STABLE COMPACTIFICATIONS OF FRAMES

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Abstract. In a classic paper, Smirnov [14] characterized the poset of compactifications of a completely regular space in terms of the proximities on the space. Later, Smyth [15] introduced the notion of a stable compactification of a T_0-space and described them in terms of quasi-proximities on the space. Banaschewski [1] formulated Smirnov's results in the pointfree setting, defining a compactification of a completely regular frame, and characterizing these in terms of the strong inclusions on the frame.

We provide an alternate description of stable compactifications of T_0-spaces as embeddings into stably compact spaces that are dense with respect to the patch topology, and relate such stable compactifications to ordered spaces. Each stable compactification of a T_0-space induces a companion topology on the space, and we show the companion topology induced by the largest stable compactification is the topology τ^* studied by Salbani [11, 12].

In the pointfree setting, we introduce a notion of a stable compactification of a frame that extends Smyth's stable compactification of a T_0-space, and Banaschewski's compactification of a frame. We characterize the poset of stable compactifications of a frame in terms of proximities on the frame, and in terms of stably compact subframes of its ideal frame. These results are then specialized to coherent compactifications of frames, and related to Smyth's spectral compactifications of a T_0-space.

1. Introduction

A classic result of Smirnov [14] shows that the poset of compactifications of a completely regular space X is isomorphic to the poset of proximities on X that are compatible with the topology on X. Banaschewski [1] generalized Smirnov's theorem to the pointfree setting by introducing the concept of a compactification of a frame. He also generalized the concept of a proximity on a space to that of a strong inclusion on a frame, and proved that the poset of compactifications of a frame L is isomorphic to the poset of strong inclusions on L. In particular, if L is the frame of open sets of a completely regular space, then Smirnov's theorem follows.

Smyth [15] generalized the theory of compactifications of completely regular spaces to that of stable compactifications of T_0-spaces. He also generalized the concept of proximity to that of quasi-proximity and proved that the poset of stable compactifications of a T_0-space X is isomorphic to the poset of quasi-proximities on X that are compatible with the topology on X. Restricting to completely regular spaces and proximities then yields Smirnov's theorem.

In this paper, we provide an alternate description of Smyth's stable compactification of a T_0-space X as an embedding of X into a stably compact space Y whose image is dense in the patch topology of Y. We then relate such stable compactifications to ordered spaces. Each stable compactification of a T_0-space X induces an ordered space structure on X whose open upsets are the given topology on X, and under which the stable compactification can be naturally viewed as an order-compactification. So each stable compactification of X yields a

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companion topology to the original, the topology of open downsets of the associated ordered space. We show the companion topology associated with the largest stable compactification of X is the topology τ^* studied by Salbani [11, 12].

We then extend Smyth's theory of stable compactifications to the pointfree setting. We introduce the concept of a stable compactification of a frame, and prove a generalization of Banaschewski's theorem, showing that the poset of stable compactifications of a frame L is isomorphic to the poset of proximities on the frame in the sense of [4], and to the poset of certain stably compact subframes of the ideal frame of L. The spatial case of this result yields Smyth's theorem.

This paper is organized in the following way. The second section provides preliminaries. In the third, we discuss stable compactifications of T_0-spaces, giving a characterization of such compactifications in terms of the patch topology, and relating such compactifications to ordered spaces. In the fourth section we define stable compactifications of frames, and provide characterizations of such stable compactifications in terms of proximities, and in terms of certain subframes of the ideal frame. The fifth section specializes the results of the fourth to coherent and spectral compactifications.

2. Preliminaries

Recall the classical notion of a compactification of a topological space X is an embedding $e : X \to Y$ into a compact Hausdorff space Y whose image is dense in Y. Here, embedding is used to mean that e is a homeomorphism from X to its image considered with the subspace topology from Y. Classical results characterize those spaces X having a compactification as the completely regular ones. It is standard to form a poset from the compactifications of a completely regular space, as in the following definition.

Definition 2.1. *For compactifications $e : X \to Y$ and $e' : X \to Y'$ of X write $e' \sqsubseteq e$ if there is a continuous map $f : Y \to Y'$ with $e' = f \circ e$.*

It is well known that $\sqsubseteq$ is a quasiorder, and the associated partially ordered set is called the poset of (inequivalent) compactifications of X. Smirnov described this poset in terms of proximities on X. Standard results show that the Stone-Čech compactification of X is the largest member of this poset, and that this poset has a least element iff X is locally compact, and in this case the least element is the one-point compactification of X (see, e.g., [6, Sec. 3.5 and 3.6]). Also standard to the theory of compactifications is the following result.

Theorem 2.2. *A compact Hausdorff space X has up to homeomorphism only itself as a compactification. Thus, classical compactifications are transitive in that a compactification of a compactification is a compactification.*

The notion of a compactification $e : X \to Y$ can be extended in an obvious way simply by dropping the requirement that the compact space Y be Hausdorff. However, this is a very poorly behaved notion, with a space X having such general compactifications of arbitrary cardinality. Smyth [15] introduced a notion of a stable compactification of a T_0-space, that although still pathological in some ways, is much better behaved. We describe these stable compactifications in detail in the following section, but remark they are certain dense embeddings into the stably compact spaces we describe next. A few basic definitions are required first.

A topological space X is *locally compact* if for each $x \in X$ and open neighborhood U of x, there is an open neighborhood V of x and a compact set K with $V \subseteq K \subseteq U$. A subset A of

X is *irreducible* if $A \subseteq B \cup C$ with B, C closed implies $A \subseteq B$ or $A \subseteq C$; and X is *sober* if each closed irreducible set is the closure of a unique singleton. Finally, a subset of X is *saturated* if it is an intersection of open sets.

Definition 2.3. *A space X is stably compact if it is compact, locally compact, sober, and the intersection of two compact saturated sets is compact.*

The theory of stably compact spaces is developed in detail in [7], where it is shown that there is a close connection between stably compact spaces and certain ordered topological spaces. To recall this connection, we need to describe two additional topologies associated to any stably compact space.

Definition 2.4. *For a stably compact space X with topology τ, the compact saturated sets are the closed sets of a topology τ^k on X called the co-compact topology. The join of the topologies τ and τ^k is called the patch topology π.*

An *ordered topological space* is a triple $(X, \leq, \pi)$ consisting of a set X with partial ordering $\leq$ and topology π. A subset U of X is an *upset* if $x \in U$ and $x \leq y$ imply $y \in U$, and it is a *downset* if $x \in U$ and $y \leq x$ imply $y \in U$. An ordered topological space $(X, \leq, \pi)$ is *order-Hausdorff* if $x \nleq y$ implies that there exist an upset neighborhood U of x and a downset neighborhood V of y such that $U \cap V = \varnothing$. It is well known (see, e.g., [9]) that $(X, \leq, \pi)$ is order-Hausdorff iff $\leq$ is a closed subset of X^2. Ordered topological spaces were introduced by Nachbin, who showed that compact order-Hausdorff spaces provide a natural generalization of compact Hausdorff spaces [10]. In honor of Nachbin, we call a compact order-Hausdorff space a *Nachbin space*. We recall that in a topological space, the specialization order $\leq$ is defined by $x \leq y$ iff the closure of y contains x. The following results are well known [7].

Theorem 2.5. *If (X, τ) is a stably compact space with specialization order $\leq$ and patch topology π, then $(X, \leq, \pi)$ is a Nachbin space whose open upsets are the τ-open sets, and whose open downsets are the τ^k-open sets. We call this the Nachbin space associated to (X, τ). Conversely, if $(X, \leq, \pi)$ is a Nachbin space, then the open upsets form a topology τ on X. The space (X, τ) is stably compact, and its associated Nachbin space is $(X, \leq, \pi)$.*

Following [7], we call a continuous map f between stably compact spaces *proper* if the inverse image of each compact saturated set is compact. This is equivalent to f being continuous with respect to both the given and co-compact topologies. Let StKSp be the category of stably compact spaces and proper maps. Let Nach be the category of Nachbin spaces and the continuous order-preserving maps between them. The above result extends as follows [7].

Theorem 2.6. *There is an isomorphism between the categories StKSp and Nach taking a stably compact space to its associated Nachbin space.*

We next turn our attention to frames.

