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Guram Bezhanishvili & John Harding

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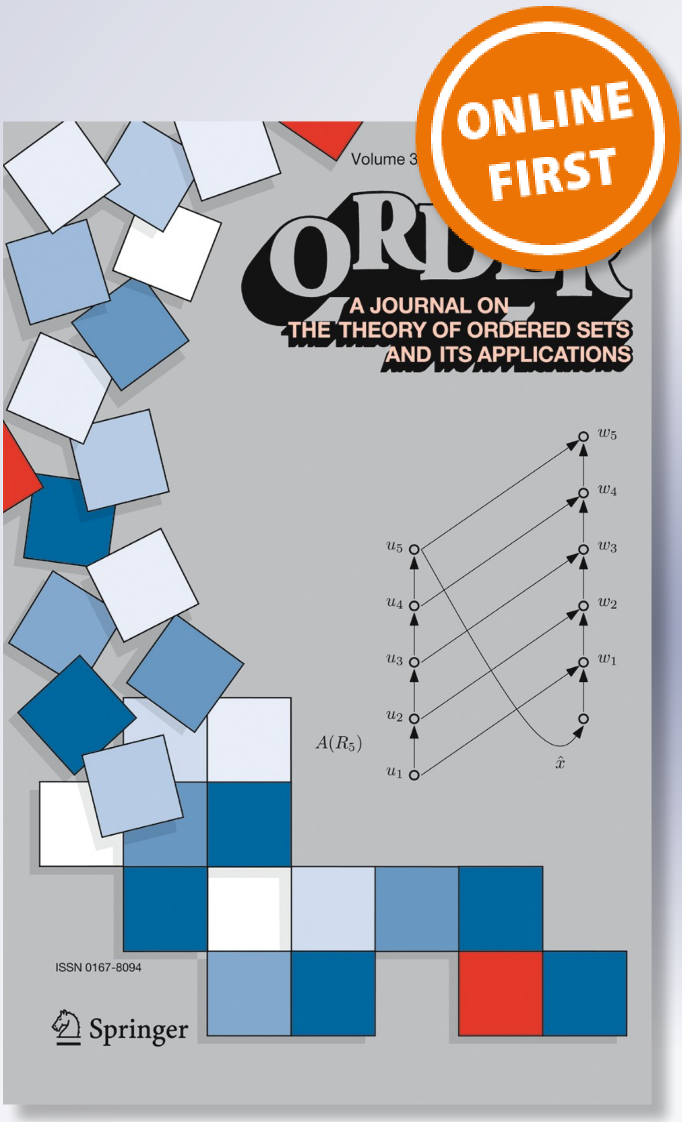

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On the Proof that Compact Hausdorff Boolean Algebras are Powersets

Guram Bezhanishvili[1] · **John Harding**[1]

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Abstract Papert Strauss (Proc. London Math. Soc. **18**(3), 217–230, 1968) used Pontryagin duality to prove that a compact Hausdorff topological Boolean algebra is a powerset algebra. We give a more elementary proof of this result that relies on a version of Bogolyubov's lemma.

Keywords Topological Boolean algebra · Pontryagin duality · Bogolyubov's lemma

1 Introduction

Papert Strauss [5] showed that every compact Hausdorff topological Boolean algebra B is isomorphic to the powerset $\mathcal{P}(X)$ of some set X. Her proof follows from Pontryagin duality, specifically from the existence of enough continuous characters to separate points. The point is that a compact Hausdorff Boolean algebra is a compact Hausdorff abelian group under the operation $x + y = (x \wedge y') \vee (x' \wedge y)$, and any character into the circle group takes only the values ± 1 since $x + x = 0$ in any Boolean algebra.

A direct proof of this result that does not rely on the considerable machinery involved in establishing Pontryagin duality turned out to be rather elusive. This was a hot topic of discussion at the international workshop TOLO 2008 (http://www.rmi.ge/tolo/), when Dito Pataraia got interested in the problem. In about a year Dito was able to design a proof of the theorem, which was presented by Mamuka Jibladze at the international conference BLAST 2009 (http://subsessile.nmsu.edu/blast/index.htm). Dito's proof was independent

To the memory of Dito Pataraia (1963–2011)

✉ Guram Bezhanishvili
guram@nmsu.edu

John Harding
jharding@nmsu.edu

1 Department of Mathematical Sciences, New Mexico State University, Las Cruces NM 88003, USA

of Pontryagin duality, but it was very involved. Since then Dito discussed at length several simplifications of his original technique, and was planning to write up his findings. Unfortunately, his untimely death did not allow the completion of the project.

