, co-compact topology, and patch topology all agree, so continuous maps between compact Hausdorff spaces are proper. For a compact regular frame, the way below relation $\ll$ and well inside relation $\prec$ agree, so each compact regular frame is stably compact, and as frame homomorphisms preserve the well inside relation, frame homomorphisms between compact regular frames are proper.

Proposition 7.4 *The category* DeV *is a full subcategory of* RPrFrm.

Proof One easily sees every de Vries algebra is a proximity frame. In fact, de Vries algebras are exactly the proximity frames L where L is Boolean and $a \prec b$ implies $\neg b \prec \neg a$. In a de Vries algebra, we see $a = \bigwedge\{b : a \prec b\}$, showing $k(a) = a$, hence $j(a) = a$. So every de Vries algebra is regular.

To see every de Vries morphism φ is a proximity morphism, we need only show $a_1 \prec b_1$ and $a_2 \prec b_2$ imply $\varphi(a_1 \vee a_2) \prec \varphi(b_1) \vee \varphi(b_2)$. But this follows from [6, Lemma 2.2]. Conversely, if φ is a proximity morphism between de Vries algebras, then for $a \prec b$ we must show $\neg\varphi(\neg a) \prec \varphi(b)$. Choose c, d with $a \prec c \prec d \prec b$. Then $\neg c \prec \neg a$ and $c \prec d$, so $1 = \varphi(\neg c \vee c) \prec \varphi(\neg a) \vee \varphi(d)$, showing $\neg\varphi(\neg a) \leq \varphi(d) \prec \varphi(b)$. So the proximity morphisms between de Vries algebras are exactly the de Vries morphisms, and as the definitions of composition $*$ agree, our result follows. □

Clearly the functors Ω and *pt* between KHaus and KRFrm are the restrictions of Ω and *pt* between StKSp and StKFrm. We show the corresponding results hold for the restrictions of the other functors from the stably compact setting.

Proposition 7.5 *The functors $\mathcal{RO}$ and $\mathcal{E}$ between* DeV *and* KHaus *are the restrictions of the functors $\mathcal{RO}$ and $\mathcal{E}$ between* RPrFrm *and* StKSp *of Section 6.*

Proof To begin, compare the definitions of the functors given at the start of this section with Theorem 6.3. The situation for $\mathcal{RO}$ is obvious from the fact that cl_π and cl agree for any compact Hausdorff space. To settle matters for $\mathcal{E}$, we have only to show the prime round filters of a de Vries algebra are exactly the maximal round filters; then the notions of ends for de Vries algebras and for proximity frames agree, the way in which the space of ends is topologized is by definition identical, and the action of the functors on morphisms agree.

Suppose F is a round filter of a de Vries algebra. Surely if F is maximal among round filters it is prime in the lattice of round filters. Suppose F is prime in the lattice of round filters and F is properly contained in the round filter G. By roundness, there are $a \prec b \prec c$ all belonging to $G - F$. Then $\neg c \prec \neg b \prec \neg a$. As $\Uparrow b \cap \Uparrow \neg b = \Uparrow 1 \subseteq F$ and $c \notin F$, primeness then gives $\Uparrow \neg b \subseteq F \subseteq G$. Then $\neg a$ and a belong to G, so F is maximal among round filters. □

Proposition 7.6 *The inclusion and round ideal functor $\mathfrak{RI}$ between* StKFrm *and* PrFrm *restrict to functors between* KRFrm *and* StrInc*.*

Proof To show the inclusion functor restricts to a functor from KRFrm to StrInc, we need only that the well inside relation on a compact regular frame is a strong inclusion. This was given in [1, p. 108, Example (2)]. To see that $\mathfrak{RI}$ restricts to a functor from StrInc to KRFrm we need that the round ideals of a frame with strong inclusion form a compact regular frame. This was established in [1, Lemma 2] where round ideals were called strongly regular. □

Proposition 7.7 *The inclusion and regularization functor $\mathfrak{R}$ between* PrFrm *and* RPrFrm *restrict to functors between* StrInc *and* DeV*. Further, $\mathfrak{R}$ takes a frame with strong inclusion to its Booleanization equipped with the restriction of the strong inclusion.*

Proof That the proximity of a de Vries algebra is a strong inclusion is given in [6, Lemma 2.4], so the inclusion functor restricts as described.