Definition 2.7. *A frame is a complete lattice L that satisfies $a \wedge \bigvee S = \bigvee \{a \wedge s : s \in S\}$. A frame homomorphism is a map $f : L \to M$ that preserves finite meets (including 1) and arbitrary joins (including 0).*

For a topological space X, its open sets $\Omega(X)$ form a frame, and for any continuous map $f : X \to Y$, the map $\Omega(f) = f^{-1} : \Omega(Y) \to \Omega(X)$ is a frame homomorphism. This gives a contravariant functor $\Omega : \mathsf{Top} \to \mathsf{Frm}$ from the category of topological spaces and continuous maps to the category of frames and frame homomorphisms. A *point* of a frame L is a frame homomorphism $p : L \to 2$ into the two-element frame. The points $\mathrm{pt}(L)$ of L are topologized by taking for all $a \in L$ the sets $\varphi(a) = \{p : p(a) = 1\}$ as open sets. For a frame homomorphism $f : L \to M$, the map $\mathrm{pt}(f) : \mathrm{pt}(M) \to \mathrm{pt}(L)$ defined by $\mathrm{pt}(f)(p) = p \circ f$ is continuous. This gives a contravariant functor $\mathrm{pt} : \mathsf{Frm} \to \mathsf{Top}$. The following results are well known [8].

Theorem 2.8. *The functors Ω and pt give a dual adjunction between Top and Frm. For each frame L and space X, this dual adjunction provides a frame homomorphism $h : L \to \Omega(\mathrm{pt}(L))$ and continuous map $s : X \to \mathrm{pt}(\Omega(X))$ called the sobrification of the space. The frame homomorphism h is always onto, and the sobrification s is a topological embedding iff the space is T_0. A frame is called spatial if h is an isomorphism, and a space is called sober if s is a homeomorphism. The functors Ω and pt restrict to give a dual equivalence between the categories of spatial frames and sober spaces. If X, Y are spaces with Y sober, then Ω gives a bijection between the homsets $\mathsf{Top}(X, Y)$ and $\mathsf{Frm}(\Omega(Y), \Omega(X))$. Finally, if X is T_0, then a continuous map $e : X \to Y$ is an embedding iff the frame homomorphism $\Omega(e) : \Omega(Y) \to \Omega(X)$ is onto.*

We turn now to finer properties of frames.

Definition 2.9. *For a, b elements of a frame L, we say a is way below b, and write $a \ll b$, if for any T with $b \le \bigvee T$, there is a finite subset $S \subseteq T$ with $a \le \bigvee S$. We say a is well inside b, and write $a \prec b$, if $\neg a \vee b = 1$, where $\neg a$ is the pseudocomplement of a in L.*

An element a of a frame L is *compact* if $a \ll a$, and a frame L is *compact* if its top element 1 is compact. We next use the way below and well inside relations to define the particular classes of frames of primary interest here.

Definition 2.10. *We say a frame L is*

1. *locally compact if $a = \bigvee\{x : x \ll a\}$ for each $a \in L$.*
2. *regular if $a = \bigvee\{x : x \prec a\}$ for each $a \in L$.*
3. *stable if $a \ll b, c$ implies $a \ll b \wedge c$ for all $a, b, c \in L$.*

We say L is compact regular if it is compact and regular, and stably compact if it is locally compact, compact, and stable.

Let KRFrm be the category of compact regular frames and frame homomorphisms between them. A frame homomorphism f is called *proper* if $a \ll b$ implies $fa \ll fb$. Let StKFrm be the category of stably compact frames and proper frame homomorphisms between them. Let KHaus be the category of compact Hausdorff spaces. The following are well known [7, 8].

Theorem 2.11. *A space X is stably compact iff the frame $\Omega(X)$ is stably compact, and a frame L is stably compact iff it is isomorphic to $\Omega(X)$ for some stably compact space X. Further, a continuous map f between stably compact spaces X and Y is proper iff the corresponding frame homomorphism between $\Omega(Y)$ and $\Omega(X)$ is proper. Thus, the functors Ω and pt restrict to give a dual equivalence between StKSp and StKFrm. Each compact Hausdorff space is stably compact, and every continuous map between compact Hausdorff spaces is proper. So KHaus is a full subcategory of StKSp, KRFrm is a full subcategory of StKFrm, and Ω and pt restrict to give a dual equivalence between KHaus and KRFrm.*

A frame is *coherent* if each element is the join of compact elements, and the meet of two compact elements is compact. A frame homomorphism h between two coherent frames L

and M is *coherent* if a compact in L implies that $h(a)$ is compact in M. Let CohFrm be the category of coherent frames and the coherent frame homomorphisms between them. A space X is a *spectral space* if it is sober, compact, and the compact open sets are closed under finite intersections and form a basis. A continuous map between spectral spaces is a *spectral map* if the inverse image of each compact open set is compact open. Let Spec be the category of spectral spaces and the spectral maps between them. We conclude the preliminaries with the following well known result [8].

Theorem 2.12. *The category* Spec *is a full subcategory of* StKSp*, the category* CohFrm *is a full subcategory of* StKFrm*, and the functors* Ω, pt *restrict to a dual equivalence between* Spec *and* CohFrm*.*

3. Stable compactifications of spaces

In this section we recall Smyth's definition of a stable compactification of a T_0-space X, and Smyth's ordering of the stable compactifications of X. We provide an alternate description of such stable compactifications in terms of the patch topology of a stably compact space, and remark on the connections between stable compactifications and ordered spaces.

Definition 3.1. *A stable compactification of a* T_0*-space* X *is a pair* (Y, e) *where* Y *is a stably compact space and* $e : X \to Y$ *is a homeomorphism from* X *onto a subspace of* Y *that satisfies* $U \ll V \Rightarrow \overline{U} \ll V$ *for all* $U, V \in \Omega(Y)$*.*

Here Smyth uses $U \ll V$ to mean U is way below V in the frame of open sets, and he uses $\overline{U}$ for the largest open set of Y whose intersection with the image of X is contained in U. If (Y, e) is a stable compactification of X, using $\varnothing \ll \varnothing$ it follows that the image $e[X]$ is dense in Y. We next recall Smyth's ordering of the stable compactifications of a T_0-space X.

Definition 3.2. *For two stable compactifications* $e : X \to Y$ *and* $e' : X \to Y'$*, define* $e' \sqsubseteq e$ *if there is a proper continuous map* $f : Y \to Y'$ *with* $e' = f \circ e$*. We let* $\textsc{Comp}\ X$ *be the poset of equivalence classes of stable compactifications of* X *under the partial order associated with the quasi-order* $\sqsubseteq$ *and denote the equivalence class of a compactification* $e : X \to Y$ *by* $[e]$*.*

Smyth characterized the poset $\textsc{Comp}\ X$ in terms of his "quasi-proximities" on X, and showed it has a largest element given by the space of prime filters of the frame $\Omega(X)$ of open sets of X.

Remark 3.3. Stable compactifications of T_0-spaces lack some of the familiar properties of classical compactifications of completely regular spaces. Smyth's result [15, Prop. 16] that the space of prime filters of $\Omega(X)$ gives the largest stable compactification of X yields an example showing that a compact Hausdorff space can have a stable compactification that is not Hausdorff. One can further show that for stable compactifications $e : X \to Y$ and $k : Y \to Z$, the composite $k \circ e : X \to Z$ need not be a stable compactification.

The condition $U \ll V \Rightarrow \overline{U} \ll V$ in Smyth's definition of a stable compactification has a strongly frame-theoretic nature. We provide a description of stable compactifications in more purely topological terms, namely as embeddings into stably compact spaces that are dense in the patch topology. We note that this is somewhat the inverse of the usual sequence of things in pointfree topology, when standard topological notions are given pointfree meaning. We begin with several standard facts from the theory of ordered spaces whose proofs can be found in [7, 10].

Proposition 3.4. *Let $(Y, \leq, \pi)$ be a Nachbin space.*

(1) *$A \ll B$ in the frame of open upsets of Y iff $\mathrm{cl}_\pi(A) \subseteq B$.*
(2) *If B is an open downset, $B = \bigcup\{A : A$ is an open downset and $\mathrm{cl}_\pi(A) \subseteq B\}$.*
(3) *If A is closed, then its downset $\downarrow A$ is closed.*

Here $\mathrm{cl}_\pi(A)$ indicates closure in the topology π.