It is our aim to provide a direct proof of Papert Strauss' result that does not rely on Pontryagin duality. Our main tool is a paper of Dikranjan and Stoyanov [3] giving a more elementary treatment of Pontryagin duality based on Prodanov's proof [6] of the Peter-Weyl theorem in the abelian case. Using basic results from Boolean algebras and topological lattices, much of the difficulty in the paper of Dikranjan and Stoyanov can be removed.

We obtain a short proof of the theorem of Papert Strauss that involves only concepts familiar in the study of topological lattices, with one exception. There is a key combinatorial lemma (a version of Bogolyubov's lemma [2, p. 6]) about finite Boolean algebras, and the only known proof of this lemma relies on the theory of group characters for finite abelian groups. It is a remarkable consequence of Prodanov's result that this combinatorial lemma is the key to understanding (infinite) compact Hausdorff topological Boolean algebras. We conjecture that this lemma can be considerably sharpened, and feel it likely has a direct combinatorial proof. In sharpened form, this lemma may be of independent interest in the study of finite posets.

2 Topological Boolean Algebras

As usual, by a *topological lattice* we mean a lattice L which is a topological space and for which the lattice operations $\wedge, \vee : L^2 \to L$ are continuous. For a subset S of L, we use $\downarrow S$ and $\uparrow S$ for the downset and upset generated by S. If S is a singleton set $\{x\}$, we simply write $\downarrow x$ and $\uparrow x$. It is well known (see, e.g., [1, Lemmas 2 and 3]) that if $S \subseteq L$ is open, then so are $\downarrow S$ and $\uparrow S$, and if L is Hausdorff and S is compact, then $\downarrow S$ and $\uparrow S$ are closed. In particular, if L is a compact Hausdorff topological lattice, then if S is closed, then so are $\downarrow S$ and $\uparrow S$. Thus, if S is open, then the largest upset $L \setminus \downarrow(L \setminus S)$ and the largest downset $L \setminus \uparrow(L \setminus S)$ contained in S are open. It is also well known (see, e.g., [4, Section VII.1.5]) that a compact Hausdorff topological lattice L is complete, and the join of every ideal I of L belongs to the closure of I.

A *topological Boolean algebra* is a Boolean algebra B which is a topological lattice and for which the complement operation $(-)' : B \to B$ is continuous. We use $+$ for the operation

$$x + y = (x \wedge y') \vee (x' \wedge y).$$

It is well known that B is an abelian group under $+$ with identity 0, and that $x + x = 0$ for all $x \in B$. For $S, T \subseteq B$, we use $S + T$ with the usual meaning, and we use $S_{(n)}$ for S added to itself n times $S + \cdots + S$. We will require the following basic facts.

Lemma 2.1 *Suppose B is a compact Hausdorff topological Boolean algebra, $x \in B$, and U is an open neighborhood of 0 in B.*

(1) *$x + U$ is an open neighborhood of x.*
(2) *The closure of U is contained in $U + U$.*
(3) *If U is a downset, then $U + U$ is the downset $U \vee U$.*
(4) *If U is an open downset, then there is an open downset neighborhood V of 0 with $V + V \subseteq U$.*