Suppose $(L, \prec)$ is a frame with strong inclusion and $a \in L$. Then $\neg a = \bigvee\{b : b \prec \neg a\}$. Using the behavior of pseudocomplement in a frame, we have $\neg\neg a = \bigwedge\{\neg b : b \prec \neg a\}$. Therefore, by Lemma 7.3.2, $\neg\neg a = \bigwedge\{\neg b : a \prec \neg b\}$. As $k(a) = \bigwedge\{c : a \prec c\}$, this shows $k(a) \leq \neg\neg a$. Lemma 7.3.1 provides $a \prec c$ implies $\neg\neg a \leq c$. So the definition of $k(a)$ provides $\neg\neg a \leq k(a)$. Thus, $k(a) = \neg\neg a$ in any frame with a strong inclusion. So in L, $j(a) = \bigwedge\{(a \to \neg\neg b) \to \neg\neg b : b \in L\}$. For $b \in L$, general properties of pseudocomplement on a frame [19, p. 60] give $a \to \neg\neg b = a \to (\neg b \to 0) = \neg b \to (a \to 0) = \neg b \to \neg a$. So $\neg\neg a \wedge (a \to \neg\neg b) = (\neg b \to \neg a) \wedge (\neg a \to 0) \leq \neg b \to 0 = \neg\neg b$. This shows $\neg\neg a \leq (a \to \neg\neg b) \to \neg\neg b$ for each $b \in L$, hence $\neg\neg a \leq j(a)$. By Proposition 5.3.1 $j(a) \leq k(a)$ always holds, and as $k(a) = \neg\neg a$, we have $j(a) = \neg\neg a$.

Thus, $\mathfrak{R}L$ is the Booleanization of L with proximity $\prec$ being the restriction of the strong inclusion on L. This Booleanization is a complete Boolean algebra, and this restriction $\prec$ is a proximity that satisfies $a \prec b \Rightarrow \neg b \prec \neg a$, hence gives a de Vries algebra. □

As an immediate corollary, we have the following.

Corollary 7.8 *The functors $\mathfrak{R}\mathfrak{I}$ and $\mathfrak{B}$ between* DeV *and* KRFrm *are the restrictions of the functors $\mathfrak{R}\mathfrak{I}$ and $\mathfrak{R}$ between* RPrFrm *and* StKFrm.

Remark 7.9 That j agrees with $\neg\neg$ for compact regular frames has a simple interpretation when viewed topologically. In the compact regular frame of open sets of a compact Hausdorff space, we have $\neg\neg U = \operatorname{int}\operatorname{cl} U$, and in Section 6 we have seen $j(U) = \operatorname{int}\operatorname{cl}_\pi U$. As the patch topology and given topology agree for any compact Hausdorff space, the result is clear.

We have established the following.

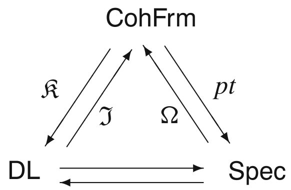
Theorem 7.10 *The functors from Section 6 giving equivalences and dual equivalences among* StKSp*,* StKFrm*,* PrFrm*, and* RPrFrm *restrict to those giving equivalences and dual equivalences among* KHaus*,* KRFrm*,* StrInc*, and* DeV *shown at the beginning of this section.*

Remark 7.11 Extra information has been added to the compact Hausdorff setting. As the category StrInc contains both KRFrm and DeV as full subcategories, it provides a common generalization of both. This provides an interesting link between these point-free versions of compact Hausdorff spaces. In fact, for a compact Hausdorff space X, Lemma 5.10 and the above result that $j = \neg\neg$ for compact regular frames, shows that the Booleanization map $\neg\neg$ is an isomorphism in StrInc from the open sets of X to the regular open sets of X. Thus, these are literally isomorphic ways to treat the space X.

Remark 7.12 As we pointed out in Remark 7.2, Frith's category ProxFrm of proximal frames also has frames with strong inclusions as objects, but morphisms are frame homomorphisms preserving strong inclusions. Frith showed that ProxFrm is isomorphic to the coreflective subcategory of the category UniFrm of uniform frames consisting of totally bounded uniform frames. He also proved that there is a dual adjunction between UniFrm and the category UniSp of uniform spaces, which restricts to a dual equivalence between the full subcategory of UniFrm consisting of spatial uniform frames and the full subcategory of UniSp consisting of separated uniform spaces. It is natural to consider the composite UnifSp $\to$ UniFrm $\to$ ProxFrm $\subseteq$ StrInc $\to$ KHaus. As expected, this is the Samuel compactification [3].

8 The Spectral and Stone Settings

In this section we restrict the equivalences and dual equivalences from the stably compact setting to the setting of spectral and Stone spaces, the Stone duals of bounded distributive lattices and Boolean algebras. We begin by recalling the familiar Stone duality between the categories of bounded distributive lattices and spectral spaces, and their point-free version, the category of coherent frames.