Theorem 3.5. *For a T_0-space X, an embedding $e : X \to Y$ into a stably compact space Y is a stable compactification of X iff the image of X is dense in the patch topology of Y.*

Proof. By identifying X with its image $e[X]$ in Y, we assume that X is a subspace of Y and Y is a stably compact space with topology τ and patch topology π. Let $(Y, \leq, \pi)$ be the Nachbin space associated to Y.

"$\Leftarrow$" Assume X is patch-dense in Y. To show the identical embedding of X into Y is a stable compactification, we must show that for U, V τ-open subsets of Y, that $U \ll V$ implies $\overline{U} \ll V$. If $U \ll V$, then by Proposition 3.4.1, $\mathrm{cl}_\pi(U) \subseteq V$. As X is patch-dense in Y, for each patch-open subset W of Y, we have $\mathrm{cl}_\pi(W) = \mathrm{cl}_\pi(W \cap X)$. Therefore, from the definition of $\overline{U}$ as the largest τ-open set whose intersection with X is contained in U, we have $\mathrm{cl}_\pi(\overline{U}) = \mathrm{cl}_\pi(\overline{U} \cap X) = \mathrm{cl}_\pi(U \cap X) = \mathrm{cl}_\pi(U)$. Thus, $\mathrm{cl}_\pi(\overline{U}) \subseteq V$, so $\overline{U} \ll V$.

"$\Rightarrow$" Assume X is not patch-dense in Y. We show that there exist τ-open sets U, V such that $U \ll V$ and $\overline{U} \not\ll V$. We recall (see Theorem 2.5) that the open upsets of the Nachbin space $(Y, \leq, \pi)$ are the topology τ, the open downsets are the co-compact topology τ^k, and the join of these topologies is the patch topology π. We also use $-T$ for the set-theoretic complement of T.

Claim 3.6.

(1) *There exist an open upset S and an open downset T with $S \cap T \neq \varnothing$ and $X \cap S \cap T = \varnothing$.*
(2) *For each open upset A, if $-T \subseteq A$, then $S \subseteq \overline{A}$.*

Proof of Claim: (1) This is a consequence of the fact that the patch topology is the join of the topologies of the open upsets and open downsets and that X is not patch-dense. (2) As $X \cap S \cap T = \varnothing$, we have $X \cap S \subseteq -T \subseteq A$. So $S \subseteq \overline{A}$. □

Claim 3.7. *Let $z \in S \cap T$. There are open downsets P, Q with*

(1) *$z \in Q$ and $\mathrm{cl}_\pi(Q) \subseteq T$.*
(2) *$z \in P$ and $\mathrm{cl}_\pi(P) \subseteq Q$.*

Further, both $U = -\downarrow\mathrm{cl}_\pi(Q)$ and $V = -\downarrow\mathrm{cl}_\pi(P)$ are open upsets.

Proof of Claim: (1) As $z \in T$ and T is an open downset, Proposition 3.4.2 gives an open downset Q with $z \in Q$ and $\mathrm{cl}_\pi(Q) \subseteq T$. (2) As $z \in Q$ and Q is an open downset, another application of Proposition 3.4.2 gives an open downset P with $z \in P$ and $\mathrm{cl}_\pi(P) \subseteq Q$. For the further comment, by Proposition 3.4.3, both $\downarrow\mathrm{cl}_\pi(Q)$ and $\downarrow\mathrm{cl}_\pi(P)$ are closed, and are clearly downsets. Thus, their complements are open upsets. □

We show the open upsets U, V satisfy $U \ll V$ and $\overline{U} \not\ll V$. As $Q \subseteq \downarrow\mathrm{cl}_\pi(Q)$, we have $U = -\downarrow\mathrm{cl}_\pi(Q) \subseteq -Q$, and as $\mathrm{cl}_\pi(P) \subseteq Q$ and Q is a downset, $\downarrow\mathrm{cl}_\pi(P) \subseteq Q$, giving $-Q \subseteq -\downarrow\mathrm{cl}_\pi(P) = V$. Thus, $U \subseteq -Q \subseteq V$, and as Q is open, $\mathrm{cl}_\pi(U) \subseteq V$. So by Proposition 3.4.1, $U \ll V$. To see that $\overline{U} \not\ll V$, note $z \in P \subseteq \downarrow\mathrm{cl}_\pi(P)$, so $z \notin -\downarrow\mathrm{cl}_\pi(P) = V$. As $\mathrm{cl}_\pi(Q) \subseteq T$ and

T is a downset, we have $\downarrow\mathrm{cl}_\pi(Q) \subseteq T$, hence $-T \subseteq -\downarrow\mathrm{cl}_\pi(Q) = U$. Since U is an open upset and $-T \subseteq U$, by Claim 3.6.2, $S \subseteq \overline{U}$. But $z \in S$, hence $z \in \overline{U}$, and $z \notin V$, so $\overline{U} \nsubseteq V$. Thus, $\overline{U} \not\prec V$. □

Corollary 3.8. *The stable compactifications of a T_0-space X determine, and are determined by, mappings of X into a Nachbin space $(Y, \leq, \pi)$ that are embeddings with respect to the topology of open upsets of Y, and are dense with respect to π.*

We next use this result to relate the poset of stable compactifications of a completely regular space X to its poset of classical compactifications. Surely any compactification of X is a stable compactification, and any stable compactification into a compact Hausdorff space is a compactification. We use $k : X \to \beta X$ for the Stone-Čech compactification of X, and recall this is the largest compactification of X.

Proposition 3.9. *The poset of classical compactifications of a completely regular space X is a retract of the downset of* Comp X *generated by βX.*

Proof. Suppose $e : X \to Y$ is a stable compactification of X that lies beneath $k : X \to \beta X$ in the poset of stable compactifications. Then there is a proper continuous map $f : \beta X \to Y$ with $e = f \circ k$. Let σ be the topology on βX and τ be the topology on Y. As βX is compact Hausdorff, its patch topology is σ. Let π be the patch topology on Y. Since f is proper with respect to σ and τ, it is continuous with respect to the patch topologies σ and π. Let $U \in \pi$. Then $f^{-1}(U) \in \sigma$, so $k^{-1}f^{-1}(U)$ is open in X, hence $e^{-1}(U)$ is open in X. Thus, $e : X \to (Y, \pi)$ is continuous. By Theorem 3.5, $e[X]$ is dense in (Y, π), and as (Y, π) is a compact Hausdorff space, this is a compactification of X. It is then routine to show that the map sending such a stable compactification $e : X \to (Y, \tau)$ to the compactification $e : X \to (Y, \pi)$ is the required retraction. □

We recall the classical result that a completely regular space has a least compactification iff it is locally compact, and in this case, its least compactification is the one-point compactification. As the construction of the one-point compactification of a locally compact Hausdorff space generalizes to any T_0-space (see, e.g., [5, Sec. 3]), every T_0-space X has a (possibly non-Hausdorff) one-point compactification. This one-point compactification does not have to be a stable compactification of X. In fact, as the next corollary shows, not every T_0-space has a least stable compactification.

Corollary 3.10. *The space of rationals $\mathbb{Q}$ with the usual topology has no least stable compactification.*

Proof. If there were a least element in the poset of stable compactifications of $\mathbb{Q}$, then by Proposition 3.9, there would be a least element in the poset of classical compactifications of $\mathbb{Q}$. This is not the case since $\mathbb{Q}$ is not locally compact. □

There are further connections between stable compactifications and ordered spaces, some of which bear fruit, while others do not. We describe some of these connections below. We start by recalling Nachbin's generalization of the concept of compactification to that of order-compactification.

Definition 3.11. *An order-compactification of an ordered space X is a pair (Y, e) such that Y is a Nachbin space, $e : X \to Y$ is both a topological embedding and an order-embedding, and the image $e[X]$ is topologically dense in Y.*

Proposition 3.12. *Let $e : X \to Y$ be a stable compactification. Then the associated Nachbin structure $(Y, \leq, \pi)$ induces an ordered space structure on X whose open upsets are the original topology of X, and whose partial ordering is the specialization order of this topology. Further, the embedding e of this ordered structure into $(Y, \leq, \pi)$ is an order-compactification.*

Proof. Let (X, τ) and (Y, δ) be our original T_0 and stably compact spaces. By Theorem 3.5, the image $e[X]$ is dense in the patch topology π of Y, so is a topologically dense subspace of the Nachbin space $(Y, \leq, \pi)$. The restriction of this Nachbin structure to $e[X]$ makes $e[X]$ into an ordered space having $(Y, \leq, \pi)$ as an order-compactification. So this induces an ordered space structure on X having $(Y, \leq, \pi)$ as an order-compactification. It remains only to show the open upsets of this ordered space structure on X are the original topology τ, and that the partial ordering on X is the specialization order.

