Houston Journal of Mathematics
© 2016 University of Houston
Volume 42, No. 4, 2016

SECTIONS IN ORTHOMODULAR STRUCTURES OF DECOMPOSITIONS

JOHN HARDING AND TAEWON YANG

Communicated by Klaus Kaiser

Abstract. There is a family of constructions to produce orthomodular structures from modular lattices, lattices that are M and M^*-symmetric, relation algebras, the idempotents of a ring, the direct product decompositions of a set or group or topological space, and from the binary direct product decompositions of an object in a suitable type of category. We show that an interval $[0, a]$ of such an orthomodular structure constructed from A is again an orthomodular structure constructed from some B built from A. When A is a modular lattice, this B is an interval of A, and when A is a set, group, topological space, or more generally an object in a suitable category, this B is a factor of A.

1. Introduction

The key fact in the quantum logic approach to quantum mechanics [1, 21] is that the closed subspaces of a Hilbert space form an orthomodular poset (abbrev.: OMP). Quantum logic formulates a portion of quantum mechanics in terms of arbitrary OMPs, and either attempts to justify the special role played by the OMP constructed from a Hilbert space, or to propose alternatives to this OMP.

A number of types of OMP arise from constructions very close to the Hilbert space one. Taking a possibly incomplete inner product space E, its splitting subspaces are those ordered pairs of orthogonal subspaces (S, T) where $E = S \oplus T$. The collection of splitting subspaces forms an OMP [5]. Moving a step further from Hilbert spaces, for any vector space V the ordered pairs (S, T) of subspaces with

2000 *Mathematics Subject Classification.* Primary 06C15, 81P10; Secondary 03G15, 06C20, 18A30.

Key words and phrases. Orthomodular poset, idempotents of a ring, relation algebra, symmetric modular lattice, direct product decomposition, direct products in a category.

$V = S \oplus T$ forms an OMP. Such OMPs have been considered by a number of authors [3, 10, 24, 25].

The OMP constructed from pairs of subspaces of a vector space V can be realized from the perspective of lattice theory. The collection of all subspaces of V forms a modular lattice, and those (S, T) with $V = S \oplus T$ are exactly the complementary pairs in this lattice. This construction can be applied to any bounded modular lattice, and even to any lattice that is both M and M^*-symmetric [4, 10, 24].

The closed subspaces of a Hilbert space correspond to orthogonal projections, which are certain idempotents of the endomorphism ring. This can be extended to show that the *-projections of any *-ring form an OMP [6], and further, that the idempotents of any ring with unit form an OMP [20]. This construction is closely related to those above. For a vector space V, the direct sum decompositions $V = S \oplus T$ correspond to the idempotents of the endomorphism ring of V, and for a ring R, the idempotents of R correspond to direct sum decompositions of the left R-module R_R.

The above constructions all have ties to linear algebra and direct sums. A different perspective, allowing movement to a broader setting, comes from the fact that finite direct sums and finite direct products of vector spaces coincide. In [10] it was shown that for A any set, group, poset, topological space, or uniform space, its direct product decompositions form an OMP called Fact A.

The direct product decompositions of a set X correspond to certain ordered pairs of equivalence relations called factor pairs [2], so this construction can be made from the algebra of relations on X. This can be extended to show that ordered pairs of permuting and complementary equivalence elements of any relation algebra form an OMP. In [14], this construction from decompositions was taken to a categorical setting, where it was shown that the direct product decompositions of any object in an honest category (essentially one where ternary product diagrams form a pushout) form a type of orthomodular structure known as an orthoalgebra (abbrev.: OA). See [10, 11, 12, 13, 14, 15, 16, 17] for further details on the orthostructures Fact A.

It is well known that an interval $[0, a]$ of an OMP or OA naturally forms an OMP or OA. It is the purpose of this note to show that such an interval of Fact A is given by Fact B for some structure B created from A. In the case that Fact A is constructed from complementary pairs (x, y) of a bounded modular lattice A, the interval $[0, (x, y)]$ of Fact A is isomorphic to Fact B where B is the interval $[0, x]$ of A considered as a modular lattice. When Fact A is built from the idempotents e of a ring R, then $[0, e]$ is isomorphic to Fact B where $B = \{x : ex = x = xe\}$.

Finally, when A is a set, or object in a strongly honest category, then the interval $[0,[A \simeq B \times C]]$ is isomorphic to Fact B.

This note is organized in the following fashion. In the second section we give the pertinent definitions. In the third, we provide proofs of our result in the case Fact A is constructed from a bounded modular lattice or ring. These are short and easy, and the results for the vector space setting, lattices that are M and M^*-symmetric, set, and relation algebra setting follow directly, or with minor modifications, from these. In the fourth section we provide the most difficult of the results, that of the construction applied to an object in an honest category.

