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Equations in Type-2 Fuzzy Sets

John Harding*, Carol Walker and Elbert Walker
Department of Mathematical Sciences, New Mexico State University,
Las Cruces, New Mexico, USA
**hardy@nmsu.edu*

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The main concern of this paper is with the equations satisfied by the algebra of truth values of type-2 fuzzy sets. That algebra has elements all mappings from the unit interval into itself with operations given by certain convolutions of operations on the unit interval. There are a number of positive results. Among them is a decision procedure, similar to the method of truth tables, to determine when an equation holds in this algebra. One particular equation that holds in this algebra implies that every subalgebra of it that is a lattice is a distributive lattice. It is also shown that this algebra is locally finite. Many questions are left unanswered. For example, we do not know whether or not this algebra has a finite equational basis, that is, whether or not there is a finite set of equations from which all equations satisfied by this algebra follow. This and various other topics about the equations satisfied by this algebra will be discussed.

Keywords: Type-2 fuzzy set; truth value algebra; convolution; locally finite; equational basis; variety; semilattice.

1. Introduction

The algebra of truth values of type-2 fuzzy sets is rather complicated with many features, but we concentrate on the equations satisfied by this algebra. This is a review of some results that have appeared.

We begin with a discussion of equations satisfied by ordinary fuzzy sets and by interval-valued fuzzy sets. The situation in these cases is much simpler.

The following basic idea of a variety and its connection to equations will be used throughout.[1]

Definition 1. A **variety** is a family of algebras of a given type that is closed under homomorphic images of subalgebras of products.

Some fundamental facts about varieties are the following.

Theorem 1. *Let* EQ *be a family of equations. Then the family of algebras of a given type that satisfy these equations is a variety.*

Definition 2. Let $\mathcal{V}(\mathbb{A})$ be the **variety generated** by an algebra $\mathbb{A}$, that is, all algebras that are homomorphic images of subalgebras of products of $\mathbb{A}$.

Theorem 2. *The variety $\mathcal{V}(\mathbb{A})$ generated by $\mathbb{A}$ is the class of all algebras that satisfy the same equations as $\mathbb{A}$.*

2. Fuzzy Subsets

A **fuzzy subset of a set** S is a mapping $f : S \to [0,1]$. The set S has no operations on it. Operations are put on $[0,1]$, and for ordinary fuzzy sets, they are

$$\begin{aligned} x \vee y &= \max\{x, y\} \\ x \wedge y &= \min\{x, y\} \\ x' &= 1 - x \end{aligned}$$

and the constants 0 and 1. Thus we get the algebra

$$\mathbb{I} = ([0,1], \vee, \wedge, ', 0, 1)$$

This is the **truth value algebra** of ordinary fuzzy sets. The **type** of this algebra $\mathbb{I}$ is $(2, 2, 1, 0, 0)$. That is, it has two binary operations, one unary operation, and two nullary operations. Unless otherwise noted, we will always be dealing with algebras of this type.

The set $\mathrm{Map}(S, [0,1])$ of all fuzzy subsets of S gets corresponding operations on its elements pointwise from those of $\mathbb{I}$, namely

$$\begin{aligned} (f \vee g)(s) &= f(s) \vee g(s) \\ (f \wedge g)(s) &= f(s) \wedge g(s) \\ f'(s) &= (f(s))' \\ 1(s) &= 1 \\ 0(s) &= 0 \end{aligned}$$

Note that we use the same symbols, namely $\vee, \wedge, ', 0, 1$, for the pointwise operations on the elements of $\mathrm{Map}(S, [0,1])$ as for the operations on $[0,1]$. Thus we get the algebra $\mathbb{F}(S) = (\mathrm{Map}(S, [0,1]), \vee, \wedge, ', 0, 1)$, the **algebra of fuzzy subsets of the set** S.

One might notice at this point that if the interval $[0,1]$ were restricted to the two element set $\{0,1\}$, the operations put on $\{0,1\}$ would result in the usual two-element Boolean algebra, and $(\mathrm{Map}(S, \{0,1\}), \vee, \wedge, ', 0, 1)$ would simply correspond to the algebra of subsets of S with $\vee$ corresponding to the usual union of subsets, and so on.