Here DL is the category of bounded distributive lattices and bound preserving lattice homomorphisms, and Spec is that of *spectral spaces* and *spectral maps*. These are compact sober spaces where compact open sets are closed under finite intersections and form a basis, and continuous maps where the inverse image of a compact open set is compact. A frame is *coherent* if its compact elements are a bounded sublattice and each element is a join of compact elements. In any coherent frame, $a \ll b$ iff there is a compact k with $a \leq k \leq b$. The category CohFrm has coherent frames as objects, and frame homomorphisms that map compact elements to compact elements as its morphisms. The following is well known [15, Chapter II.3].

Theorem 8.1 *Let $\mathfrak{I}$ be the functor taking a bounded distributive lattice to its ideal lattice and a lattice homomorphism f to $\downarrow f[\cdot]$; and $\mathfrak{K}$ be the functor taking a coherent frame to its lattice of compact elements and a coherent frame homomorphism f to its restriction to the compact elements. Then $\mathfrak{I}, \mathfrak{K}$, together with pt, Ω and the functors from Stone duality provide a circle of equivalences and dual equivalences among* DL*,* CohFrm*, and* Spec*.*

We next provide the proximity frame counterparts to these categories.

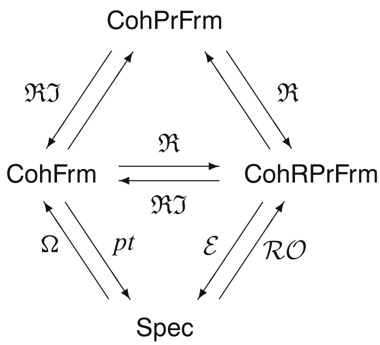
Definition 8.2 An element r of a proximity frame is reflexive if $r \prec r$, and a proximity frame is defined to be coherent if $a \prec b$ implies there is a reflexive r with $a \prec r \prec b$. CohPrFrm is the category of coherent proximity frames and the proximity morphisms between them.

Definition 8.3 The category CohRPrFrm of coherent regular proximity frames is the full subcategory of RPrFrm whose objects are additionally coherent.

As the compact saturated subsets of a spectral space are exactly the intersections of compact open sets, it follows that any spectral space is stably compact, and that the spectral maps between spectral spaces are exactly the proper continuous maps. In a coherent frame, we have $a \ll b$ iff there is a compact k with $a \leq k \leq b$, and it follows that each coherent frame is stably compact, and that the coherent frame homomorphisms are exactly the proper frame homomorphisms between coherent frames. We then have the following.

Proposition 8.4 *The categories* Spec, CohFrm, CohPrFrm, *and* CohRPrFrm *are respectively full subcategories of* StKSp, StKFrm, PrFrm, *and* RPrFrm.

Our aim is to show the equivalences and dual equivalences from Section 6 restrict to give the equivalences and dual equivalences shown below.



Proposition 8.5 *Suppose L is a coherent proximity frame and M is a coherent frame.*

(1) *The regularization $\mathfrak{R}L$ is a coherent frame.*
(2) *The frame $\mathfrak{RI}L$ of round ideals of L is coherent.*
(3) *With its way below relation $\ll$ as proximity, M is a coherent proximity frame.*

Proof

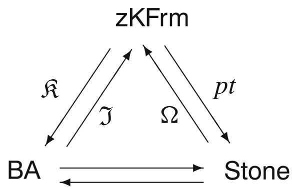
(1) If $r \in L$ is reflexive, then as $k(r) = \bigwedge\{b : r \prec b\}$ we have $k(r) = r$, and as $r \leq j(r) \leq k(r)$, it follows that $r = j(r)$, hence is regular. For any regular a, b in L, there is a reflexive $r \in L$ with $a \prec r \prec b$, and then $a \prec r \prec b$ in L_j as well.
(2) Proposition 4.8 shows that $\mathfrak{RI}L$ is a stably compact frame. By Proposition 4.6.2 we have $I \ll J$ in $\mathfrak{RI}L$ iff $I \subseteq \Downarrow a$ for some $a \in J$, and it follows that the compact elements of $\mathfrak{RI}L$ are exactly the $\Downarrow r$ where r is reflexive in L. (Note, $\Downarrow r = \downarrow r$ for r reflexive.) Suppose I is a round ideal of L and $a \in I$. By roundness there is $b \in I$ with $a \prec b$, and as L is coherent, there is a reflexive r with $a \prec r \prec b$. This shows $I = \bigcup\{\Downarrow r : r \text{ is reflexive and } r \in I\}$. So each element of $\mathfrak{RI}L$ is a join of compact elements. As the meet of reflexive elements is reflexive, the above description of compact elements as the $\downarrow r$ for r reflexive shows the compact elements of $\mathfrak{RI}L$ are a sublattice. Thus, $\mathfrak{RI}L$ is a coherent frame.
(3) An element of M is reflexive under $\ll$ iff it is compact. Suppose $a \ll b$ in M. As M is coherent, b is a join of compact elements, and as $a \ll b$ there is a compact element c with $a \leq c \leq b$. Then $c \ll c$ and $a \ll c \ll b$.