2. Preliminaries

Definition 1. An orthocomplemented poset is a bounded poset P equipped with a unary operation $'$ that is period two and order inverting, where each x, x' have only the bounds as lower and upper bounds. An orthocomplemented poset is an orthomodular poset (abbrev. OMP) if

(1) $x \leq y'$ implies x, y have a least upper bound $x \oplus y$.
(2) $x \leq y$ implies $x \oplus (x \oplus y')' = y$.

Definition 2. An orthoalgebra (abbrev.: OA) is a set with partially defined binary operation $\oplus$ that is commutative and associative and constants $0, 1$ such that

(1) For each a there is a unique element a' with $a \oplus a' = 1$.
(2) If $a \oplus a$ is defined, then $a = 0$.

In an orthocomplemented poset, we say a, b are orthogonal if $a \leq b'$, hence $b \leq a'$. Each OMP naturally forms an OA under the partial binary operation of orthogonal joins. Conversely, in an OA define $a \leq b$ if there is c with $a \oplus c = b$. Then with the obvious operation $'$ this forms an orthocomplemented poset in which $a \oplus b$ is a minimal, but not necessarily least, upper bound of a, b. This orthocomplemented poset constructed from an OA is an OMP iff $a \oplus b$ gives least upper bounds for all a, b. Importantly, the operation $\oplus$ in an OA is cancellative, so if $a \oplus b = a \oplus c$, then $b = c$. For further details on OMPs and for OAs, including the following, see [7, 19, 26].

Proposition 2.1. *If P is an OMP, then an interval $a\!\downarrow$ of P forms an OMP under the induced partial ordering and the orthocomplementation defined by $b^{\#} = a \wedge b'$.*

Proposition 2.2. *If A is an OA, then an interval $a\!\downarrow$ of A forms an OA with constants $0, a$ under the restriction of $\oplus$ to this interval.*

We turn next to a description of methods to construct OMPs from various types of structures. See [10] for further details.

Theorem 2.3. *For a bounded modular lattice L, let $L^{(2)}$ be the set of all ordered pairs of complementary elements of L and define $\leq$ and $'$ on $L^{(2)}$ as follows:*

(1) $(x_1, x_2) \leq (y_1, y_2)$ *iff* $x_1 \leq y_1$ *and* $y_2 \leq x_2$,
(2) $(x_1, x_2)' = (x_2, x_1)$.

Then $L^{(2)}$ is an OMP.

Theorem 2.4. *For R a ring with unit, let $E(R)$ be its idempotents and define a relation $\leq$ and unary operation $'$ on $E(R)$ as follows:*

(1) $e \leq f$ *iff* $ef = e = fe$,
(2) $e' = 1 - e$.

Then $E(R)$ is an OMP.

Theorem 2.5. *For X a set let Fact X be the set of all ordered pairs of equivalence relations (θ_1, θ_2) of X such that $\theta_1 \cap \theta_2 = \Delta$ and $\theta_1 \circ \theta_2 = \nabla$, where Δ and ∇ are the smallest and largest equivalence relations. Define $\leq$ and $'$ on Fact X as follows:*

(1) $(\theta_1, \theta_2) \leq (\phi_1, \phi_2)$ *iff* $\theta_1 \subseteq \phi_1$, $\phi_2 \subseteq \theta_2$, *and all relations involved permute,*
(2) $(\theta_1, \theta_2)' = (\theta_2, \theta_1)$.

Then Fact X is an OMP.

There are a number of extensions to these results that we briefly describe.

Remark. *Let (a, b) be an ordered pair of elements in a lattice. We say (a, b) is a modular pair, written $(a, b)M$, if $c \leq b$ implies $c \vee (a \wedge b) = (c \vee a) \wedge b$; and (a, b) is a dual-modular pair, written $(a, b)M^*$, if $b \leq c$ implies $c \wedge (a \vee b) = (c \wedge a) \vee b$. A lattice is M-symmetric if $(a, b)M$ implies $(b, a)M$, M^*-symmetric if $(a, b)M^*$ implies $(b, a)M^*$, and symmetric if it is both M and M^*-symmetric. The result in Theorem 2.3 extends to a bounded symmetric lattice L if $L^{(2)}$ is defined to be all complementary pairs of elements that are both modular and dual-modular pairs.*

Remark. *The result in Theorem 2.4 has an extension to more general structures known as orthomodular partial semigroups* [8, 17]*. These are structures with a partially defined multiplication that behaves in a similar way to the multiplication of a ring restricted to pairs of commuting elements.*

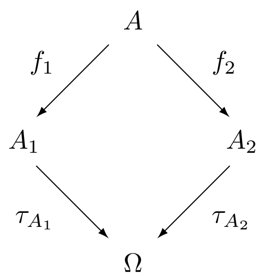
Remark. *The result in Theorem 2.5 is formulated in terms of the algebra of relations of a set. This can be generalized in an obvious way to apply to any relation* *algebra, in the sense of Tarski* [10]. *For a relation algebra R, an equivalence element is one with $1' \leq a = a\,;a = a^{\smile}$. These are equivalence relations in the relation algebra of all relations on a set. Define $R^{(2)}$ to be the set of all ordered pairs (a, b) of equivalence elements whose meet is $1'$ and with $a\,;b = 1 = b\,;a$. Then $R^{(2)}$ forms an* OMP *under an ordering and orthocomplementation defined in an obvious manner as a generalization of those in Theorem 2.5.*