A main point here is that the same equations hold in $\mathbb{F}(S)$ as in $\mathbb{I}$. The algebra $\mathbb{I}$ is a **Kleene algebra**, that is, a bounded distributive lattice with a negation $'$ that satisfies De Morgan's laws and the Kleene inequality $x \wedge x' \leq y \vee y'$. This inequality may be stated as the equation $(x \wedge x') \wedge (y \vee y') = x \wedge x'$. The defining equations for a Kleene algebra are

(1) $x \wedge x = x$; $x \vee x = x$ (idempotent)

(2) $x \wedge y = y \wedge x$; $x \vee y = y \vee x$ (commutative)

(3) $x \wedge (y \wedge z) = (x \wedge y) \wedge z$; $x \vee (y \vee z) = (x \vee y) \vee z$ (associative)

(4) $x \wedge (x \vee y) = x$; $x \vee (x \wedge y) = x$ (absorption laws)

(5) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$; $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ (distributive)

(6) $0 \vee x = x$; $1 \wedge x = x$ (identities)

(7) $x'' = x$ (involution)

(8) $(x \wedge y)' = x' \vee y'$; $(x \vee y)' = x' \wedge y'$ (De Morgan's laws)

(9) $x \wedge x' \leq y \vee y'$ (Kleene inequality)

Thus $(\textsc{Map}(S, [0,1]), \vee, \wedge, {}', 0, 1)$ is also a Kleene algebra. It is well known that every equation that is satisfied by $\mathbb{I}$ is a consequence of these.[2] That is, this set of equations is an **equational basis** for Kleene algebras.

It is also well known that the same equations hold in $\mathbb{I}$ as in the algebra in Fig. 1 with the obvious lattice operations and negation defined by $0' = 1$, $1' = 0$ and $a' = a$.[2]

Fig. 1. A 3-element Kleene algebra.

Thus, to see whether an equation holds in all Kleene algebras, it is enough to see whether it holds in the 3-element Kleene algebra in Fig. 1. This amounts to trying all possible combinations of $0, a, 1$ for the variables in the equation, and amounts to a 3-valued version of the usual **truth tables** of classical propositional logic.

3. Interval-Valued Fuzzy Sets

There is an extensive theory of classical fuzzy sets and there have been many varied applications. But an increasingly prevalent view is that models based on $[0,1]$ are inadequate. Many believe that assigning an exact number to an expert's opinion is too restrictive, and that the assignment of an interval of values is more realistic. This leads to the notion of **interval-valued fuzzy sets**. The interval $[0,1]$ is replaced by the set

$$[0,1]^{[2]} = \{(a,b) : a, b \in [0,1],\ a \leq b\}.$$

The element (a,b) is just the ordered pair with $a \leq b$. For interval-valued fuzzy sets, the algebra of truth values is $\mathbb{I}^{[2]} = ([0,1]^{[2]}, \vee, \wedge, {}', 0, 1)$ with

$$\begin{aligned} [0,1]^{[2]} &= \{(a,b) : a,b \in [0,1], a \le b\} \\ (a,b) \vee (c,d) &= (a \vee c, b \vee d) \\ (a,b) \wedge (c,d) &= (a \wedge c, b \wedge d) \\ (a,b)' &= (b', a') \\ 0 &= (0,0) \\ 1 &= (1,1) \end{aligned}$$

The algebra $\mathbb{I}^{[2]}$ satisfies the same equations as the algebra $\mathbb{I}$, except that it does not satisfy the Kleene inequality. That is, $\mathbb{I}^{[2]}$ is a **De Morgan algebra**, a bounded distributive lattice with a negation $'$ satisfying De Morgan's laws.[3]

The same equations hold in $\mathbb{I}^{[2]}$ as in the De Morgan algebra in Fig. 2 with negation $'$ satisfying $0' = 1$, $1' = 0$, $a' = a$ and $b' = b$.[2]

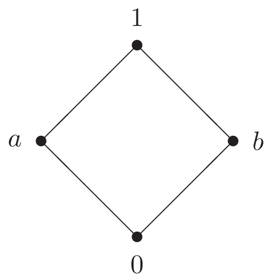


Fig. 2. A 4-element De Morgan algebra.

To see whether an equation holds in $\mathbb{I}^{[2]}$, it is sufficient to see whether it holds in the 4-element algebra of Fig. 2. This is accomplished by a 4-value version of the familiar method of truth tables.