□

Theorem 8.6 *The equivalences and dual equivalences from Section 6 among the categories* StKSp, StKFrm, PrFrm, *and* RPrFrm *restrict to give equivalences and dual equivalences among the categories* Spec, CohFrm, CohPrFrm, *and* CohRPrFrm.

Proof It is known [15, Section II.3] that Ω and pt restrict to a dual equivalence between CohFrm and Spec. Proposition 8.5.3 shows the inclusion functor restricts to a functor from CohFrm to CohPrFrm, and Proposition 8.5.2 shows $\mathfrak{R}\mathfrak{I}$ restricts to a functor from CohPrFrm to CohFrm. As these are full subcategories, these restrictions give an equivalence. Clearly inclusion is a functor from CohRPrFrm to CohPrFrm, and Proposition 8.5.1 shows regularization restricts to a functor from CohPrFrm to CohRPrFrm. Thus, these restrictions give an equivalence. By definition, $\mathcal{RO} = \mathfrak{R} \circ \Omega$, hence restricts to a functor from Spec to CohRPrFrm. For a proximity frame L, Proposition 4.15 shows the space $\mathcal{E}L$ of ends of L is homeomorphic to $pt\,\mathfrak{R}\mathfrak{I}L$, so by the above, $\mathcal{E}L$ is a spectral space. Thus, $\mathcal{RO}$ and $\mathcal{E}$ restrict to a dual equivalence between CohRPrFrm and Spec. From these results $\mathfrak{R}$ and $\mathfrak{R}\mathfrak{I}$ restrict to functors between CohFrm and CohRPrFrm, hence provide an equivalence. □

We next restrict further to the setting of Stone spaces.



Here BA is the category of Boolean algebras and their homomorphisms. A *Stone space* is a compact Hausdorff space having a basis of clopen sets, and Stone is the category of Stone spaces and continuous maps between them. A frame is *zero-dimensional* if each element is a join of complemented elements, and zKFrm is the category of zero-dimensional compact frames and the frame homomorphisms between them. It is known that BA, Stone, and zKFrm are full subcategories of DL, Spec, and CohFrm, and that the equivalences and dual equivalences among DL, Spec, and CohFrm restrict to ones among BA, Stone and zKFrm [15]. We next provide proximity frame counterparts of these categories.

Definition 8.7 A strong inclusion $\prec$ on a frame L is zero-dimensional if for each $a \prec b$ there is a reflexive r with $a \prec r \prec b$. We let zStrInc be the full subcategory of StrInc whose objects are frames with zero-dimensional strong inclusions.

Definition 8.8 Let zDeV be the full subcategory of DeV whose objects are de Vries algebras whose proximity is a zero-dimensional strong inclusion.

Proposition 8.9

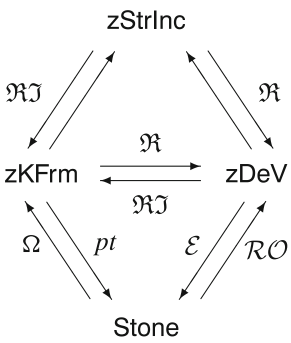
(1) Stone = KHaus ∩ Spec.
(2) zKFrm = KRFrm ∩ CohFrm.
(3) zStrInc = StrInc ∩ CohPrFrm.
(4) zDeV = DeV ∩ CohRPrFrm.

Proof (1) A spectral space has a basis of compact open sets, and if it is a Hausdorff space these will be closed, hence clopen. Conversely, any Stone space is clearly

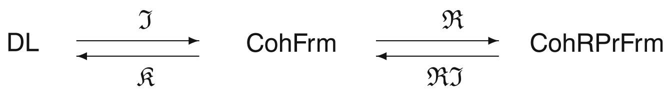
Hausdorff and spectral. (2) It is well known that each zero-dimensional compact frame is regular. Conversely, in a compact regular frame, the compact elements are exactly the complemented elements, hence if it is coherent, it is zero-dimensional. (3) This is direct from the definition. (4) A zero-dimensional de Vries algebra is defined to be a de Vries algebra that is a coherent proximity frame. The result follows as we have shown every de Vries algebra is a regular proximity frame. □

The results from the spectral setting and the compact Hausdorff setting then immediately give the following.