Remark. One might ask about the functoriality of the above constructions. Consider OMP *as a category with morphisms those that preserve order, orthocomplementation, and orthogonal joins. It is easily established that $E(R)$, $L^{(2)}$, and $R^{(2)}$ provide functors from the categories of rings, bounded modular lattices, and relation algebras to* OMP. *This is simply because operations in $E(R)$, are defined through those in R, and similarly for the other cases. But the more interesting level, a lower level of sorts, does not provide functoriality. It is easily seen that Fact X cannot be extended to a functor from* SET *to* OMP *via any obvious candidate. More complex arguments using* [9] *show that no such functor can exist. The situation can be viewed roughly as follows: there is a base level of structures such as sets or vector spaces, and a higher level of structures that can be used to represent their morphisms such as relation algebras or modular lattices. Functoriality is obtained from the higher level to* OMP, *but not from the lower level to* OMP.

We turn to our final constructions of orthomodular structures, those from the direct product decompositions of an object in a certain type of category. For general background on category theory we refer the reader to [18, 22], and for more details of the specifics of this construction [14].

We consider categories with finite products, hence a terminal object Ω, and we use τ_A for the unique morphism $\tau_A : A \to \Omega$. Define an equivalence relation $\simeq$ on the morphisms in a category by setting $f \simeq g$ if there is an isomorphism u with $u \circ f = g$, and use this to define an equivalence relation $\approx$ on the collection of all finite product diagrams by setting $(f_1, \ldots, f_m) \approx (g_1, \ldots, g_n)$ if $m = n$ and $f_i \simeq g_i$ for each $i = 1, \ldots, n$. We let the equivalence class of $\approx$ containing $(f_1, \ldots, f_n)$ be $[\,f_1, \ldots, f_n\,]$ and call this equivalence class an n-ary decomposition of A. Also, for an n-tuple of morphisms with common domain $f_i : A \to A_i$, we use $\langle f_1, \ldots, f_n \rangle$ for the morphism from A into the product of their codomains. See [14] for further details.

Definition 3. In a category with finite products, a binary product diagram (f_1, f_2) where $f_i : A \to A_i$, is called a disjoint binary product if $(f_1, f_2, \tau_{A_1}, \tau_{A_2})$ is a pushout.



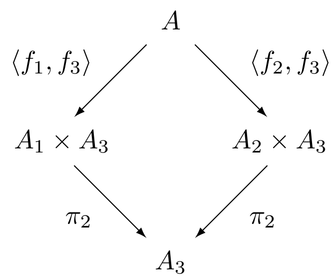
A ternary product (f_1, f_2, f_3) is disjoint if the binary products, such as $(f_1, \langle f_2, f_3 \rangle)$, one can build from it are disjoint.

Remark. This notion of disjointness is not something that holds of binary products in an arbitrary category with products. Consider a lattice as a category where each element is an object, and there is a unique morphism from a to b whenever $b \leq a$. In this category, products are given by joins, and the product of a pair x, y is disjoint iff the meet of x, y is the least element of the lattice, i.e. if the pair x, y is disjoint in the sense usually used in lattice theory. However, many categories have the property that all binary products are disjoint. This is the case in the category of non-empty sets, groups, rings, topological spaces, and so forth.

Remark. The diagram in Definition 3 is obviously a pullback since it is built from a product, hence is both a pullback and a pushout. Such diagrams are called pulations *by Ademek, Herrlich, and Strecker, and* Doolittle squares *by Freyd.*

The essential property in constructing an orthomodular structure from the direct product decompositions of an object in a category seems to be that a certain diagram built from a ternary direct product diagram forms a pushout, in fact a pulation. This was introduced in [14] under the name *honest category.*

Definition 4. A category is honest if it has finite products; all projections are epic; and for each disjoint ternary product diagram (f_1, f_2, f_3), where $f_i : A \to A_i$, the following diagram is a pushout.



In [14] it was shown that the disjoint binary decompositions of an object A in an honest category form an OA. We would like to show that an interval $[f_1, f_2]\!\downarrow$, where $f_i : A \to A_i$, is isomorphic to the OA of disjoint binary decompositions of the factor A_1. For a disjoint decomposition of A in the interval $[f_1, f_2]\!\downarrow$ we can produce a decomposition of A_1, but cannot show this decomposition is disjoint. We bypass this difficulty by considering only those honest categories where every binary product diagram is disjoint. As mentioned in Remark 2, this includes many natural examples.