The open upsets of $e[X]$ are the restrictions of the open upsets of $(Y, \leq, \pi)$, hence are the restrictions of members of δ to $e[X]$. So the open upsets of the induced structure on X are the inverse images under e of members of δ, and as e is an embedding with respect to τ and δ, these are exactly the members of τ. As the partial ordering of $(Y, \leq \pi)$ is the specialization order of δ, the partial ordering of $e[X]$ is the specialization order of the open upsets of $e[X]$, hence the partial ordering on X is the specialization order of τ. $\square$

Proposition 3.12 shows that every stable compactification can be viewed as an order-compactification. The following example shows the converse of this does not hold. The difficulty, roughly speaking, is in the fact that an order-compactification must be an embedding with respect to the patch topology, while a stable compactification must be an embedding with respect to the topology of open upsets.

Example 3.13. Let $(X, \leq, \pi)$ be the natural numbers $\mathbb{N}$ with discrete topology, ordered as an antichain, and let $(Y, \leq, \pi)$ be the one-point compactification $\mathbb{N} \cup \{\infty\}$ ordered as an antichain on $\mathbb{N}$ and with $n \leq \infty$ for each $n \in \mathbb{N}$. The identical embedding is an order-compactification of $(X, \leq, \pi)$ into $(Y, \leq, \pi)$. But the open upsets of Y are the cofinite ones containing ∞, while all subsets of X are open upsets. So the identical embedding is not a stable compactification with respect to the topologies of open upsets.

Proposition 3.12 can be viewed in another light. Each stable compactification of a T_0-space (X, τ) induces an ordered space structure on X having τ as its open upsets, and giving a companion topology τ' of open downsets, so that the join of the topologies $\pi = \tau \vee \tau'$ is a completely regular topology. Salbani [11, 12] has considered a method to associate with any T_0 topology τ on X a companion topology he calls τ^*, where τ^* has the members of τ as a basis for the closed sets.

Proposition 3.14. *For a T_0-space (X, τ), Salbani's topology τ^* is the companion topology to τ arising from the largest stable compactification of X.*

Proof. Smyth [15, Prop. 16] showed that the largest stable compactification of (X, τ) is the space Y of prime filters of $\Omega(X)$, i.e. the spectral space of the distributive lattice $\Omega(X)$. Here Y has as a basis for its topology all sets $\varphi(U) = \{F : U \in F\}$, where $U \in \Omega(X)$, and the embedding $e : X \to Y$ is given by $e(x) = \{U : x \in U\}$. Note $e^{-1}\varphi(U) = U$ for each $U \in \Omega(X)$. The Nachbin space associated to Y is the Priestley space of $\Omega(X)$. The closed sets of the topology of open downsets of Y are the closed upsets of the Priestley space. These are the intersections of the clopen upsets, hence of the sets $\varphi(U)$, where $U \in \Omega(X)$. So the companion topology on X induced by this stable compactification has the sets $U = e^{-1}\varphi(U)$

for $U \in \Omega(X)$ as a basis for its closed sets. Thus, this companion topology is Salbani's topology τ^*. □

Remark 3.15. We may consider these results in one further context. We recall that a *bitopological space* is a set X equipped with two topologies τ_1 and τ_2. For a bispace (X, τ_1, τ_2), let $\pi = \tau_1 \vee \tau_2$ be the patch topology. Following [11], we call a bispace (X, τ_1, τ_2) *compact* if (X, π) is compact, T_0 if (X, π) is T_0, and *regular* if it is T_0 and for each $U \in \tau_i$, we have $U = \bigcup\{V \in \tau_i : \mathrm{cl}_k(V) \subseteq U\}$ $(i \neq k,\ i, k = 1, 2)$. The correspondence of Theorem 2.5 between stably compact spaces and Nachbin spaces extends to also include compact regular bispaces. Indeed, if $(X, \leq, \pi)$ is a Nachbin space, then the open upsets and open downsets form a compact regular bispace, and each compact regular bispace arises this way (see, e.g., [7]).

Salbany [11] generalized the notion of compactification to that of bicompactification. A *bicompactification* of a bispace (X, τ_1, τ_2) is a bispace embedding $e : (X, \tau_1, \tau_2) \to (Y, \delta_1, \delta_2)$ into a compact regular bispace (Y, δ_1, δ_2) such that $e[X]$ is dense in the patch topology $\pi = \delta_1 \vee \delta_2$. For any stable compactification $e : (X, \tau_1) \to (Y, \delta_1)$ of a T_0-space, letting δ_2 be the co-compact topology of δ_1, produces a compact regular bispace (Y, δ_1, δ_2). This induces a completely regular bispace structure (X, τ_1, τ_2) on X with $e : (X, \tau_1, \tau_2) \to (Y, \delta_1, \delta_2)$ a bispace compactification. (Note that τ_2 is not determined by τ_1, but is dependent on the specific stable compactification of (X, τ_1).) This is the bispace analogue of Proposition 3.12, and indicates that every stable compactification can be viewed as a bicompactification. On the other hand, translating Example 3.13 into the language of bispaces shows that not every bicompactification can be viewed as a stable compactification.

4. Stable compactifications of frames

In this section we extend the notion of stable compactifications to the setting of frames, and describe the poset of inequivalent compactifications of a frame in several ways. To begin, we recall Banascewski's definition of a compactification of a frame [1].

Definition 4.1. *A compactification of a frame L is a dense frame homomorphism $f : M \to L$ from a compact regular frame M onto L. Here, a frame homomorphism is dense if for all $x \in M$ we have $f(x) = 0$ implies $x = 0$.*

Banaschewski showed that a frame L has a compactification iff it is a completely regular frame, and that for a completely regular space X, the compactifications of the frame $\Omega(X)$ correspond to the compactifications of the space X. So compactifications of spatial frames amount to a translation of the notion of compactification to the frame language. However, while every compact regular frame M is spatial, there are completely regular frames L that are not spatial, and for these the notion of compactification is new. We now extend these ideas to stable compactifications of frames.

Definition 4.2. *For a frame homomorphism $f : M \to L$ we let $r_f : L \to M$ be the right adjoint of f, namely the map $r_f(a) = \bigvee\{x : f(x) \leq a\}$.*

The map r_f preserves finite meets, but need not preserve finite joins. When the map f is clear from the context, we often use r in place of r_f.

Definition 4.3. *A stable compactification of a frame L is a pair (M, f) where M is a stably compact frame and $f : M \to L$ is an onto frame homomorphism that satisfies*

$$(*) \qquad x \ll y \Rightarrow r(f(x)) \ll y.$$

The reader may notice that as with stable compactifications of spaces, density is not specifically required in the definition. As with spaces, it is a consequence of the definition.

Lemma 4.4. *If $f : M \to L$ is a stable compactification of L, then f is dense.*

Proof. As $0 \ll 0$, we have $r(0) \ll 0$, giving $r(0) = 0$. □

Proposition 4.5. *Every compactification of L is a stable compactification of L.*

Proof. Suppose $f : M \to L$ is a compactification. Then f is a dense onto frame homomorphism. To show it is a stable compactification, we must verify condition $(*)$ of Definition 4.3. The way below relation $\ll$ and well inside relation $\prec$ agree in any compact regular frame, so it is sufficient to show that if $x, y, z \in M$ with $x \prec y$ and $f(z) \le f(x)$, then $z \prec y$. From $f(z) \le f(x)$ it follows that $f(z) \wedge \neg f(x) = 0$, so $f(z) \wedge f(\neg x) = 0$, and the density of f yields $z \wedge \neg x = 0$, hence $\neg x \le \neg z$. But $x \prec y$ means $\neg x \vee y = 1$, hence $\neg z \vee y = 1$, giving $z \prec y$. □

We next show that the stable compactifications of the frame of open sets of a T_0-space X correspond to Smyth's stable compactifications of the space X. We recall (see Theorem 2.11) that the stably compact frames are, up to isomorphism, exactly the frames $\Omega(Y)$ for a stably compact space Y. Also, by Theorem 2.8, if Y is a stably compact space, hence a sober space, then Ω provides a bijection between the homsets $\mathsf{Top}(X, Y)$ and $\mathsf{Frm}(\Omega(Y), \Omega(X))$.