Theorem 8.10 *The equivalences and dual equivalences from Section 6 among the categories* StKSp*,* StKFrm*,* PrFrm*, and* RPrFrm *restrict to give equivalences and dual equivalences among the categories* Stone*,* zKFrm*,* zStrInc*, and* zDeV.



Before concluding this section, we consider the relationship of the categories DL and BA to categories of regular proximity frames. In the distributive lattice setting, this situation is shown below.



The equivalence between DL and CohRPrFrm is given by $\mathfrak{N} \circ \mathfrak{I}$ and $\mathfrak{K} \circ \mathfrak{N}\mathfrak{I}$. It is worthwhile to describe these functors directly. For L a coherent regular proximity frame, we saw in the proof of Proposition 8.5 that the compact elements of $\mathfrak{N}\mathfrak{I}L$ are exactly the $\twoheaddownarrow a = {\downarrow} a$ where a is reflexive in L. So up to isomorphism, $\mathfrak{K} \circ \mathfrak{N}\mathfrak{I}$ takes L to the distributive lattice of its reflexive elements, and a proximity morphism φ to its restriction to the reflexive elements. We let $\mathfrak{X} = \mathfrak{K} \circ \mathfrak{N}\mathfrak{I}$ and call this the *reflexive element functor*. To consider the composite $\mathfrak{N} \circ \mathfrak{I}$ we need the following.

Definition 8.11 For a bounded distributive lattice L we say $M \subseteq L$ is admissible if

(1) $\bigvee M$ exists.
(2) For each $a \in L$, $\bigvee\{a \wedge m : m \in M\}$ exists and equals $a \wedge \bigvee M$.

An ideal I of L is called a D-ideal if for each $M \subseteq I$ with M admissible we have $\bigvee M \in I$.

These D-ideals were introduced by Bruns and Lakser in [9] in the setting of semilattices, rather than just for distributive lattices as suits our need here. They showed the set $\mathcal{I}_D S$ of D-ideals of a semilattice S is the injective hull of S in the category of semilattices, and characterized this as the unique join-dense extension of S preserving joins of admissible sets.

Proposition 8.12 *For a bounded distributive lattice L, the fixed points under k in the ideal frame $\mathfrak{I}L$ are the normal ideals of L, and the fixed points under j are the D-ideals of L.*

Proof In $\mathfrak{I}L$ we have $I \ll J$ iff $I \subseteq \downarrow a$ for some $a \in J$. So $k(I)$ is the intersection of the principal ideals containing I, hence is the smallest normal ideal containing I. It follows that $j(I) = \bigcap\{(I \to N) \to N : N$ is a normal ideal of $L\}$. Use $D(I)$ for the smallest D-ideal containing I and note [9, Lemma 3] shows $D(I) = \{\bigvee M : M \subseteq I$ and M is admissible$\}$. We will show $j(I) = D(I)$.

Let $M \subseteq I$ be admissible, N be a normal ideal, and note $I \to N = \{a : a \wedge b \in N$ for all $b \in I\}$. So if $a \in I \to N$ we have $a \wedge m \in N$ for all $m \in M$. As normal ideals are closed under joins, the admissibility of M gives $a \wedge \bigvee M = \bigvee\{a \wedge m : m \in M\} \in N$. So $\bigvee M \in (I \to N) \to N$. This shows $D(I) \subseteq j(I)$.

For $a \in L$ let $M_a = \{a \wedge b : b \in I\}$. We first show that $a \in j(I)$ implies $a = \bigvee M_a$. Take any upper bound u of M_a and note $a \in I \to \downarrow u$. If $a \in j(I)$ we have $a \in (I \to \downarrow u) \to \downarrow u$, so $a \wedge a \leq u$. Thus, $a = \bigvee M_a$. We next show $a \in j(I)$ implies M_a is admissible. So for $c \in L$ we must show $c \wedge \bigvee M_a = \bigvee\{c \wedge m : m \in M_a\}$. But this follows trivially as $a \wedge c \in j(I)$ and $M_{a \wedge c} = \{c \wedge m : m \in M_a\}$. So, by [9, Lemma 3] $j(I) \subseteq D(I)$. □

So, making use of the least distributive ideal operation $D(\cdot)$, we have the following.

Theorem 8.13 *Use $\mathfrak{D}$ for the composite $\mathfrak{R} \circ \mathfrak{I}$. Then $\mathfrak{D}$ takes a bounded distributive lattice to its frame $\mathcal{I}_D L$ of D-ideals where $I \prec J$ iff $I \subseteq \downarrow a \subseteq J$ for some $a \in L$; and $\mathfrak{D}$ takes a bounded lattice homomorphism f to $D(f[\cdot])$.*

Collecting the above results gives the following.