Definition 5. A category is strongly honest if it is honest and all binary product diagrams are disjoint.

The following is then an immediate reformulation of a result of [14] to apply to strongly honest categories.

Theorem 2.6. *Let A be an object in a strongly honest category and $\mathcal{D}(A)$ be the collection of binary decompositions of A. Define a partial binary operation $\oplus$ on $\mathcal{D}(A)$ where $[f_1, f_2] \oplus [g_1, g_2]$ is defined if there is a ternary decomposition $[c_1, c_2, c_3]$ with*

$$[f_1, f_2] = [c_1, \langle c_2, c_3\rangle] \quad \textit{and} \quad [g_1, g_2] = [c_2, \langle c_1, c_3\rangle]$$

In this case, define $[f_1, f_2] \oplus [g_1, g_2] = [\langle c_1, c_2\rangle, c_3]$. Then, with this operation and constants $0 = [\tau_A, 1_A]$ and $1 = [1_A, \tau_A]$, the decompositions $\mathcal{D}(A)$ form an OA.

To conclude this section we note there are obviously many relationships between these constructions. For instance, the category of non-empty sets is strongly honest, and the construction of an orthostructure from a set X given by Theorem 2.5 agrees with that given by Theorem 2.6. Similar comments hold for the many ways to create an orthostructure from a vector space. However, there is so far no unifying setting that includes all the results described in the above theorems, and their extensions discussed in the remarks. A categorical approach seems the most promising avenue to a general result, but results of [16] indicate there may be approaches to building orthomodular structures from objects in certain categories that do not rely on honesty. There seem to be interesting open problems in this direction.

3. The main result in the lattice and ring setting

Here we prove our main result in the setting of bounded modular lattices and rings, and indicate extensions to other settings. The result in the setting of strongly honest categories is given in the following section.

Theorem 3.1. *Let L be a bounded modular lattice and (a,b) be a complementary pair in L. Then the interval $(a,b)\!\downarrow$ of the* OMP $L^{(2)}$ *is isomorphic to the* OMP $a\!\downarrow^{(2)}$.

Proof. Define maps $\Gamma : (a,b)\!\downarrow\ \to a\!\downarrow^{(2)}$ and $\Phi : a\!\downarrow^{(2)} \to (a,b)\!\downarrow$ as follows.

$$\begin{aligned} \Gamma(x,y) &= (x, y\wedge a) \\ \Phi(u,v) &= (u, v\vee b) \end{aligned}$$

To see these are well defined, we must show they result in complementary pairs in the appropriate lattice. Suppose $(x,y)\in(a,b)\!\downarrow$. Then x,y are complementary in L, $x\leq a$ and $b\leq y$. Then $x\wedge(y\wedge b)=0$, and as $x\leq a$ modularity gives $x\vee(y\wedge a)=(x\vee y)\wedge a=a$, so $x, y\wedge a$ are complements in $a\!\downarrow$. Conversely, if u,v are complements in $a\!\downarrow$, then $u\vee(v\vee b)=a\vee b=1$, and as $v\leq a$ modularity gives $u\wedge(v\vee b)\leq a\wedge(v\vee b)=v\vee(a\wedge b)=v$, so $u\wedge(v\vee b)\leq u\wedge v=0$.

So Γ and Φ are well defined. It is obvious they preserve order. To see they are inverses of one another, note $\Phi\Gamma(x,y)=(x,(y\wedge a)\vee b)$ and $\Gamma\Phi(u,v)=(u,(v\vee b)\wedge a)$. As $b\leq y$ modularity gives $(y\wedge a)\vee b=y\wedge(a\vee b)=y$, and as $v\leq a$ modularity gives $(v\vee b)\wedge a=v\vee(a\wedge b)=v$. So $\Phi\Gamma$ and $\Gamma\Phi$ are identity maps. It remains only to show Γ and Φ are compatible with orthocomplementations. We use $(x,y)^{\#}$ to denote orthocomplement in $(a,b)\!\downarrow$ and $'$ for orthocomplementation in both $L^{(2)}$ and $a{\downarrow}^{(2)}$.

$$\begin{aligned}\Gamma((x,y)^{\#}) &= \Gamma((x,y)' \wedge (a,b)) \\ &= \Gamma(y \wedge a, x \vee b) \\ &= (y \wedge a, (x \vee b) \wedge a) \\ &= (y \wedge a, x) \\ &= (\Gamma(x,y))'\end{aligned}$$

$$\begin{aligned}\Phi((u,v)') &= (v, u \vee b) \\ &= ((v \vee b) \wedge a, u \vee b) \\ &= (u, v \vee b)' \wedge (a,b) \\ &= (\Phi(u,v))^{\#}\end{aligned}$$

□

Theorem 3.2. *For e an idempotent of a ring R, let $R_e = \{x : ex = x = xe\}$. Then R_e is a ring with unit e under the multiplication and addition of R, and the interval $e{\downarrow}$ of the* OMP *$E(R)$ is equal to $E(R_e)$.*