Proposition 4.6. *For X a T_0-space, Y a stably compact space, and $e : X \to Y$ continuous, e is a stable compactification of X iff $\Omega(e)$ is a stable compactification of $\Omega(X)$.*

Proof. Theorem 2.8 states that e is an embedding iff $\Omega(e)$ is onto. For U and V open subsets of Y we have $\Omega(e)(V) \subseteq \Omega(e)(U)$ iff $e^{-1}(V) \subseteq e^{-1}(U)$ iff $V \cap e[X] \subseteq U \cap e[X]$. It follows that $\overline{U} = r(\Omega(e))(U)$ for all open $U \subseteq Y$. Thus, e is a stable compactification iff $\Omega(e)$ is a stable compactification. □

We turn next to providing internal ways to describe stable compactifications of a frame. This is similar in spirit to Smirnov's result providing an internal characterization of the compactifications of a completely regular space, and Banaschewski's result [1] characterizing compactifications of a frame. Our first approach, via proximities, is closely related to Banaschewski's characterization using strong inclusions.

Definition 4.7. [4] *Let L be a frame. A proximity on L is a binary relation $\prec$ on L satisfying:*

1. $0 \prec 0$ *and* $1 \prec 1$.
2. $a \prec b$ *implies* $a \le b$.
3. $a \le b \prec c \le d$ *implies* $a \prec d$.
4. $a, b \prec c$ *implies* $a \vee b \prec c$.
5. $a \prec b, c$ *implies* $a \prec b \wedge c$.
6. $a \prec b$ *implies there exists* $c \in L$ *with* $a \prec c \prec b$.
7. $a = \bigvee\{b \in L : b \prec a\}$.

Example 4.8. Some examples of proximity frames are the following. (1) The partial ordering of any frame is a proximity. (2) A strong inclusion on a frame [1] is a proximity. (3) The way below relation on a stably compact frame is a proximity. (4) The well inside relation on any regular frame is a proximity. (5) The really inside relation on any completely regular frame [8, Sec. IV.1] is a proximity.

Definition 4.9. *For $f : M \to L$ a stable compactification of L, define a relation $\prec_f$ on L by setting $a \prec_f b \Leftrightarrow r_f(a) \ll r_f(b)$.*

To make notation nicer, we often use $\prec$ in place of $\prec_f$ and r in place of r_f.

Lemma 4.10. *$a \prec b$ iff $x \ll y$ for some x, y with $f(x) = a$ and $f(y) = b$.*

Proof. $\Rightarrow$: This is trivial as $f(r(a)) = a$ and $f(r(b)) = b$ since f is onto.

$\Leftarrow$: Suppose $x \ll y$, where $f(x) = a$ and $f(y) = b$. Then as $y \leq r(b)$, we have $x \ll r(b)$. But part of the definition of a stable compactification says $p \ll q \Rightarrow r(f(p)) \ll q$. Thus, as $r(f(x)) = r(a)$, we have $r(a) \ll r(b)$. □

Proposition 4.11. *If $f : M \to L$ is a stable compactification, then $\prec$ is a proximity on L.*

Proof. For (1) note $0 \ll 0$ always holds, and as M is compact $1 \ll 1$. By Lemma 4.10, $f(0) \prec f(0)$ and $f(1) \prec f(1)$, giving $0 \prec 0$ and $1 \prec 1$. For (2) suppose $a \prec b$. Then $r(a) \ll r(b)$, hence $r(a) \leq r(b)$, giving $a = fr(a) \leq fr(b) = b$. For (3) suppose $a \leq b \prec c \leq d$. Then $r(a) \leq r(b) \ll r(c) \leq r(d)$, so $r(a) \ll r(d)$, hence $a \prec d$. For (4) suppose $a, b \prec c$. Then $r(a), r(b) \ll r(c)$, hence by general properties of the way below relation, $r(a) \vee r(b) \ll r(c)$. Then as $f(r(a) \vee r(b)) = a \vee b$ and $f(r(c)) = c$, Lemma 4.10 gives $a \vee b \prec c$. For (5) suppose $a \prec b, c$. Then $r(a) \ll r(b), r(c)$ and as M is stable, $r(a) \ll r(b) \wedge r(c)$, giving $r(a) \ll r(b \wedge c)$, hence $a \prec b \wedge c$. For (6) suppose $a \prec b$. Then $r(a) \ll r(b)$. As M is stably compact, we may interpolate to find z with $r(a) \ll z \ll r(b)$. Then letting $f(z) = c$, Lemma 4.10 shows $a \prec c \prec b$. For (7) as M is stably compact, $r(a) = \bigvee\{x : x \ll r(a)\}$, so $f(r(a)) = \bigvee\{f(x) : x \ll r(a)\}$. By Lemma 4.10, if $x \ll r(a)$ then $f(x) \prec a$. It follows that $a = \bigvee\{b : b \prec a\}$. □

Definition 4.12. *For a proximity $\prec$ on L, we say an ideal I of L is $\prec$-round if for each $a \in I$ there is $b \in I$ with $a \prec b$. We let $\mathfrak{I}_\prec L$ be the collection of all $\prec$-round ideals of L.*

Definition 4.13. *For a stably compact frame M, we say $N \subseteq M$ is a stably compact subframe of M if*

(1) *N is a subframe of M.*
(2) *N is a stably compact frame.*
(3) *The identical embedding of N in M is proper, so $a \ll_N b \Rightarrow a \ll_M b$.*

A stably compact subframe M of the ideal frame $\mathfrak{I}L$ is called dense if $\bigvee\cdot : M \to L$ is onto.

We note that if N is a stably compact subframe of M, then $a \ll b$ in N iff $a \ll b$ in M. One direction is provided by the definition of stably compact subframe, the other as $a \ll b$ in a frame implies $a \ll b$ in any subframe. We also point the reader to [4, Sec. 4], where a number of results were established for the frame of round ideals of a proximity frame. In [4], this frame was called $\mathfrak{R}\mathfrak{I}L$ rather than $\mathfrak{I}_\prec L$ as above because there was no need to consider more than one proximity on a given frame, as there shall be here.

Proposition 4.14. *For a proximity $\prec$ on L, the set $\mathfrak{I}_\prec L$ of $\prec$-round ideals is a dense stably compact subframe of the frame $\mathfrak{I}L$ of ideals of L. Further, the join map $\bigvee\cdot : \mathfrak{I}_\prec L \to L$ is a stable compactification of L.*

Proof. For $a \in L$ let $\twoheaddownarrow a = \{b \in L : b \prec a\}$. In [4, Prop. 4.6] it is shown that $\mathfrak{I}_\prec L$ is a subframe of $\mathfrak{I}L$, and $I \ll J$ in $\mathfrak{I}_\prec L$ iff $I \subseteq \twoheaddownarrow a$ for some $a \in J$. This second condition shows $I \ll J$ in $\mathfrak{I}_\prec L$ implies $I \ll J$ in $\mathfrak{I}L$. That $\mathfrak{I}_\prec L$ is a stably compact frame is given in [4, Prop. 4.8]. Together, these show $\mathfrak{I}_\prec L$ is a stably compact subframe of $\mathfrak{I}L$.

