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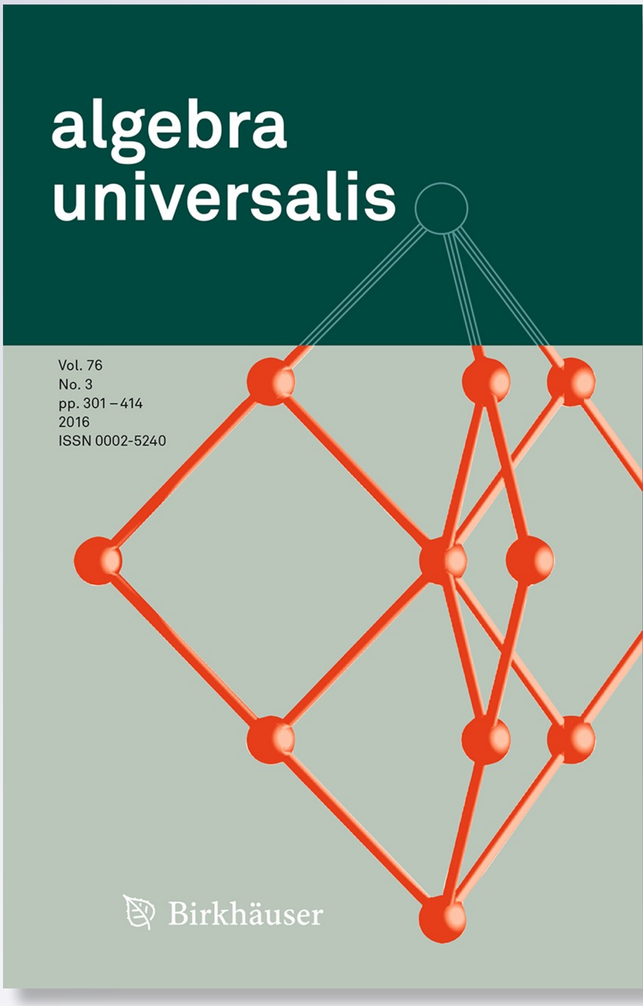

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Abstract. We prove that the topology of a compact Hausdorff topological Heyting algebra is a Stone topology. It then follows from known results that a Heyting algebra is profinite iff it admits a compact Hausdorff topology that makes it a compact Hausdorff topological Heyting algebra.

1. Introduction

A *topological algebra* is an algebra A together with a topology on A for which all of the basic operations of A are continuous. A *compact Hausdorff topological algebra* is a topological algebra where the topology is compact Hausdorff, and a *Stone topological algebra* is one where the topology is a Stone topology (a compact Hausdorff zero-dimensional topology). An algebra is *profinite* if it is the inverse limit of an inverse system of finite algebras. Since the inverse limit of an inverse family of finite discrete spaces is a Stone space, each profinite algebra is naturally a Stone topological algebra. In good cases, the converse is also true. For example, it is well known that a group is profinite iff it admits a topology making it a Stone topological group. It was proved in [4] that the same is also true for semigroups and distributive lattices. For further results in this direction, consult [3, Thm. VI.2.9].

For Boolean algebras, a stronger result is true. Namely, a Boolean algebra B is profinite iff it admits a topology making it a compact Hausdorff topological Boolean algebra, which happens iff B is isomorphic to a powerset algebra. Further, when these conditions occur, B admits exactly one compact Hausdorff topology making it a topological Boolean algebra. This topology is the interval topology, which in this case is a Stone topology with a subbasis of clopen sets given by the upsets of atoms and the downsets of coatoms. For proofs of these results, see [5] or [3, Prop. VII.1.16], and [2] for a different proof.

The results for Boolean algebras cannot be generalized to (bounded) distributive lattices (see, e.g., [3, Sec. VII.1.15]), but we show that they can be generalized to Heyting algebras. For an illustrative example, we consider the unit interval $[0, 1]$ with the interval topology. This is a compact Hausdorff distributive lattice whose topology is not Stone. The interval $[0, 1]$ is also a Heyting algebra, but the Heyting implication $\to$ is not continuous under

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the interval topology since the sequence $1/n$ converges to 0, but the sequence $1/n \to 0$ is constantly 0 and so does not converge to $0 \to 0 = 1$.