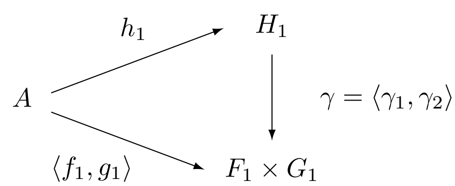
Proof. It is trivial that R_e is a ring with unit e. Suppose f belongs to the interval $e{\downarrow}$ of $E(R)$. Then $ef = f = fe$, so f belongs to R_e and is idempotent in R_e, so f belongs to $E(R_e)$. Conversely, suppose g belongs to $E(R_e)$. Then g is idempotent in R_e, hence also in R, and $eg = g = ge$. So g belongs to the interval $e{\downarrow}$ of $E(R)$. Thus as sets $e{\downarrow}$ is equal to $E(R_e)$. The definition of $\leq$ in both structures is $gh = g = hg$, so they coincide. It remains only to show their orthocomplementations agree. The orthocomplementation # in the interval $e{\downarrow}$ is given by $f^{\#} = f' \wedge e = (1-f)e$. Then as $fe = f$ this evaluates to $e - f$, which is the orthocomplementation in $E(R_e)$. □

Remark. Theorem 3.1 has several generalizations. First, one notices it applies to the case of symmetric lattices as each step only involves basic properties of M and M^*-symmetry found in [23]. *The main part is in showing that the images of the isomorphisms are modular and dual modular pairs. Next, and somewhat surprising, one notices it applies to the relation algebra setting described in the third remark after Theorem 2.5. Here, the interval $(a,b){\downarrow}$ in $R^{(2)}$ is isomorphic to $a{\downarrow}^{(2)}$ where $a{\downarrow}$ is naturally considered as a relation algebra. The key point is that small fragments of modularity hold in any relation algebra. These were discovered* *by Chin and Tarski, and their role in the current context is described in detail in* [10]. *Theorem 3.2 also has generalizations to situations described in the second remark after Theorem 2.5.*

4. The main result in the categorical setting

Theorem 4.1. *Suppose A is an object in a strongly honest category $\mathcal{C}$ and $[h_1, h_2]$ is a binary decomposition of A where $h_i : A \to H_i$. Then the interval $[h_1, h_2]{\downarrow}$ of the* OA $\mathcal{D}(A)$ *is isomorphic to the* OA $\mathcal{D}(H_1)$.

Proof. We first define a map $\Gamma : [h_1, h_2]{\downarrow} \to \mathcal{D}(H_1)$. Suppose $[f_1, f_2] \leq [h_1, h_2]$. By definition of $\leq$ there is $[g_1, g_2]$ with $[f_1, f_2] \oplus [g_1, g_2]$ defined and equal to $[h_1, h_2]$, and as every OA is cancellative this $[g_1, g_2]$ is unique. By definition of $\oplus$ there is a ternary decomposition $[c_1, c_2, c_3]$ of A with $[f_1, f_2] = [c_1, \langle c_2, c_3\rangle]$, $[g_1, g_2] = [c_2, \langle c_1, c_3\rangle]$ and $[h_1, h_2] = [\langle c_1, c_2\rangle, c_3]$. So this ternary decomposition is equal to $[f_1, g_1, h_2]$. As $h_1 \simeq \langle f_1, g_1\rangle$ there is an isomorphism $\gamma : H_1 \to F_1 \times G_1$ with $\gamma \circ h_1 = \langle f_1, g_1\rangle$, and this γ is unique since h_1 is a projection and projections are epic. Basic properties of products show this γ can be written $\langle \gamma_1, \gamma_2\rangle$ where $[\gamma_1, \gamma_2]$ is a decomposition of H_1. We define $\Gamma([f_1, f_2]) = [\gamma_1, \gamma_2]$.



The above discussion established the following.

Claim 4.2. *If $[f_1, f_2] \oplus [g_1, g_2] = [h_1, h_2]$, then $\Gamma([f_1, f_2]) = [\gamma_1, \gamma_2]$ iff $\gamma_1 \circ h_1 = f_1$ and $\gamma_2 \circ h_1 = g_1$.*

We next define $\Phi : \mathcal{D}(H_1) \to [h_1, h_2]{\downarrow}$. Let $[m_1, m_2]$ be a decomposition of H_1. Then (m_1h_1, m_2h_1, h_2) is a ternary decomposition of A. By the definition of $\oplus$ we have $[m_1h_1, \langle m_2h_1, h_2\rangle] \oplus [m_2h_1, \langle m_1h_1, h_2\rangle] = [\langle m_1h_1, m_2h_1\rangle, h_2]$. Basic properties of products give $\langle m_1h_1, m_2h_1\rangle \simeq h_1$, so this latter term is $[h_1, h_2]$. This shows $[m_1h_1, \langle m_2h_1, h_2\rangle] \leq [h_1, h_2]$. We define $\Phi([m_1, m_2]) = [m_1h_1, \langle m_2h_1, h_2\rangle]$.