An **interval-valued fuzzy subset** of a set S is a function

$$f : S \to [0,1]^{[2]}$$

The set $\text{MAP}(S, [0,1]^{[2]})$ of all interval-valued fuzzy subsets of S gets corresponding operations on its elements pointwise from those of $\mathbb{I}^{[2]}$. For example, let $f(x) = (f_1(x), f_2(x))$ and $g(x) = (g_1(x), g_2(x))$. Then

$$\begin{aligned} (f \wedge g)(x) &= (f_1(x) \wedge g_1(x), f_2(x) \wedge g_2(x)) \\ (f \vee g)(x) &= (f_1(x) \vee g_1(x), f_2(x) \vee g_2(x)) \\ f'(x) &= (1 - f_2(x), 1 - f_1(x)) \end{aligned}$$

$\mathbb{IF}(S) = (\text{MAP}(S, [0,1]^{[2]}), \vee, \wedge, {}', 0, 1)$ is the **algebra of interval-valued fuzzy subsets of the set** S. The same equations hold in $\mathbb{IF}(S)$ as in $\mathbb{I}^{[2]}$, and these are exactly the equations true in all De Morgan algebras.

4. The Truth Value Algebra of Type-2 Fuzzy Sets

Some find that models based on $[0,1]^{[2]}$ are inadequate, believing that assigning a pair of exact numbers to an expert's opinion is still too restrictive, and that the assignment of fuzzy subsets of $[0,1]$ is more useful. This leads to the notion of **type-2 fuzzy sets**. The set $[0,1]^{[2]}$ is replaced by the set

$$[0,1]^{[0,1]} = \{f : [0,1] \to [0,1]\}$$

Type-2 fuzzy sets were introduced by Zadeh in 1975,[4] extending the notions both of ordinary fuzzy sets and of interval-valued fuzzy sets. They seem destined to play an increasingly important role in applications.[5,6]

A type-2 fuzzy subset A of a set S is a function $A : S \to [0,1]^{[0,1]}$, where $A(x)$ is the **degree of membership**, or **truth value** of the element x in S. The operations on elements of $[0,1]^{[0,1]}$ are convolutions of the operations of $\mathbb{I}$ in the domain, using the complete join and binary meet on the range $[0,1]$.[7–9]

Definition 3. The **truth value algebra of type-2 fuzzy sets** is the algebra $\mathbb{M} = ([0,1]^{[0,1]}, \sqcup, \sqcap, {}', \bar{0}, \bar{1})$ with

$$\begin{aligned}(f \sqcup g)(x) &= \bigvee\{f(y) \wedge g(z) : y \vee z = x\}\\ (f \sqcap g)(x) &= \bigvee\{f(y) \wedge g(z) : y \wedge z = x\}\\ f'(x) &= f(1-x)\end{aligned}$$

and the constants $\bar{0}(x) = \begin{cases} 1 \text{ if } x = 0 \\ 0 \text{ if } x \neq 0 \end{cases}$ and $\bar{1}(x) = \begin{cases} 1 \text{ if } x = 1 \\ 0 \text{ if } x \neq 1 \end{cases}$

Note that these operations are all **convolutions** of a sort, in the sense used, for example, in ordinary multiplication of polynomials. This is clear in the definitions of $\sqcup$ and $\sqcap$, which are convolutions of the operations $\vee$ and $\wedge$ of $\mathbb{I}$, respectively. This can also be seen for the negation by noting $f'(x)$ is the convolution $\bigvee\{f(y) : 1-y = x\}$ of the negation $1-y$ of $\mathbb{I}$. In fact, $\bar{0}$ and $\bar{1}$ are convolutions of the constants 0 and 1 of $\mathbb{I}$ as well.

First we remark that $\mathbb{M}$ is indeed a generalization of $\mathbb{I}$ and $\mathbb{I}^{[2]}$.

Theorem 3. *Functions in $[0,1]^{[0,1]}$ that are 1 at one point and 0 elsewhere are called **singletons**. They form a subalgebra $\mathbb{S}$ isomorphic to $\mathbb{I} = ([0,1], \vee, \wedge, {}', 0, 1)$.*

Singletons are characteristic functions of 1-element subsets of $[0,1]$, where the **characteristic function** of a subset U of $[0,1]$ is the function taking value 1 when $x \in U$ and 0 otherwise.