Theorem 8.14 *There is an equivalence between* DL *and* CohRPrFrm *given by the reflexive element functor $\mathfrak{X}$ and the D-ideal functor $\mathfrak{D}$.*

Remark 8.15 Our results from the Stone setting show $\mathfrak{X}$ and $\mathfrak{D}$ restrict to an equivalence between BA and zDeV. Using the fact that the distributive ideals of a Boolean algebra are exactly the normal ideals, in the Boolean setting, the description of the functor $\mathfrak{D}$ can be simplified. For a Boolean algebra B, $\mathfrak{D}B$ is the de Vries algebra of normal ideals of B where $I \prec J$ if there is $a \in B$ with $I \subseteq \downarrow a \subseteq J$, and for a homomorphism f, $\mathfrak{D}(f)$ is the de Vries morphism $LU(f[\cdot])$, the function taking a normal ideal to the normal ideal generated by its image under f. The functor $\mathfrak{X}$ takes a zero-dimensional de Vries algebra to its Boolean algebra of reflexive elements. Thus, we arrive at the situation described in [5].

9 The Extremally Disconnected Setting

Recall that a compact Hausdorff space is extremally disconnected if the closure of each open set is clopen. The category of extremally disconnected compact Hausdorff spaces and continuous maps is known to be dually equivalent to the category of complete Boolean algebras and the Boolean homomorphisms between them [23]. Here we consider categories of proximity frames related to the notion of extremal disconnectedness.

Definition 9.1 A proximity frame is extremally disconnected if its regular and reflexive elements coincide. A stably compact frame is extremally disconnected if it is extremally disconnected as a proximity frame with its way below relation as its proximity. A stably compact space is extremally disconnected if its frame of open sets is extremally disconnected.

Definition 9.2 Let ePrFrm be the full subcategory of PrFrm whose members are extremally disconnected, and use eStKFrm, eKRFrm, eRPrFrm, eDeV, eStKSp, and eKHaus with similar meanings. Let frm be the full subcategory of DL whose members are frames, and cBA be the full subcategory of BA whose members are complete Boolean algebras.

These definitions have a number of equivalents.

Proposition 9.3 *A proximity frame L is extremally disconnected iff the proximity of its regularization L_j is its partial ordering. So a regular proximity frame is extremally disconnected iff its proximity is its partial ordering.*

Proof If L is extremally disconnected, then each regular element is reflexive, so the proximity on L_j must be the partial ordering $\leq$. Conversely, if the proximity on L_j is its partial ordering, then each regular element is reflexive; and if a is reflexive, then $j(a) = a$, so a is regular. Thus, L is extremally disconnected. It follows that a regular proximity frame is extremally disconnected iff its proximity is its partial ordering. □

Proposition 9.4 *A stably compact frame is extremally disconnected iff its regular elements under the way below proximity $\ll$ are exactly its compact elements.*

Proof Reflexive elements under $\ll$ are exactly compact elements. □

Proposition 9.5 *A stably compact space is extremally disconnected iff the closure in the patch topology of any open set is open.*

Proof Suppose X is a stably compact space. By Theorem 6.2 and the comments that precede it, the nucleus j on $\Omega(X)$ is given by $U = \operatorname{int}\operatorname{cl}_\pi U$ where cl_π is closure in the patch topology. Suppose $\Omega(X)$ is extremally disconnected and U is open in X. Then $\operatorname{int}\operatorname{cl}_\pi U$ is regular, hence compact. This means it is compact open, so closed in the co-compact topology, hence also closed in the patch topology. As $U \subseteq \operatorname{int}\operatorname{cl}_\pi U$ and this is a patch closed set, $\operatorname{cl}_\pi U \subseteq \operatorname{int}\operatorname{cl}_\pi U$, showing $\operatorname{cl}_\pi U$ is open. Conversely, suppose the patch closure of any open set of X is open. To show $\Omega(X)$ is extremally disconnected, we must show its regular and reflexive elements coincide. As reflexive elements are always regular, and reflexive coincides with compact, we must show that if $U = \operatorname{int} \operatorname{cl}_\pi U$, then U is compact. But we assumed the patch closure of each open set is open, so $U = \operatorname{cl}_\pi U$. So U is closed in the patch topology, which is Hausdorff, hence U is compact in the patch topology. As the patch topology is finier than the original, U is compact in the original topology as well. □

The following result shows extremally disconnected proximity frames are coherent, with similar results for the other categories. So ePrFrm could equally have been called eCohPrFrm, and so forth. Choice of name aside, the point is that our discussion is framed entirely within the spectral space setting.