The map $\bigvee\cdot : \mathfrak{I}L \to L$ is known to be a frame homomorphism. So its restriction to $\mathfrak{I}_{\prec}L$ is a frame homomorphism. To see it is onto, if $a \in L$, then $\Downarrow a$ is a $\prec$-round ideal and by properties of a proximity, $\bigvee \Downarrow a = a$. Thus, $\mathfrak{I}_{\prec}L$ is dense. Finally, we show $\bigvee\cdot$ satisfies condition $(*)$. Suppose I, J are $\prec$-round ideals with $I \ll J$. Then there is $a \in J$ with $I \subseteq \Downarrow a$. Suppose $\bigvee I = b$. Then the largest round ideal mapped by $\bigvee\cdot$ to b is $\Downarrow b$. As $I \subseteq \downarrow a$ and $b = \bigvee I$, we have $b \le a$. Then $\Downarrow b \subseteq \downarrow a$ and $a \in J$ imply $\Downarrow b \ll J$. This shows $\bigvee\cdot$ satisfies condition $(*)$, so is a stable compactification. ☐

Proposition 4.15. *If $f : M \to L$ is a stable compactification, then there are mutually inverse frame isomorphisms $g : M \to \mathfrak{I}_{\prec_f}L$ and $h : \mathfrak{I}_{\prec_f}L \to M$ defined by $g(m) = \{f(n) : n \ll m\}$ and $h(I) = \bigvee r_f[I]$. Further, $(\bigvee\cdot) \circ g = f$ and $f \circ h = \bigvee\cdot$.*

Proof. For $m \in M$ we first show $I = g(m)$ is a $\prec_f$-round ideal of L. Suppose $n \ll m$ and $a \le f(n)$. Then $r(a) \le rf(n)$, and by condition $(*)$ on f we have $n \ll m \Rightarrow rf(n) \ll m$, so $r(a) \ll m$, thus $a = fr(a) \in I$. So I is a downset. If $n_1 \ll m$ and $n_2 \ll m$, then $n_1 \vee n_2 \ll m$, and as f is a frame homomorphism, it follows that I is closed under finite joins, so I is an ideal of L. Say $n \ll m$. Then there is p with $n \ll p \ll m$. Therefore, by Lemma 4.10, $f(n) \prec_f f(p)$ and $f(p) \in I$. So I is $\prec_f$-round.

We have shown g is well-defined. Clearly h is also well-defined, and it is obvious that both g and h are order-preserving. For $m \in M$ we have $hg(m) = \bigvee\{rf(n) : n \ll m\}$. Condition $(*)$ on f shows $n \ll m \Rightarrow rf(n) \ll m$, hence $n \ll m \Rightarrow n \le rf(n) \le m$. As M is stably compact, $m = \bigvee\{n : n \ll m\}$, and it follows that $m = \bigvee\{rf(n) : n \ll m\}$. Thus, $h \circ g$ is the identity map on M.

Suppose I is a $\prec_f$-round ideal of L. If $a \in I$, then there is $b \in I$ with $a \prec_f b$. By the definition of $\prec_f$ we have $r(a) \ll r(b)$, hence $r(a) \ll r(b) \le \bigvee r[I] = h(I)$. As $gh(I) = \{f(n) : n \ll h(I)\}$ we have $a = fr(a) \in gh(I)$. Thus, $I \subseteq gh(I)$. Conversely, suppose $a \in gh(I)$. Then $a = f(n)$ for some $n \ll h(I)$. As $h(I) = \bigvee r[I]$, the definition of way below and the fact that $r[I]$ is up-directed gives $n \le r(b)$ for some $b \in I$. Therefore, $a = f(n) \le fr(b) = b$, and as I is an ideal, we have $a \in I$. Thus, $I = gh(I)$, showing $g \circ h$ is the identity map on $\mathfrak{I}_{\prec_f}L$. So we have shown g and h are mutually inverse frame isomorphisms between M and $\mathfrak{I}_{\prec_f}L$.

For the further comment, suppose $m \in M$. As M is stably compact, $m = \bigvee\{n : n \ll m\}$, and as f is a frame homomorphism, $f(m) = \bigvee\{f(n) : n \ll m\}$. Thus, $(\bigvee\cdot) \circ g = f$. Then $f \circ h = (\bigvee\cdot) \circ g \circ h = \bigvee\cdot$ as g and h are mutually inverse isomorphisms. ☐

Proposition 4.16. *If L is a frame and M is a dense stably compact subframe of $\mathfrak{I}L$, then $\bigvee\cdot : M \to L$ is a stable compactification of L, and M is equal to $\mathfrak{I}_{\prec_{\bigvee\cdot}}L$.*

Proof. By the definition of a stably compact subframe, we have M is a stably compact frame. Also, this definition implies M is a subframe of $\mathfrak{I}L$, and as the join map from $\mathfrak{I}L$ to L is a frame homomorphism, its restriction to M is also a frame homomorphism. We have assumed the join map from M to L is an onto mapping, so to show $\bigvee\cdot : M \to L$ is a stable compactification we need only show this map satisfies condition $(*)$. Suppose $I, J \in M$ with $I \ll J$ in M, hence by the definition of a stably compact subframe, $I \ll J$ in $\mathfrak{I}L$. Let $r(I)$ be the largest element of M mapped by $\bigvee\cdot$ to $\bigvee I$, and let $\hat{r}(I)$ be the largest element of $\mathfrak{I}L$ mapped by $\bigvee\cdot$ to $\bigvee I$. As $\bigvee\cdot : \mathfrak{I}L \to L$ is a stable compactification, we have $\hat{r}(I) \ll J$ in $\mathfrak{I}L$, so $r(I) \le \hat{r}(I) \ll J$ in $\mathfrak{I}L$, giving $r(I) \ll J$ in $\mathfrak{I}L$, hence $r(I) \ll J$ in M. Thus, $\bigvee\cdot : M \to L$ is a stable compactification.

We now show $M = \mathfrak{I}_{\prec_{\bigvee\cdot}}$. Suppose I is an element of M. Surely I is an ideal of L, we must show it is $\prec_{\bigvee\cdot}$-round. Let $a \in I$. Then as $\bigvee\cdot : M \to L$ is assumed to be onto, there is some $J \in M$ with $\bigvee J = a$. So $J \subseteq {\downarrow}a$ and $a \in I$ give $J \ll I$. As M is stably compact, $\ll$ is interpolating, so we can find K in M with $J \ll K \ll I$. Setting $b = \bigvee K$, the definition of $\prec_{\bigvee\cdot}$ gives $a \prec_{\bigvee\cdot} b$ since $J \ll K$ and both $a = \bigvee J$ and $b = \bigvee K$. Now $K \ll I$ gives $K \subseteq {\downarrow}c$ for some $c \in I$, hence $b \in I$. So I is indeed $\prec_{\bigvee\cdot}$-round. Conversely, suppose I is a $\prec_{\bigvee\cdot}$-round ideal of L. As $\bigvee\cdot : M \to L$ is an onto frame homomorphism, for each $a \in I$ there is a largest ideal J_a in M with $a = \bigvee J_a$. Let J be the join in the ideal frame of $\{J_a : a \in I\}$. Then as M is a subframe of $\mathfrak{I}L$, we have $J \in M$. For each $a \in I$ we have $J_a \subseteq {\downarrow}a$, so each J_a is contained in I, hence $J \subseteq I$. Suppose $a \in I$. As I is $\prec_{\bigvee\cdot}$-round, there is $b \in I$ with $a \prec_{\bigvee\cdot} b$. This means there are ideals $P \ll Q$ in M with $a = \bigvee P$ and $b = \bigvee Q$. As $P \ll Q$, there is $c \in Q$ with $P \subseteq {\downarrow}c$. Clearly $a \leq c$, and $c \in Q \subseteq J_b$. Thus, $a \leq c \in J$, so $a \in J$. So $J = I$, showing I belongs to M. □

Definition 4.17. *For stable compactifications $f : M \to L$ and $f' : M' \to L$ of a frame L, define $f \sqsubseteq f'$ if there is a proper frame homomorphism $g : M \to M'$ with $f' \circ g = f$. Then $\sqsubseteq$ is reflexive and transitive, so is a quasi-order on the class of stable compactifications of L. Let* Comp L *be the poset of equivalence classes of stable compactifications under the partial order associated with $\sqsubseteq$, and denote the equivalence class of $f : M \to L$ by $[f]$.*

Remark 4.18. Proposition 4.15 shows every equivalence class of Comp L contains a member of the form $\bigvee\cdot : \mathfrak{J} \to L$ for some stably compact subframe $\mathfrak{J}$ of $\mathfrak{I}L$. So Comp L is a set with a partial ordering even though there is a proper class of compactifications.

Definition 4.19. *For a frame L, let* Prox L *be the poset of proximities on L, partially ordered by set inclusion, and* Sub $\mathfrak{I}L$ *be the poset of dense stably compact subframes M of the ideal frame $\mathfrak{I}L$, partially ordered by set inclusion.*

We next see that the posets Comp L, Prox L, and Sub $\mathfrak{I}L$ are isomorphic.

Theorem 4.20. *For a frame L there are isomorphisms*

$\Phi :$ Comp $L \to$ Prox L *where* $\Phi([f]) = \prec_f$
$\Psi :$ Prox $L \to$ Sub $\mathfrak{I}L$ *where* $\Psi(\prec) = \mathfrak{I}_\prec L$
$\Pi :$ Sub $\mathfrak{I}L \to$ Comp L *where* $\Pi(M)$ *is the equivalence class of* $\bigvee\cdot : M \to L$.