Theorem 4. *The set of characteristic functions of closed intervals $[a,b]$ of the reals, with $a \leq b$, forms a subalgebra $\mathbb{D}$ of $\mathbb{M}$ isomorphic to $\mathbb{I}^{[2]} = ([0,1]^{[2]}, \vee, \wedge, {}', 0, 1)$.*

A first step in making the algebra $\mathbb{M}$ more tractable is to realize the operations $\sqcup$ and $\sqcap$ in a more computationally convenient way.

Definition 4. For $f : [0,1] \to [0,1]$ define

$$f^L(x) = \bigvee \{f(y) : y \leq x\}$$
$$f^R(x) = \bigvee \{f(y) : x \leq y\}$$

Note that f^L is the least increasing function that is pointwise greater than f, and that f^R is the least decreasing function that is pointwise greater than f.

Using the operations L and R, the operations $\sqcap$ and $\sqcup$ may be expressed as pointwise operations, as follows.

Theorem 5. *For $f, g \in [0,1]^{[0,1]}$,*

$$f \sqcap g = (f \vee g) \wedge (f^R \wedge g^R) = (f \wedge g^R) \vee (f^R \wedge g)$$
$$f \sqcup g = (f \vee g) \wedge (f^L \wedge g^L) = (f \wedge g^L) \vee (f^L \wedge g)$$

Using this theorem, it is fairly straightforward to verify the following.[9]

Corollary 1. *Let $f, \ g, \ h \in [0,1]^{[0,1]}$.*

(1) $f \sqcup f = f$; $f \sqcap f = f$
(2) $f \sqcup g = g \sqcup f$; $f \sqcap g = g \sqcap f$
(3) $f \sqcup (g \sqcup h) = (f \sqcup g) \sqcup h$; $f \sqcap (g \sqcap h) = (f \sqcap g) \sqcap h$
(4) $f \sqcup (f \sqcap g) = f \sqcap (f \sqcup g)$
(5) $\bar{0} \sqcup f = f, \bar{1} \sqcap f = f$
(6) $f'' = f$
(7) $(f \sqcup g)' = f' \sqcap g'$; $(f \sqcap g)' = f' \sqcup g'$

Several comments about these equations are in order here. Notice that neither of $\sqcup$ and $\sqcap$ distributes over the other, and there are easy examples that illustrate this. In item 4, if $f \sqcup (f \sqcap g) = f \sqcap (f \sqcup g)$ were f, which in general it is not, then the absorption law would hold and we would have a (nondistributive) lattice. Again there are examples illustrating this.

Since each of the operations $\sqcup$ and $\sqcap$ is idempotent, commutative and associative, they each induce partial orders as given by the following definition.

Definition 5. $f \sqsubseteq_{\sqcup} g$ if $f \sqcup g = g$, and $f \sqsubseteq_{\sqcap} g$ if $f \sqcap g = f$.

We often call $\sqsubseteq_{\sqcup}$ the **join order** and $\sqsubseteq_{\sqcap}$ the **meet order**. It is easy to see that the operations $\sqcup$ and $\sqcap$ do not give the same partial orders.[9]

Theorem 6. *Each partial order $\sqsubseteq_{\sqcup}$ and $\sqsubseteq_{\sqcap}$ induces a **semilattice**. That is, $f \sqcup g$ is the supremum of the two elements f and g under the partial order $\sqsubseteq_{\sqcup}$, and $f \sqcap g$ is the infimum of f and g under the partial order $\sqsubseteq_{\sqcap}$.*

An algebra with two binary operations, with each a semilattice operation, is a **bisemilattice**. A bisemilattice that satisfies item 4 above is aa **Birkhoff system**. Item 4 is known as **Birkhoff's equation**. See[11] for further details.

5. The Variety Generated by $\mathbb{M}$

The main object of this paper is to discuss the equations satisfied by the algebra $\mathbb{M}$. It has been shown that there is a finite algebra that generates the same variety as $\mathbb{M}$, providing a method of generalized truth tables to decide if an equation holds in $\mathbb{M}$. See[10] for details.

Definition 6. Let E be the set of all elements of M taking values in the 2-element set $\{0, 1\}$. In other words, E is the set of all characteristic functions of subsets of $[0, 1]$.