Proposition 9.6 *The categories* ePrFrm, eRPrFrm, eStKFrm, *and* eStKSp *are full subcategories of* CohPrFrm, CohRPrFrm, CohFrm, *and* Spec *respectively.*

Proof Suppose L is a proximity frame. Proposition 5.3.2 gives $a \prec b \Rightarrow j(a) \prec b$. So if L is extremally disconnected $a \prec j(a) \prec j(a) \prec b$, showing L is coherent. Suppose L is a stably compact frame. Then 1 is regular, a finite meet of regular elements is regular, and each element of L is a join of regular elements, since this is true of regular elements in any proximity frame. Therefore, if L is extremally disconnected, by Proposition 9.4, these are true of its compact elements. Since a finite join of compact elements is always compact, L is a coherent frame. Finally, for an extremally disconnected stably compact space X, by definition $\Omega(X)$ is an extremally disconnected stably compact frame, hence is coherent. This shows X is a spectral space. That each of these subcategories is a full subcategory follows from Section 8. □

The following basic lemma will simplify our discussion. Its proof is obvious.

Lemma 9.7 *Suppose F, G are an equivalence between categories* C *and* D *and* C′ *and* D′ *are full subcategories of* C *and* D *that are closed under isomorphisms. Then if for all $A \in$* C *we have $A \in$* C′ *iff $F(A) \in$* D′*, then F, G restrict to an equivalence between* C′ *and* D′*.*

We use this to show the results of Section 6 restrict to the extremally disconnected setting.

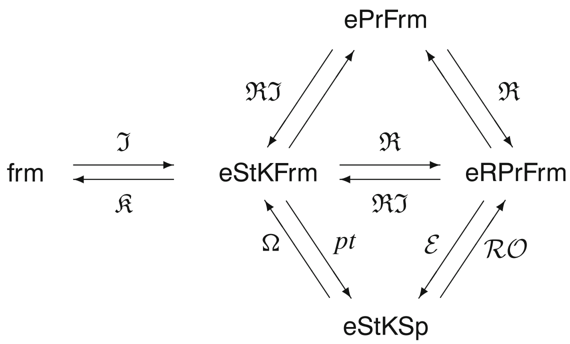
Theorem 9.8 *The functors of Section 6 restrict to equivalences and dual equivalences among* eStKSp*,* eStKFrm*,* ePrFrm*, and* eRPrFrm*. Also the equivalence of* DL *and* CohFrm *of Section 8 restricts to an equivalence of* frm *and* eStKFrm*.*

Proof The form of Definition 9.1 is exactly suited to apply Lemma 9.7. We first show the subcategory ePrFrm of PrFrm is closed under isomorphisms. Suppose L and M are isomorphic proximity frames and L is extremally disconnected. Then the regularizations L_j and M_j are isomorphic and the definition of extremally disconnected gives the proximity of L_j is its ordering. As isomorphisms between regular proximity frames are structural, the proximity on M_j is also its partial ordering, and this shows M is extremally disconnected. As isomorphisms in RPrFrm, StKFrm, and

StKSp are structural, the subcategories eRPrFrm, eStKFrm, and eStKSp are closed under isomorphisms as well.

The inclusion functor i and round ideal functor $\mathfrak{R}\mathfrak{I}$ give an equivalence between StKFrm and PrFrm. By Definition 9.1, $L \in$ eStKFrm iff $i(L) \in$ ePrFrm. Thus, $\mathfrak{R}\mathfrak{I}$ and i restrict to an equivalence between eStKFrm and ePrFrm. Identical arguments show pt and Ω restrict to a dual equivalence between eStKSp and eStKFrm, and that inclusion and regularization $\mathfrak{R}$ restrict to an equivalence between eStKSp and eRPrFrm. The remainder of the equivalences follow as the equivalences in Section 6 commute up to natural isomorphism.

Surely frm and eStKFrm are full subcategories of DL and CohFrm that are closed under isomorphisms. The ideal functor $\mathfrak{I}$ and compact element functor $\mathfrak{K}$ give an equivalence between DL and CohFrm. Suppose L is a frame and I is an ideal of L. By Proposition 8.12 we have I is regular iff I is a D-ideal of L. In a frame every subset is admissible, so the D-ideals of L are the principal ideals, which are the compact elements of $\mathfrak{I}L$. This shows $\mathfrak{I}L$ is extremally disconnected. Suppose M is an extremally disconnected stably compact frame. Then regular elements of M are exactly the compact elements of M, so $\mathfrak{K}M$ is a frame. Thus, $\mathfrak{I}$ and $\mathfrak{K}$ restrict to an equivalence between frm and eStKFrm.