Further, $\Phi^{-1} = \Pi \circ \Psi$, $\Psi^{-1} = \Phi \circ \Pi$, *and* $\Pi^{-1} = \Psi \circ \Phi$.

Proof. To see Φ is well-defined, suppose $f : M \to L$ and $f' : M' \to L$ are equivalent stable compactifications, so there are proper frame homomorphisms $g : M \to M'$ and $g' : M' \to M$ with $f'g = f$ and $fg' = f'$. If $a \prec_f b$, then by Lemma 4.10, there are $x \ll y$ in M with $f(x) = a$ and $f(y) = b$. As g is proper, we have $g(x) \ll g(y)$, and as $f'g = f$, we have $a = f'g(x) \prec_{f'} f'g(y) = b$. So $\prec_f \subseteq \prec_{f'}$, and by symmetry $\prec_{f'} \subseteq \prec_f$, hence equality. So the definition of Φ does not depend on the member f of the equivalence class $[f]$ chosen. That $\Phi([f])$ is indeed a member of Prox L is given by Proposition 4.11. That Ψ is a map into Sub $\mathfrak{I}L$ is given by Proposition 4.14, and that Π is a map into Comp L is given by Proposition 4.16.

To see Φ is order-preserving, suppose $[f] \leq [f']$ where $f : M \to L$ and $f' : M' \to L$ are stable compactifications. Then there is a proper frame homomorphism $g : M \to M'$ with $f'g = f$. We have just seen that this implies $\prec_f \subseteq \prec_{f'}$, so $\Phi([f]) \leq \Phi([f'])$. To see Ψ is order-preserving, suppose $\prec \subseteq \prec'$. Then $\mathfrak{I}_\prec L$ is a subset of $\mathfrak{I}_{\prec'} L$, so $\Psi(\prec) \subseteq \Psi(\prec')$. Finally, to show Π is order-preserving, suppose M and M' are dense stably compact subframes of $\mathfrak{I}L$ with $M \subseteq M'$. Let $g : M \to M'$ be the identical embedding. As both M and M' are subframes of $\mathfrak{I}L$, we have finite meets and arbitrary joins in M and M' agree with those in $\mathfrak{I}L$, so g is a frame homomorphism. To see g is proper, we note that the definition of a stably compact subframe implies that the way below relations in M and M' are the restrictions of the way below relation in $\mathfrak{I}L$. Finally, for $I \in M$ we have $(\bigvee\cdot) \circ g(I)$ is simply the join of I in L, which is equal to $(\bigvee\cdot)I$. This shows $\bigvee\cdot : M \to L$ is $\sqsubseteq$ related to $\bigvee\cdot : M' \to L$, hence the equivalence class of the first compactification is beneath that of the second in the partial ordering of COMP L, showing $\Pi(M) \le \Pi(M')$.

To show that Φ, Ψ, Π are isomorphisms and the further remarks describing their inverses, it is enough to show $\Pi\Psi\Phi$, $\Psi\Phi\Pi$, and $\Phi\Pi\Psi$ are the identity maps on COMP L, SUB $\mathfrak{I}L$, and PROX L, respectively.

To see $\Pi\Psi\Phi$ is the identity on COMP L, let $f : M \to L$ be a stable compactification. Then $\Pi\Psi\Phi([f]) = \Pi\Psi(\prec_f) = \Pi(\mathfrak{I}_{\prec_f}L)$, and this final item is the equivalence class of the compactification $\bigvee\cdot : \mathfrak{I}_{\prec_f}L \to L$. Proposition 4.15 shows $f : M \to L$ and $\bigvee\cdot : \mathfrak{I}_{\prec_f} \to L$ are equivalent, so $\Pi\Psi\Phi([f]) = [f]$.

To see $\Psi\Phi\Pi$ is the identity map on SUB $\mathfrak{I}L$, suppose M belongs to SUB $\mathfrak{I}L$. Proposition 4.16 shows M is equal to $\mathfrak{I}_{\prec_{\bigvee\cdot}}L$, hence $\Psi\Phi\Pi(M) = M$.

Finally, we show $\Phi\Pi\Psi$ is the identity on PROX L. Suppose $\prec$ is a proximity on L and let $\prec'$ be the proximity $\Phi\Pi\Psi(\prec)$. Suppose $a \prec b$. Then there is c with $a \prec c \prec b$. The ideals $\twoheaddownarrow a$ and $\twoheaddownarrow b$ are $\prec$-round and as $\prec$ is a proximity, we have $(\bigvee\cdot)\twoheaddownarrow a = a$ and $(\bigvee\cdot)\twoheaddownarrow b = b$. As $\twoheaddownarrow a \subseteq \downarrow c$ and $c \in \twoheaddownarrow b$, we have $\twoheaddownarrow a \ll \twoheaddownarrow b$, and it follows from Lemma 4.10 and the definition of $\prec' = \prec_{\bigvee\cdot}$ that $a \prec' b$. Conversely, suppose $a \prec' b$. Then the definition of $\prec_{\bigvee\cdot}$ gives $r(a) \ll r(b)$, where r is the right adjoint of $\bigvee\cdot : \mathfrak{I}_{\prec}L \to L$. Clearly the largest $\prec$-round ideal of L mapped by $\bigvee\cdot$ to a is $\twoheaddownarrow a$, so $r(a) = \twoheaddownarrow a$, and $r(b) = \twoheaddownarrow b$. So $\twoheaddownarrow a \ll \twoheaddownarrow b$. This means there is $c \in \twoheaddownarrow b$ with $\twoheaddownarrow a \subseteq \downarrow c$. As $\prec$ is a proximity, $a = \bigvee \twoheaddownarrow a$, so $a \le c$, and as $c \in \twoheaddownarrow b$, we have $c \prec b$, hence $a \prec b$. So $\prec = \prec'$, thus $\prec = \Phi\Pi\Psi(\prec)$. □

We conclude this section with a discussion of matters related to the poset of stable compactifications of a frame. We begin with a comparison to Smyth's poset COMP X of stable compactifications of a T_0-space described in Definition 3.2.

Proposition 4.21. *For a T_0-space X, the poset* COMP X *of stable compactifications of X is isomorphic to the poset* COMP $\Omega(X)$ *of stable compactifications of the frame $\Omega(X)$.*

Proof. Proposition 4.6, and the discussion before it, show that each equivalence class of stable compactifications of the frame $\Omega(X)$ contains an element of the form $\Omega(e) : \Omega(Y) \to \Omega(X)$ for some stable compactification $e : X \to Y$ of the space X. The result then follows as the proper frame homomorphisms from the frame $\Omega(Y)$ to the frame $\Omega(Z)$ of open sets of stably compact spaces Y and Z are exactly the $\Omega(f)$ where $f : Z \to Y$ is proper. □

Corollary 4.22. *For a T_0-space X with sobrification $s(X)$, the poset* COMP X *is isomorphic to the poset* COMP $s(X)$.

Proof. This follows from Proposition 4.21 as the frames $\Omega(X)$ and $\Omega(sX)$ are isomorphic. □

Remark 4.23. The poset of stable compactifications of L always has a largest element. In terms of the poset of proximities on L, this corresponds to the largest proximity, namely the partial ordering on L, and in terms of the dense stably compact subframes of the ideal frame, this corresponds to the largest such subframe, namely the ideal frame $\mathfrak{I}L$ itself. As we discuss in the next section, this largest stable compactification is coherent. We also point to Smyth's results on the largest stable compactification of a T_0-space and its connection to

Salbani's companion topology discussed in Proposition 3.14. We further note that as shown in Corollary 3.10, the poset of stable compactifications of L need not have a least element, even in the case when L is a spatial frame.

Remark 4.24. In [1] Banaschewski showed that for a completely regular frame L, there is an isomorphism between the poset of compactifications of L and the poset of strong inclusions on L. Strong inclusions are proximities $\prec$ on L that are contained in the well inside relation on L and satisfy $a \prec b$ implies $\neg b \prec \neg a$. He also showed that each compactification of L is equivalent to one of the form $\bigvee\cdot : M \to L$, where M is a compact regular subframe of the regular coreflection $\mathfrak{R}L$. It follows that the poset of compactifications of L is isomorphic to the poset of dense compact regular subframes of $\mathfrak{R}L$. In particular, $\mathfrak{R}L$ gives the largest element of the poset of compactifications of L. The above results form extensions of these to the setting of stable compactifications. Note, the largest stable compactification of L given by $\mathfrak{I}L$ need not be a compactification of L.