Theorem 7. $\mathbb{E} = (E, \sqcap, \sqcup, ', \bar{0}, \bar{1})$ *is a subalgebra of* $\mathbb{M}$*, and the algebras* $\mathbb{M}$ *and* $\mathbb{E}$ *generate the same variety.*

There is an alternate way to view $\mathbb{E}$ that is of interest. The **complex algebra** of an algebra $\mathbb{A}$ is an algebra of the same type as $\mathbb{A}$ whose underlying set is the collection of all subsets of the underlying set of $\mathbb{A}$. Each n-ary operation f of $\mathbb{A}$ gives an n-ary operation of the complex algebra taking n subsets of the underlying set of $\mathbb{A}$ to the set of their images under f. A familiar example is the complex algebra of a group $\mathbb{G}$ where multiplication of subsets S and T of the underlying set of $\mathbb{G}$ is the set $S \cdot T$ of multiples of their elements.

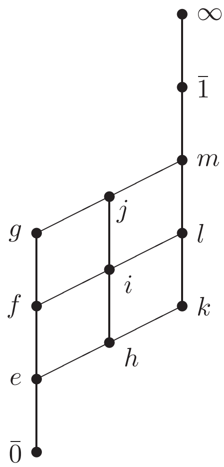
Theorem 8. *The algebra* $\mathbb{E}$ *is isomorphic to the complex algebra of the algebra* $\mathbb{I} = ([0, 1], \wedge, \vee, ', 0, 1)$.

The algebra $\mathbb{E}$ is more tractable than $\mathbb{M}$, but it is still an infinite algebra and has a complicated structure. Let C be the 5-element chain consisting of multiples of 1/5 in the unit interval [0,1]. This produces a subalgebra $\mathbb{C} = (C, \wedge, \vee, ', 0, 1\}$ of $\mathbb{I}$. There are an abundance of homomorphisms from $\mathbb{I}$ into $\mathbb{C}$, and these can be lifted to homomorphisms from the complex algebra of $\mathbb{I}$ to the complex algebra of $\mathbb{C}$. This provides the following.

Theorem 9. *The complex algebra of* $\mathbb{I}$ *and the complex algebra of* $\mathbb{C}$ *generate the same variety. Thus, the complex algebra of* $\mathbb{C}$ *generates the same variety as* $\mathbb{E}$*, and most vitally, as* $\mathbb{M}$.

The complex algebra of the 5-element algebra $\mathbb{C}$ has $2^5 = 32$ elements. So it can be determined whether an equation holds in $\mathbb{M}$ by checking whether it holds in this 32-element algebra by a method of generalized truth tables. Techniques from universal algebra show that the variety generated by the complex algebra of $\mathbb{C}$ is generated by the 12-element algebra shown in Fig. 3. In this figure, only the operations $\sqcup, *, \bar{0}$ and $\bar{1}$ are described. The operation $\sqcup$ is described by giving the poset of the order $\sqsubseteq_{\sqcup}$. The negation $*$ and constants $\bar{0}$ and $\bar{1}$ are obvious. The operation $\sqcap$ is determined by De Morgan's equation $x \sqcap y = (x' \sqcup y')'$. Clearly, the method of truth tables can be applied using this smaller algebra.

Theorem 10. *The variety $\mathcal{V}(\mathbb{M})$ is generated by the twelve-element algebra shown in Fig.* 3.



$$\infty^* = \infty$$
$$\bar{1}^* = \bar{0}$$
$$m^* = h$$
$$l^* = e$$
$$k^* = k$$
$$j^* = j$$
$$i^* = g$$
$$h^* = m$$
$$g^* = i$$
$$f^* = f$$
$$e^* = l$$
$$\bar{0}^* = \bar{1}$$

Fig. 3. The 12-element algebra.

Definition 7. An algebra is **locally finite** if every subalgebra generated by a finite subset is finite. A variety is locally finite if every member of it is locally finite.

It is known that every variety generated by a finite algebra is locally finite.[1]

Corollary 2. *The variety $\mathcal{V}(\mathbb{M})$ generated by $\mathbb{M}$ is locally finite.*

6. Varieties Generated by Reducts of $\mathbb{M}$

A **reduct** of an algebra $\mathbb{A}$ is an algebra obtained from $\mathbb{A}$ by retaining the same underlying set as $\mathbb{A}$ and some of the operations of $\mathbb{A}$, but removing from consideration other operations of $\mathbb{A}$. For example, each ring has a reduct to an abelian group obtained by removing the ring multiplication and constant 1. We consider varieties generated by reducts of $\mathbb{M}$ obtained by removing the negation, and then by removing the negation and constants. See[10] for details.