The goal of this note is to prove the following:

Theorem 1.1. *For a Heyting algebra H, the following are equivalent:*

(1) *H admits a topology making it a compact Hausdorff topological Heyting algebra.*
(2) *H admits a topology making it a Stone topological Heyting algebra.*
(3) *H is profinite.*
(4) *H is isomorphic to the lattice of all upsets of an image-finite poset X.*

Further, when these equivalent conditions hold, there is exactly one topology on H making H a compact Hausdorff topological Heyting algebra. This topology is the interval topology, which in this case is a Stone topology with a subbasis of clopen sets given by the upsets of completely join prime elements and the downsets of completely meet prime elements.

Proof. For (2)⇒(3), see [3, Prop. VI.2.10]; for (3)⇔(4), see [1, Thm. 3.6]; and (3)⇒(1) is obvious. We will prove (1)⇒(2) and the further remarks describing the topology in the next section. □

2. Main result

Let H be a compact Hausdorff Heyting algebra with implication $\to$ and negation $\neg$. Basic facts about topological lattices (see, e.g., [3, Sec. VII.1]) show that H is complete, hence $a \wedge \bigvee S = \bigvee\{a \wedge s : s \in S\}$, the *join infinite distributive law*, holds in H. Moreover, the closed ideals of H are exactly the principal downsets $\downarrow a$ and the closed filters are exactly the principal upsets $\uparrow a$ of H. An element $d \in H$ is *dense* provided $\neg d = 0$. Let D be the set of all dense elements of H. Then, as in any Heyting algebra, D is a filter. Also, since $D = \neg^{-1}(0)$ is the inverse image of a closed set under the continuous map $\neg$, it is closed. So there is a least dense element $d \in H$ such that $D = \uparrow d$.

For $p, q \in H$ with $p \leq q$, the interval $[p, q]$ of H is a Heyting algebra whose meet and join are the restrictions of those of H and whose implication is given by $x \Rightarrow y = (x \to y) \wedge q$. Since $[p, q] = \uparrow p \cap \downarrow q$ is a closed subset of H, it follows that $[p, q]$ is also a compact Hausdorff topological Heyting algebra, hence has a least dense element. Specializing this to intervals $\uparrow d$ allows the following definition.

Definition 2.1. *Define a transfinite sequence of elements $d_\alpha \in H$ as follows.*

(1) $d_0 = 0$.
(2) *If $\alpha = \beta + 1$ is a successor ordinal, then d_α is the least dense element in $\uparrow d_\beta$.*
(3) *If α is a limit ordinal, then $d_\alpha = \bigvee\{d_\beta : \beta < \alpha\}$.*

Since the bottom element of a Heyting algebra is dense iff the Heyting algebra is trivial, if $d_\alpha \neq 1$, then $d_\alpha < d_{\alpha+1}$. Since the sequence $\{d_\alpha\}$ must eventually stabilize, this immediately gives the following.

Lemma 2.2. *The sequence $\{d_\alpha\}$ is strictly increasing until it stabilizes at* 1.

An ordered pair (j, m) of elements in a Heyting algebra H is a *splitting pair* if $\uparrow j$ and $\downarrow m$ are disjoint and their union is H. It is well known that in a complete Heyting algebra H, for each completely join prime element j there is a completely meet prime element m with (j, m) a splitting pair, and that for each completely meet prime element m there is a completely join prime j with (j, m) a splitting pair. We let $J^\infty(H)$ be the set of completely join prime elements of H and $M^\infty(H)$ be the set of completely meet prime elements of H.

Lemma 2.3. *Suppose H is a compact Hausdorff Heyting algebra.*

(1) *If c is a coatom of H, then $c \in M^\infty(H)$.*
(2) *If $p \in H$ and $j \in J^\infty(\downarrow p)$, then $j \in J^\infty(H)$.*

Proof. (1): For any $x \in H$, we have $x \not\leq c$ iff $x \vee c = 1$. Let $F = \{x : x \vee c = 1\}$. Clearly, F is an upset and distributivity shows that it is closed under finite meets, hence is a filter. Since $F = (\cdot \vee c)^{-1}(1)$, it is closed, hence is equal to $\uparrow j$ for some $j \in H$. Then j is the least element of H that does not lie under c, showing that (j, c) is a splitting of H. Consequently, $\bigwedge S \leq c$ implies $j \not\leq \bigwedge S$, so $S \not\subseteq F$, hence $S \cap \downarrow c \neq \varnothing$. Thus, $c \in M^\infty(H)$.