Remark 4.25. In [3] Banaschewski, Brümmer, and Hardie introduced biframes as a pointfree version of bitopological spaces, much as frames are a pointfree version of topological spaces. A *biframe* is a triple $M = (M_0, M_1, M_2)$, where M_1, M_2 are subframes of the frame M_0 and M_0 is generated by $M_1 \cup M_2$, and a *biframe homomorphism* $h : M \to L$ is a frame homomorphism $h : M_0 \to L_0$, where $h(M_i) \subseteq L_i$ for $i = 1, 2$.

The notions of compactness and regularity for biframes were introduced in [3], and in [2] Banaschewski and Brümmer constructed for any stably compact frame M_1, a compact regular biframe (M_0, M_1, M_2). Their technique involved representing M_1, and the stably compact frame M_2 of Scott open filters of M_1, in the congruence frame $\textsc{Con}(M_1)$ of M_1, and then constructing M_0 from the subframe of this congruence frame generated by the images of M_1 and M_2. It follows that the category of compact regular biframes is equivalent to the category of stably compact frames, hence dually equivalent to the category of stably compact spaces, and also to the category of Nachbin spaces.

In [13] Schauerte studied bicompactifications of biframes. She defined a bicompactification of a biframe L to be a pair (M, f), where M is a compact regular biframe and $f : M \to L$ is a dense onto biframe homomorphism. Here density is used in the usual sense with respect to M_0 and L_0, while onto means that the restrictions to M_i are onto L_i for $i = 1, 2$. Schauerte [13] generalized Banaschewski's theorem by proving that the poset of bicompactifications of a biframe is isomorphic to the poset of "strong inclusions" on L.

Our results on stable compactifucations and ordered spaces can be placed in the context of biframes. Suppose $f : M_1 \to L_1$ is a stable compactification of a frame L_1. As f is an onto frame homomorphism, there is a frame homomorphism $\overline{f} : \textsc{Con}(M_1) \to \textsc{Con}(L_1)$ taking a congruence θ on M_1 to the congruence on L associated with $\theta \vee \ker f$. For the compact regular biframe $M = (M_0, M_1, M_2)$ constructed in [2], the frames M_0, M_1, M_2 were realized inside the congruence frame $\textsc{Con}(M_1)$, and we obtain a biframe $L = (L_0, L_1, L_2)$ with L_i determined by the image of $\overline{f}(M_i)$ for $i = 0, 1, 2$. This gives a biframe compactification $\overline{f} : M \to L$. So every stable compactification naturally yields a biframe compactification. On the other hand, translating Example 3.13 into the language of biframes shows that not every bicompactification of a biframe arises this way. So the correspondence between stable compactifications of frames and bicompactifications of biframes is similar to that between stable compactifications of T_0-spaces and bicompactifications of bispaces discussed in Remark 3.15.

5. Coherent and spectral compactifications

Recall that a frame is coherent if its compact elements are a bounded sublattice, and each element is a join of compact elements. A space is spectral if it is the space of prime filters of a bounded distributive lattice. Every coherent frame is stably compact, and every spectral space is stably compact. Here we consider stable compactifications in the context of coherent frames and spectral spaces. This is closely related to Smyth's characterization [15, Prop. 20] of spectral compactifications of a T_0-space X in terms of lattice bases of the frame of open sets $\Omega(X)$, where we call a stable compactification (Y, e) of a T_0-space X a *spectral compactification* if Y is a spectral space.

Definition 5.1. *Let L be a frame and $f : M \to L$ be a stable compactification of L. We call f a coherent compactification of L if M is a coherent frame. Let* Coh L *be the sub-poset of* Comp L *whose equivalence classes consist of coherent compactifications of L. A proximity $\prec$ on L is called coherent if $a \prec b$ implies there is c with $c \prec c$ and $a \prec c \prec b$.*

Proposition 5.2. Coh L *is isomorphic to the sub-poset of* Prox L *consisting of coherent proximities on L, and to the sub-poset of* Sub $\mathfrak{I}L$ *consisting of dense stably compact subframes that are additionally coherent.*

Proof. Consider the isomorphism $\Phi : \text{Comp } L \to \text{Prox } L$ of Theorem 4.20 and suppose that $f : M \to L$ is a stable compactification of L. By Lemma 4.10, $a \prec_f b$ iff $f(x) = a$ and $f(y) = b$ for some $x \ll y$ in M. If M is coherent, the proximity $\ll$ on M is coherent, and it follows that $\prec_f$ is coherent as well. Next, consider the isomorphism $\Psi : \text{Prox } L \to \text{Sub } \mathfrak{I}L$ and suppose that $\prec$ is a coherent proximity. Then the frame $\mathfrak{I}_\prec L$ of $\prec$-round ideals of L is coherent. Indeed, if I, J are $\prec$-round ideals with $I \ll J$, then there is $a \in J$ with $I \subseteq \twoheaddownarrow a$. As J is round, there is $b \in J$ with $a \prec b$. Then as $\prec$ is coherent, there is $c \prec c$ with $a \prec c \prec b$. Therefore, $I \ll \twoheaddownarrow c \ll J$. Finally, consider the isomorphism $\Pi : \text{Sub } \mathfrak{I}L \to \text{Comp } L$. Clearly if M is a dense stably compact subframe of $\mathfrak{I}L$ that is coherent, then the stable compactification $\bigvee \cdot : M \to L$ is by definition coherent. ☐

In the coherent setting, there is an alternate path to a description of compactifications that is convenient. We call a bounded sublattice S of a frame L a *lattice basis* if S is join-dense in L. Let Lat L be the poset of lattice bases of L, where the ordering is set inclusion.

Proposition 5.3. Coh L *is isomorphic to* Lat L.

Proof. By Proposition 5.2, Coh L is isomorphic to the poset CSub $\mathfrak{I}L$ of dense stably compact subframes of $\mathfrak{I}L$ that are themselves coherent.

If M belongs to CSub $\mathfrak{I}L$, then as $\ll$ in M is the restriction of $\ll$ in $\mathfrak{I}L$, the compact elements of M are those principal ideals $\downarrow a$ belonging to M. As M is coherent, we have $S = \{a \in L : \downarrow a \in M\}$ is a sublattice of L, and as each element of M is the join of compact elements and the join map $\bigvee \cdot : M \to L$ is onto, S is a dense sublattice of L, hence a lattice basis. Setting $\Gamma(M) = \{a : \downarrow a \in M\}$ gives an order-preserving map from CSub $\mathfrak{I}L$ to Lat L.

If S is a lattice basis of L, set $\mathfrak{I}_S L$ to be the set of ideals of L generated by S, and note that this is the subframe of $\mathfrak{I}L$ generated by $\{\downarrow a : a \in S\}$. The compact elements of $\mathfrak{I}_S L$ are exactly the $\downarrow a$ where $a \in S$, and it follows that $\mathfrak{I}_S L$ is a coherent frame. If $I \ll J$ in $\mathfrak{I}_S L$, then $I \subseteq \downarrow a$ for some $a \in J$ with $a \in S$, hence $I \ll J$ in $\mathfrak{I}L$. So $\mathfrak{I}_S L$ is a stably compact subframe of $\mathfrak{I}L$ that is coherent, and it is dense as S is a dense sublattice of L. Setting $\Lambda(S) = \mathfrak{I}_S L$ then gives an order-preserving map from Lat L to CSub $\mathfrak{I}L$.

Our constructions show that $\Lambda\Gamma(M) = M$ for each $M \in \text{CSub}\,\mathfrak{I}L$ and $\Gamma\Lambda(S) = S$ for each $S \in \text{Lat}\ L$, so Γ and Λ establish an isomorphism of $\text{CSub}\,\mathfrak{I}L$ and $\text{Lat}\ L$. □

Remark 5.4. Smyth [15, Prop. 20] showed that the poset of spectral compactifications of X is isomorphic to the poset of lattice bases of $\Omega(X)$. The above result is an obvious extension of this to the setting of coherent compactifications of frames.

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