Proof (1) The function $h(y) = x + y$ is continuous and its own inverse, hence is a homeomorphism. (2) Suppose x is in the closure of U. By (1), $x + U$ is an open neighborhood of x, hence intersects U nontrivially. If y belongs to this intersection, then $y = x + u$ for some $u \in U$, giving $x = y + u \in U + U$. (3) Suppose $x, y \in U$. As U is a downset, $x \wedge y', x' \wedge y \in U$, so $x + y = (x \wedge y') \vee (x' \wedge y) \in U \vee U$. This yields $U + U \subseteq U \vee U$. Conversely, $x \vee y = x + (x' \wedge y)$ and $x' \wedge y \in U$ as U is a downset. Thus, $x \vee y \in U + U$, so $U \vee U \subseteq U + U$, hence the equality. (4) Since $\vee$ is continuous, $0 \vee 0 = 0 \in U$, and U is open, there is an open neighborhood W of 0 with $W \vee W \subseteq U$. Let $V = \downarrow W$. Then V is an open downset neighborhood of 0, and as U is a downset, by (3), $V + V = V \vee V \subseteq \downarrow(W \vee W) \subseteq U$. □

3 Boolean Bogolyubov Lemma

The key result we require is a version of the Bogolyubov Lemma for Boolean algebras.

Lemma 3.1 (Boolean Bogolyubov Lemma) *Let F be a finite Boolean algebra, S be a subset of F, and U be a downset of F such that $S + U = F$. Then there is an element of $U_{(4)}$ that is the meet of a set of at most $|S|^2$ coatoms of F.*

The proof of our version of Bogolyubov's Lemma for Boolean algebras is based on the proof of Bogolyubov's Lemma for finite abelian groups found in [2, pp. 6–7] and relies entirely on properties of characters of finite abelian groups. We note that Lemma 3.1 is not a direct reformulation of the Bogolyubov Lemma from [2, Lem. 1.2.3], and requires some small additional considerations.

We begin with a basic result about characters of a finite Boolean algebra. We recall that a *character* of a group G is a group homomorphism from G to the circle group. A character of a Boolean algebra takes only the values ± 1 because $x + x = 0$ in any Boolean algebra.

Lemma 3.2 *Suppose the Boolean algebra F is the powerset of a finite set X. Then for each $A \subseteq X$ the map $\chi_A : F \to \{-1, 1\}$ defined by $\chi_A(B) = (-1)^{|A \cap B|}$ is a character of F, and all characters of F arise this way.*

Proof For $B, C \subseteq X$, we have

$$\begin{aligned}\chi_A(B + C) &= (-1)^{|(B+C)\cap A|} = (-1)^{|(B-C)\cap A|}(-1)^{|(C-B)\cap A|}\\ &= (-1)^{|(B-C)\cap A|} \cdot (-1)^{|(B\cap C)\cap A|} \cdot (-1)^{|(B\cap C)\cap A|} \cdot (-1)^{|(C-B)\cap A|}\\ &= (-1)^{|B\cap A|} \cdot (-1)^{|C\cap A|} = \chi_A(B) \cdot \chi_A(C).\end{aligned}$$

Therefore, χ_A is a character of F. Moreover, it is easy to see that if $A \neq B$, then $\chi_A \neq \chi_B$. Since each finite group is isomorphic to its dual group of characters, F has as many characters as there are subsets of X. Thus, each character of F arises as χ_A for some $A \subseteq X$. □

We are ready to prove the Boolean Bogolyubov Lemma. Let F be the powerset of a finite set X, S be a subset of F, U be a downset of F, and $S + U = F$. We also let F' be the group of characters of F. Suppose f is the characteristic function of U and $B \subseteq X$. It follows from [2, Prop. 1.2.2] that

$$f(B) = \sum_{\chi \in F'} c_\chi \cdot \chi(B),$$

where

$$c_\chi = \frac{1}{|F|} \sum_{B \in F} f(B) \cdot \chi(B)$$

is the *Fourier coefficient* of f corresponding to the character χ. By Lemma 3.2, each character χ is of the form χ_A for some $A \subseteq X$. We use c_A for the Fourier coefficient of the character χ_A. In the notation of [2] this is c_{χ_A}. Since f is the characteristic function of U, Lemma 3.2 yields

$$c_A = \frac{1}{|F|} \sum_{B \in U} (-1)^{|A \cap B|}$$

Lemma 3.3 *If $a \in A$, then $|c_A| \leq c_{\{a\}}$.*

Proof For each $B \in U$ with $a \in B$ we have $B - \{a\} \in U$ since U is a downset. The contributions of B and $B - \{a\}$ to the sum for c_A negate one another, as do their contributions to the sum for $c_{\{a\}}$. So both c_A and $c_{\{a\}}$ are given by the sum over all $B \in U$ with $a \notin B$ and $B \cup \{a\}$ not belonging to U. The contribution of such an element to the sum for c_A may be 1 or -1, but its contribution to the sum for $c_{\{a\}}$ is always 1. □