Claim 4.3. *Γ and Φ are mutually inverse bijections.*

PROOF OF CLAIM: Suppose $[f_1, f_2] \in [h_1, h_2]\downarrow$ and $[g_1, g_2]$ is the decomposition with $[f_1, f_2] \oplus [g_1, g_2] = [h_1, h_2]$. Then $\Gamma([f_1, f_2]) = [\gamma_1, \gamma_2]$ where $\langle \gamma_1, \gamma_2 \rangle \circ h_1 = \langle f_1, g_1 \rangle$, hence $\gamma_1 \circ h_1 = f_1$ and $\gamma_2 \circ h_1 = g_1$. Then $\Phi\Gamma([f_1, f_2]) = [\gamma_1 h_1, \langle \gamma_2 h_1, h_2 \rangle] = [f_1, \langle g_1, h_2 \rangle]$. In the discussion of the definition of Γ, we saw that $\langle g_1, h_2 \rangle \simeq f_2$. It follows that $\Phi \circ \Gamma$ is the identity. For the other composite, suppose $[m_1, m_2] \in \mathcal{D}(H_1)$. Then $\Phi([m_1, m_2]) = [m_1 h_1, \langle m_2 h_1, h_2 \rangle]$ and in the discussion of the definition of Φ we saw that $[m_1 h_1, \langle m_2 h_1, h_2 \rangle] \oplus [m_2 h_1, \langle m_1 h_1, h_2 \rangle] = [h_1, h_2]$. So $\Gamma\Phi([m_1, m_2])$ is the unique isomorphism $\langle \gamma_1, \gamma_2 \rangle$ with $\langle \gamma_1, \gamma_2 \rangle \circ h_1 = \langle m_1 h_1, m_2 h_1 \rangle$, which is $\langle m_1, m_2 \rangle$. So $\Gamma \circ \Phi$ is also the identity. □

We say that a map Π between OAs preserves $\oplus$ if $x \oplus y$ being defined implies $\Pi(x) \oplus \Pi(y)$ is defined and $\Pi(x \oplus y) = \Pi(x) \oplus \Pi(y)$.

Claim 4.4. Γ *preserves* $\oplus$.

PROOF. Note that the operation $\oplus$ in the OA $[h_1, h_2]\downarrow$ is the restriction of the operation $\oplus$ of $\mathcal{D}(A)$. Suppose $[e_1, e_2]$ and $[f_1, f_2]$ belong to $[h_1, h_2]\downarrow$ and $[e_1, e_2] \oplus [f_1, f_2]$ is defined. Then there is $[g_1, g_2]$ with $([e_1, e_2] \oplus [f_1, f_2]) \oplus [g_1, g_2] = [h_1, h_2]$. Let (c_1, c_2, c_3) and (d_1, d_2, d_3) be the ternary decompositions of A realizing $[e_1, e_2] \oplus [f_1, f_2]$ and $([e_1, e_2] \oplus [f_1, f_2]) \oplus [g_1, g_2]$ respectively. Then

$$
\begin{aligned}
[c_1, \langle c_2, c_3 \rangle] &= [e_1, e_2] \\
[c_2, \langle c_1, c_3 \rangle] &= [f_1, f_2] \\
[\langle c_1, c_2 \rangle, c_3] &= [e_1, e_2] \oplus [f_1, f_2]
\end{aligned}
$$

$$
\begin{aligned}
[d_1, \langle d_2, d_3 \rangle] &= [e_1, e_2] \oplus [f_1, f_2] \\
[d_2, \langle d_1, d_3 \rangle] &= [g_1, g_2] \\
[\langle d_1, d_2 \rangle, d_3] &= [h_1, h_2]
\end{aligned}
$$

It follows that $c_1 \simeq e_1$, $c_2 \simeq f_1$, $c_3 \simeq \langle d_2, d_3 \rangle$, $d_1 \simeq \langle c_1, c_2 \rangle$, $d_2 \simeq g_1$ and $d_3 \simeq h_2$. As $[d_1, d_2, d_3] = [\langle e_1, f_1 \rangle, g_1, h_2]$ it follows from basic properties of products that $[e_1, f_1, g_1, h_2]$ is a decomposition of A and $[\langle e_1, f_1, g_1 \rangle, h_2] = [\langle d_1, d_2 \rangle, d_3] = [h_1, h_2]$. Thus $\langle e_1, f_1, g_1 \rangle \simeq h_1$, so there is an isomorphism $\gamma : H_1 \to E_1 \times F_1 \times G_1$ with $\gamma \circ h_1 = \langle e_1, f_1, g_1 \rangle$. Basic properties of products show $\gamma = \langle \gamma_1, \gamma_2, \gamma_3 \rangle$ where $[\gamma_1, \gamma_2, \gamma_3]$ is a ternary decomposition of H_1 and $\gamma_1 h_1 = e_1$, $\gamma_2 h_1 = f_1$ and $\gamma_3 h_1 = g_1$.