(2): If $S \subseteq H$ and $j \leq \bigvee S$, then by the join infinite distributive law, $j \leq p \wedge \bigvee S = \bigvee\{p \wedge s : s \in S\}$. Since j is completely join prime in $\downarrow p$, there is $s \in S$ with $j \leq p \wedge s$. Thus, $j \leq s$, and so $j \in J^\infty(H)$. □

Proposition 2.4. *For $a, b \in H$ with $a \not\leq b$, there is $j \in J^\infty(H)$ such that $j \leq a$ and $j \not\leq b$.*

Proof. Since the sequence $\{d_\alpha\}$ stabilizes at 1, there is a least ordinal α with $a \wedge d_\alpha \not\leq b \wedge d_\alpha$. If α is a limit ordinal, then

$$a \wedge d_\alpha = a \wedge \bigvee\{d_\beta : \beta < \alpha\} = \bigvee\{a \wedge d_\beta : \beta < \alpha\}.$$

But $a \wedge d_\beta \leq b \wedge d_\alpha$ for each $\beta < \alpha$. Therefore, $a \wedge d_\alpha \leq b \wedge d_\alpha$, a contradiction. Thus, α is a successor ordinal. Let β be such that $\alpha = \beta + 1$.

Let $a' = a \wedge d_\alpha$ and $b' = b \wedge d_\alpha$. Then our definitions of α and β give $a' \not\leq b'$, and $a' \wedge d_\beta = a \wedge d_\beta \leq b \wedge d_\beta = b' \wedge d_\beta$. We claim that $a' \vee d_\beta \not\leq b' \vee d_\beta$. Indeed, if this were not the case, then

$$a' = a' \wedge (b' \vee d_\beta) = (a' \wedge b') \vee (a' \wedge d_\beta) \leq (a' \wedge b') \vee (b' \wedge d_\beta) = b' \wedge (a' \vee d_\beta).$$

But this would imply $a' \leq b'$, a contradiction.

The interval $[d_\beta, d_\alpha]$ is a compact Hausdorff Heyting algebra with its implication given by $x \Rightarrow y = (x \to y) \wedge d_\alpha$. Suppose x is dense in $[d_\beta, d_\alpha]$. Then $d_\beta = (x \Rightarrow d_\beta) = (x \to d_\beta) \wedge d_\alpha$. But d_α is the least dense element in $\uparrow d_\beta$, so d_β is the only element of $\uparrow d_\beta$ whose meet with d_α gives d_β. Therefore,

$d_\beta = (x \to d_\beta) \wedge d_\alpha$ implies $x \to d_\beta = d_\beta$. This gives that x is dense in $\uparrow d_\beta$, yielding $d_\alpha \leq x$. Thus, d_α is the only dense element of $[d_\beta, d_\alpha]$, and this implies that this interval is Boolean (see, e.g., [6, p. 132]). Since every compact Hausdorff Boolean algebra is a powerset algebra, this interval is a complete and atomic Boolean algebra.

The elements $a' \vee d_\beta$ and $b' \vee d_\beta$ belong to the interval $[d_\beta, d_\alpha]$. Since $a' \vee d_\beta \not\leq b' \vee d_\beta$, there is a coatom c in this interval with $a' \vee d_\beta \not\leq c$ and $b' \vee d_\beta \leq c$. It follows that $a' \not\leq c$ and $b' \leq c$. Since c is a coatom of $[d_\beta, d_\alpha]$, it is also a coatom of $\downarrow d_\alpha$. So by Lemma 2.3.1, $c \in M^\infty(\downarrow d_\alpha)$. Thus, there is $j \in J^\infty(\downarrow d_\alpha)$ with (j, c) a splitting pair of the Heyting algebra $\downarrow d_\alpha$. Then $a' \not\leq c$ and $b' \leq c$ imply $j \leq a'$ and $j \not\leq b'$. Lemma 2.3.2 gives $j \in J^\infty(H)$. Since $j \leq a'$ and $a' = a \wedge d_\alpha$, clearly $j \leq a$. As $j \leq d_\alpha$ and $j \not\leq b' = b \wedge d_\alpha$, we have $j \not\leq b$. □

We now prove our main result, completing the proof of Theorem 1.1.

Theorem 2.5. *Let H be a compact Hausdorff Heyting algebra. Then the topology of H has a subbasis of clopen sets given by $\uparrow j$, where $j \in J^\infty(H)$, and $\downarrow m$, where $m \in M^\infty(H)$.*

Proof. For each $j \in J^\infty(H)$, there is an $m \in M^\infty(H)$ with (j, m) a splitting pair of H, and conversely. If (j, m) are a splitting pair of H, then $\uparrow j$ and $\downarrow m$ are complementary closed sets of H, hence are clopen sets of H. The topology generated by these clopen sets is contained in the topology of H, hence is compact, and Proposition 2.4 shows that it is Hausdorff. Therefore, this topology is a compact Hausdorff topology that is contained in the given topology; hence they are equal. □

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