Following the proof of the Bogolyubov Lemma [2, Lem. 1.2.3], we arrange the Fourier coefficients in some decreasing order $|c_{A_1}| \geq \cdots \geq |c_{A_k}| \geq \cdots$ so that in case of ties, $c_{\{a\}}$ comes before c_A for each $a \in A$. Consider the first $m = (|F|/|U|)^2$ entries in this sequence. Note that $S + U = F$ implies $|S| \cdot |U| \geq |F|$, so $m \leq |S|^2$. The Bogolyubov Lemma [2, Lem. 1.2.3] then yields that $\{C \in F : \chi_{A_i}(C) = 1 \text{ for all } i \leq m\} \subseteq U_{(4)}$.

Let $D = \{\{a\} : a \in A_i \text{ for some } i \leq m\}$. It follows from the construction of our sequence and Lemma 3.3 that each $\{a\}$ in D is equal to A_i for some $i \leq m$, so D has at most m elements. Let

$$B = \bigcap \{X - \{a\} : \{a\} \in D\}$$

be the meet of a set of at most m coatoms of F. Clearly $B \cap A_i = \varnothing$, so $\chi_{A_i}(B) = 1$ for every $i \leq m$. Therefore, $B \in \{C \in F : \chi_{A_i}(C) = 1 \text{ for all } i \leq m\} \subseteq U_{(4)}$. Thus, B is an element of $U_{(4)}$ that is the meet of a set of at most $|S|^2$ coatoms of F, completing the proof of the Boolean Bogolyubov Lemma.

We next conjecture a strengthened version of the Bogolyubov Lemma for Boolean algebras, and phrase it in the language of sets. In this formulation the key differences are greater control over the number and behavior of the coatoms, and we allow only two members from U rather than four.

Conjecture 3.4 (Strong Boolean Bogolyubov Lemma) *Suppose X is a finite set that is partitioned into n disjoint pieces $S_1, \ldots, S_n$ that are not necessarily of equal size. For any $A \subseteq X$ let $A_i = A \cap S_i$. There are 2^n sets that can be built from A by choosing for each $i \leq n$ either the set A_i or its complement in S_i, and then taking the union of these n pieces. Suppose $\mathcal{U}$ is a collection of subsets of X so that for any $A \subseteq X$, at least one of the 2^n sets* *built from A belongs to $\mathcal{U}$. Then there are two members of $\mathcal{U}$ whose union contains all but at most one element from each S_i.*

Remark 3.5 The case for $n = 1, 2$ and several special cases for general n have simple proofs. However, we do not know if this conjecture holds in general.

4 Proof of the Theorem

We are ready to give a more direct proof of the Papert Strauss theorem.

Theorem 4.1 (Papert Strauss) *If B is a compact Hausdorff topological Boolean algebra, then there is a set X with B isomorphic to the powerset $\mathcal{P}(X)$.*

Proof It is enough to show that B is atomic. Once this is established, let X be the atoms of B. Since B is also complete, it is well known that B is isomorphic to $\mathcal{P}(X)$.

To see B is atomic, let x be a non-zero element of B. Then $\uparrow x$ and $\{0\}$ are disjoint closed sets of B. Therefore, there are disjoint open sets C and D with $\uparrow x \subseteq C$ and $0 \in D$. Let $E = B \setminus \downarrow(B \setminus C)$ be the largest upset contained in C and $H = B \setminus \uparrow(B \setminus D)$ be the largest downset contained in D. Then E and H are disjoint open sets with $\uparrow x \subseteq E$ and $0 \in H$. By Lemma 2.1(4), there is an open downset neighborhood V of 0 with $V + V \subseteq H$. Applying Lemma 2.1(4) to V yields an open downset neighborhood U of 0 with $U + U \subseteq V$, hence $U_{(4)} \subseteq V + V \subseteq H$. Therefore, $U_{(4)}$ is disjoint from E, and hence $\uparrow x$ is disjoint from the closure of $U_{(4)}$.