□

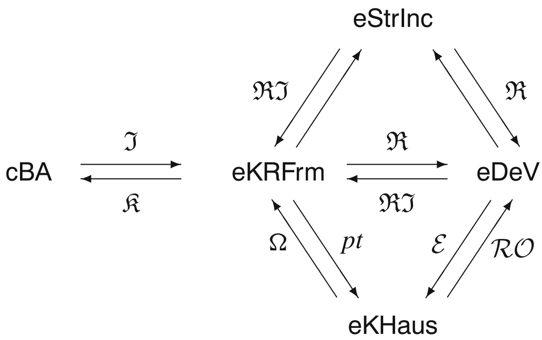
Remark 9.9 Consider directly the functors $\mathcal{RO}$, $\mathcal{E}$, and the reflexive element and distributive ideal functors $\mathfrak{X}$, $\mathfrak{D}$ from Theorem 8.14, in this restricted setting. For an extremally disconnected regular proximity frame M each filter is round, so $\mathcal{E}M$ is the usual spectral space of M. Also, as each element of M is reflexive, $\mathfrak{X}M$ is the frame M with the proximity forgotten. For a frame L, the D-ideals of L are exactly the principal ideals, so $\mathfrak{D}L$ is the frame of principal ideals with set containment as proximity. Finally, for an extremally disconnected stably compact space X and U an open subset of X, we have $\mathrm{cl}_\pi U$ is open. So U is regular open, meaning $U = \mathrm{int}\,\mathrm{cl}_\pi U$, iff $U = \mathrm{cl}_\pi U$, which occurs iff U is compact open. Therefore, $\mathcal{RO}X$ is the distributive lattice (frame in this case) of compact open sets. So essentially, $\mathfrak{X}$ and $\mathfrak{D}$ are trivial, while $\mathcal{RO}$ and $\mathcal{E}$ become restrictions of the functors from Stone duality for bounded distributive lattices.

Finally, we restrict further to the Stone setting. The following is trivial from the definitions.

Proposition 9.10

(1) eStrInc = ePrFrm ∩ StrInc.
(2) eKRFrm = eStKFrm ∩ KRFrm.
(3) eDeV = eRPrFrm ∩ DeV.
(4) eKHaus = eStKSp ∩ KHaus.
(5) cBA = frm ∩ BA.

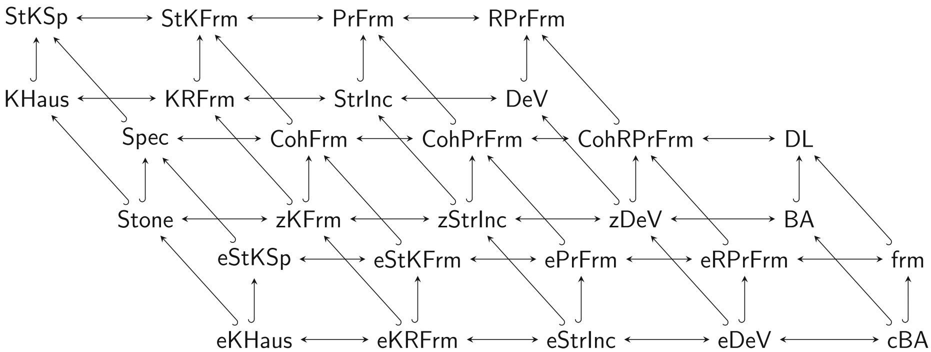
The following is immediate from results of Section 7 for the compact Hausdorff setting, and the earlier results of this section for the extremally disconnected setting.



Theorem 9.11 *The functors of Section 6 restrict to equivalences and dual equivalences among* eKHaus, eKRFrm, eStrInc, *and* eDeV. *Also the equivalence of* BA *and* zKFrm *of Section 8 restricts to an equivalence of* cBA *and* eKRFrm.

10 Conclusion

We summarize in the following diagram the equivalences and dual equivalences among the categories we have considered. The arrows ↪ mean is a full subcategory of, and the arrows ↔ indicate equivalence or dual equivalence. Horizontal rows correspond to equivalent situations, and going vertically from top to bottom gives more special situations.



The results obtained here have application to Smyth's theory [22] of stable compactifications of T_0-spaces. We plan to address this in a forthcoming paper [8]. In this forthcoming paper we will present a theory of stable compactifications of frames that generalizes Banaschewski's theory of compactifications of frames [1], and we will connect this to the theory of biframe compactifications given in [20].

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