As $([e_1, e_2] \oplus [f_1, f_2]) \oplus [g_1, g_2]$ is defined and equal to $[h_1, h_2]$, the associativity condition in any OA implies $[f_1, f_2] \oplus [g_1, g_2]$ is defined and $[e_1, e_2] \oplus ([f_1, f_2] \oplus$

$[g_1, g_2])$ is defined and is equal to $[h_1, h_2]$. As $[e_1, f_1, g_1, h_2]$ is a decomposition of A, so is $[f_1, g_1, \langle e_1, h_2\rangle]$, and this ternary decomposition realizes $[f_1, f_2] \oplus [g_1, g_2]$ being defined, hence being equal to $[\langle f_1, g_1\rangle, \langle e_1, h_2\rangle]$. Thus $[e_1, e_2] \oplus [\langle f_1, g_1\rangle, \langle e_1, h_2\rangle] = [h_1, h_2]$. Since $\gamma_1 h_1 = e_1$ and $\langle \gamma_2, \gamma_3\rangle h_1 = \langle f_1, g_1\rangle$, Claim 4.2 shows $\Gamma([e_1, e_2]) = [\gamma_1, \langle \gamma_2, \gamma_3\rangle]$.

A similar calculation making use of the commutativity of $\oplus$ in any OA shows that $\Gamma([f_1, f_2]) = [\gamma_2, \langle \gamma_1, \gamma_3\rangle]$, and another shows that $\Gamma([e_1, e_2] \oplus [f_1, f_2]) = [\langle \gamma_1, \gamma_2\rangle, \gamma_3]$. Thus $(\gamma_1, \gamma_2, \gamma_3)$ realizes $\Gamma([e_1, e_2]) \oplus \Gamma([f_1, f_2])$ is defined and equal to $\Gamma([e_1, e_2] \oplus [f_1, f_2])$. This concludes the proof of the claim. □

Claim 4.5. Φ *preserves* $\oplus$.

Proof of Claim: Suppose that $[m_1, m_2]$ and $[n_1, n_2]$ are decompositions of H_1 with $[m_1, m_2] \oplus [n_1, n_2]$ defined and $[p_1, p_2, p_3]$ is a ternary decomposition realizing this. So $[m_1, m_2] = [p_1, \langle p_2, p_3\rangle]$, $[n_1, n_2] = [p_2, \langle p_1, p_3\rangle]$ and $[m_1, m_2] \oplus [n_1, n_2] = [\langle p_1, p_2\rangle, p_3]$. The definition of Φ gives

$$\begin{aligned} \Phi([m_1, m_2]) &= [m_1 h_1, \langle m_2 h_1, h_2\rangle] \\ \Phi([n_1, n_2]) &= [n_1 h_1, \langle n_2 h_1, h_2\rangle] \\ \Phi([m_1, m_2] \oplus [n_1, n_2]) &= [\langle m_1, n_1\rangle h_1, \langle p_3 h_1, h_2\rangle] \end{aligned}$$

As $[p_1, p_2, p_3]$ is a decomposition of H_1 and $[h_1, h_2]$ is a decomposition of A, general properties of products show that $[p_1 h_1, p_2 h_1, p_3 h_1, h_2]$ is a decomposition of A.

Consider the ternary decomposition $[p_1 h_1, p_2 h_1, \langle p_3 h_1, h_2\rangle]$ of A. We note that $p_1 h_1 \simeq m_1 h_1$, $\langle p_2 h_1, \langle p_3 h_1, h_2\rangle\rangle \simeq \langle\langle p_2, p_3\rangle h_1, h_2\rangle \simeq \langle m_2 h_1, h_2\rangle$. Similarly $p_2 h_1 \simeq n_1 h_1$, and $\langle p_1 h_1, \langle p_3 h_1, h_2\rangle\rangle \simeq \langle n_2 h_1, h_2\rangle$. This shows $\Phi([m_1, m_2]) \oplus \Phi([n_1, n_2])$ is defined and equal to $[\langle p_1 h_1, p_2 h_1\rangle, \langle p_3 h_1, h_2\rangle]$. As $\langle p_1 h_1, p_2 h_1\rangle \simeq \langle m_1, n_1\rangle h_1$, it follows that $\Phi([m_1, m_2]) \oplus \Phi([n_1, n_2]) = \Phi([m_1, m_2] \oplus [n_1, n_2])$. □

To show Γ and Φ are mutually inverse OA isomorphisms, it remains only to show they preserve bounds. As we know they are inverses, it suffice to show one of them preserves bounds. In the OA $[h_1, h_2]\downarrow$ we have $0 = [\tau_A, 1_A]$ and $1 = [1_A, \tau_A]$. Note $[\tau_A, 1_A] \oplus [h_1, h_2] = [h_1, h_2]$, and it follows from Claim 4.2 that $\Gamma([\tau_A, 1_A]) = [\tau_{H_1}, 1_{H_1}]$ and $\Gamma([h_1, h_2]) = [1_{H_1}, \tau_{H_1}]$. So Γ preserves bounds. This concludes the proof of the theorem. □

Remark. We do not know if this result in the strongly honest setting extends to the honest setting. As mentioned above, the difficulty is in establishing disjointness of $\Gamma([f_1, f_2])$.