For each $b \in B$, Lemma 2.1(1) shows $b + U$ is an open neighborhood of b. By compactness, there is a finite subcollection that covers B, say $b_1 + U, \dots, b_p + U$. Let S be the subalgebra of B generated by $b_1, \dots, b_p$. Since Boolean algebras are locally finite, S is finite. Let $\mathfrak{F}$ be the collection of all finite subalgebras of B that contain S. Clearly $\mathfrak{F}$ is a directed set.

Suppose $F \in \mathfrak{F}$. Since $b_1 + U, \dots, b_p + U$ cover B, they also cover F. Therefore, the Boolean Bogolyubov Lemma is applicable to F, S, and the restriction of U to F. Thus, there is an element of $U_{(4)}$ that is the meet of a family of at most $k = |\,S\,|^2$ coatoms of F. Consequently, as U is a downset, there is a family $Q_1^F, \dots, Q_k^F$ of maximal ideals of F whose intersection is contained in U.

Extend each Q_i^F to a maximal ideal P_i^F of B. Then for each $i \leq k$ the family $\{P_i^F : F \in \mathfrak{F}\}$ is a net in the Stone space Y of B. As Y is compact Hausdorff, there is a cofinal subfamily $\mathfrak{E}$ of $\mathfrak{F}$ with the subnets $\{P_i^F : F \in \mathfrak{E}\}$ converging, say to $P_1, \dots, P_k$. Suppose b belongs to the intersection of $P_1, \dots, P_k$. Then the clopen subset $\varphi(b) := \{x : b \in x\}$ of Y corresponding to b is a neighborhood of each P_i. Using the convergence of $\{P_i^F : F \in \mathfrak{E}\}$ to P_i, there is some common $F \in \mathfrak{E}$ with $P_1^F, \dots, P_k^F$ belonging to $\varphi(b)$, and this F can be chosen to contain b. This means b belongs to each of $P_1^F, \dots, P_k^F$, and as $b \in F$, we have b belongs to the restrictions $Q_1^F, \dots, Q_k^F$ of $P_1^F, \dots, P_k^F$ to F. But $Q_1^F, \dots, Q_k^F$ were chosen so that their intersection was contained in $U_{(4)}$. Thus, $b \in U_{(4)}$. So we have produced maximal ideals $P_1, \dots, P_k$ of B whose intersection I is contained in $U_{(4)}$.

The join $\bigvee I$ of the ideal I belongs to the closure of $U_{(4)}$. Therefore, since $\uparrow x$ is disjoint from the closure of $U_{(4)}$, we see that $x \not\leq \bigvee I$. The infinite distributive law for complete Boolean algebras gives $\bigvee I = (\bigvee P_1) \wedge \cdots \wedge (\bigvee P_k)$. So there is some $i \leq k$ with $x \not\leq \bigvee P_i$.

This implies $\bigvee P_i$ is a coatom of B that does not lie above x, so its complement is an atom of B beneath x. Thus, B is atomic. □

Remark 4.2 In the above proof, the argument using the maximal ideal theorem and nets in the Stone space Y to produce prime ideals $P_1, \ldots, P_k$ of B whose intersection is contained in $U_{(4)}$ can be replaced by a compactness argument from first order logic.

Consider a language for Boolean algebras enriched with constants c_a for each $a \in B$, and unary predicates U and $P_1, \ldots, P_k$. Consider the set Σ of sentences consisting of all $c_a \leq c_b$ for constants c_a, c_b with $a \leq b$ in B, sentences $U(c_a)$ for all $a \in U$, as well as sentences saying each P_i is a prime ideal and that the intersection of $P_1, \ldots, P_k$ is contained in $U_{(4)}$. Let Σ' be a finite subset of these sentences. Taking the finite subalgebra F of B generated by the constants appearing in Σ' and S produces a model of Σ' where the predicates P_i are the sets Q_i^F. So the compactness theorem says Σ has a model, and this gives prime ideals $P_1, \ldots, P_k$ of B whose intersection is contained in $U_{(4)}$.

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