Acknowledgement

We thank the referee for a number of helpful suggestions in both the presentation and content of the paper.

References

[1] G. Birkhoff and J. von Neumann, *The logic of quantum mechanics*, Annals of Mathematics, **37** (1936), no. 4, 823-843.

[2] S. Burris and H. P. Sankappanavar, *A First Course in Universal Algebra*, Springer, 1981.

[3] G. Chevalier, *Automorphisms of an orthomodular poset of projections*, Internat. J. of Theoret. Phys., **44** (2005), no. 7, 985-998.

[4] G. Chevalier, *The orthomodular poset of projections of a symmetric lattice*, Internat. J. of Theoret. Phys., **44** (2005), no. 11, 2073-2089.

[5] A. Dvurečenskij, *Gleason's Theorem and its Applications*, Kluwer, Dordrecht/Boston/London 1993.

[6] D. J. Foulis, *Baer *-semigroups*, Proc. Amer. Math. Soc. **11** (1960), 648-654.

[7] D. J. Foulis, R. J. Greechie, and G. Rüttimann, *Filters and supports in orthoalgebras*, Int. J. of Math. Phys. **31** (1992), 787-807.

[8] S. P. Gudder and R. H. Schelp, *Coordinatization of orthocomplemented and orthomodular posets*, Proc. Amer. Math. Soc. **25** (1970), 229-237.

[9] T. Hannan and J. Harding, *Automorphisms of decompositions*, to appear in Mathematica Slovaca.

[10] J. Harding, *Decompositions in quantum logic*, Trans. Amer. Math. Soc. **348** (1996), no. 5, 1839-1862.

[11] J. Harding, *Regularity in quantum logic*, Internat. J. of Theoret. Phys. **37** (1998), no. 4, 1173–1212.

[12] J. Harding, *Axioms af an experimental system*, Internat. J. of Theoret. Phys. **38** (1999), no. 6, 1643-1675.

[13] J. Harding, *States on orthomodular posets of decompositions*, Internat. J. of Theoret. Phys. **40** (2001), no. 6, 1061-1069.

[14] J. Harding, *Orthomodularity of decompositions in a categorical setting*, Internat. J. of Theoret. Phys. **45** (2006), no. 6, 1117-1127.

[15] J. Harding, *The source of the orthomodular law*, a book chapter in The Handbook of Quantum Logic and Quantum Structures, Elsevier 2007.

[16] J. Harding, *A link between quantum logic and categorical quantum mechanics*, Internat. J. of Theoret. Phys. **48** (2009), no. 3, 769-802.

[17] J. Harding, *Daggers, kernels, Baer *-semigroups, and orthomodularity*, J. Philos. Logic, **42** (2013), no. 3, 535-549.

[18] H. Herrlich and G. Strecker, *Category Theory: an Introduction*, Allyn and Bacon, 1973.

[19] G. Kalmbach, *Orthomodular Lattices*, Academic Press 1983.

[20] F. Katrnoška, *Logics of idempotents of rings*, Proc. Second Winter School on Measure Theory, Liptovský Ján, (1990) Jednota Slovensk. Mat. Fyz., Bratislava, 1990, 100-104.

[21] G. Mackey, *The Mathematical Foundations of Quantum Mechanics*, Benjamin, N.Y., 1963.

[22] S. MacLane, *Categories for the Working Mathematician*, Springer, 1998.

[23] F. Maeda and S. Maeda, *Theory of Symmetric Lattices*, Springer, 1970.
[24] D. Kh. Mushtari, *Projection logics in Banach spaces*, Soviet Math. (Iz. VUZ) **33** (1989), 59-70.
[25] P. Ovchinnikov, *Automorphisms of the poset of skew projections*, J. Funct. Anal., **115** (1993), 184-189.
[26] P. Pták and S. Pulmannová, *Orthomodular Structures as Quantum Logics*, Fundamental Theories of Physics, **44**, Kluwer, Dordrecht, 1991.

Received April 5, 2014

John Harding, Department of Mathematical Sciences, New Mexico State University, Las Cruces, NM USA 88003
URL: math.nmsu.edu/~jharding@nmsu.edu
E-mail address: jharding@nmsu.edu

Taewon Yang, Department of Mathematical Sciences, New Mexico State University, Las Cruces, NM USA 88003
E-mail address: yangtjong@gmail.com