Theorem 11. $\mathcal{V}(([0,1]^{[0,1]}, \sqcap, \sqcup, \bar{0}, \bar{1}))$ *is generated by the 5-element algebra shown in Fig.* 4.

In Fig. 4, the algebra depicted has binary operations $\sqcap$, $\sqcup$, and constants $\bar{0}$ and $\bar{1}$. The operation $\sqcap$ is shown by describing the poset given by the order $\sqsubseteq_\sqcap$ at the left of the figure, and $\sqcup$ is shown by describing the poset given by the order $\sqsubseteq_\sqcup$ at right. The constants are indicated directly.

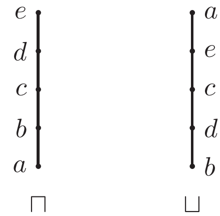


Fig. 4. A five-element "chain".

Theorem 12. *The variety* $\mathcal{V}(([0,1]^{[0,1]}, \sqcap, \sqcup))$ *is generated by the 4-element algebra shown in Fig.* 5*.*

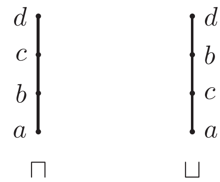


Fig. 5. A four-element "chain".

As pointed out earlier, we have the following corollary.

Corollary 3. *The varieties*

$$\mathcal{V}([0,1]^{[0,1]}, \sqcup, \sqcap, \bar{1}, \bar{0})$$
$$\mathcal{V}([0,1]^{[0,1]}, \sqcup, \sqcap)$$

are locally finite.

An equation involving only the operations $\sqcap, \sqcup$ and constants $\bar{0}, \bar{1}$ will hold in $\mathbb{M}$ if, and only if, it holds in the reduct $([0,1]^{[0,1]}, \sqcup, \sqcap, \bar{1}, \bar{0})$. So to determine if such an equation holds in $\mathbb{M}$ the method of truth tables may be applied to the 5-element algebra of Fig. 4. Similarly, to determine if an equation that uses only the operations $\sqcap, \sqcup$ holds in $\mathbb{M}$, it is sufficient to test it in the 4-element algebra of Fig. 5. This provides an easy route to the following.

Proposition 1. *The following equation, called the* ***generalized distributive law****, holds in* $\mathbb{M}$*.*

$$(x \sqcap (y \sqcup z) \sqcap ((x \sqcap y) \sqcup (x \sqcap z)) = (x \sqcap (y \sqcup z)) \sqcup ((x \sqcap y) \sqcup (x \sqcap z))\,.$$

This equation is $p \sqcap q = p \sqcup q$ where $p = x \sqcap (y \sqcup z)$ and $q = (x \sqcap y) \sqcup (x \sqcap z)$. Note that $p = q$ is the usual distributive law, and this is the origin of the name the generalized distributive law. In any lattice, $p \wedge q = p \vee q$ if, and only if, $p = q$. Thus, the generalized distributive law holds in a lattice if, and only if, the lattice is distributive. This gives the following.

Proposition 2. *Any subalgebra of* $\mathbb{M}$ *that is a lattice under the operations* $\sqcap$ *and* $\sqcup$ *is a distributive lattice under these operations.*

7. Equational Bases

It is natural to consider whether the equations (1)–(7) above are sufficient to imply all equations that hold in $\mathbb{M}$. In this section we will see this is not the case. That is, these equations are not an equational basis for $([0,1]^{[0,1]}, \sqcup, \sqcap, {}', \bar{0}, \bar{1})$.

Definition 8. An **equational base** for a variety $\mathcal{V}$ is a set EQ of equations that hold in an an algebra $\mathbb{A}$ if, and only if, the algebra $\mathbb{A}$ belongs to $\mathcal{V}$.

An **ortholattice** is a bounded lattice with a unary operation $'$ that order inverting, period two, and is a complementation, meaning that $x \wedge x' = 0$ and $x \vee x' = 1$.[1] The following is immediate.

Theorem 13. *The variety of all algebras satisfying the equations* (1)–(7) *above contains all ortholattices.*

Since there are ortholattices that are not locally finite, and the variety $\mathcal{V}(\mathbb{M})$ is locally finite, there must be ortholattices that do not belong to $\mathcal{V}(\mathbb{M})$. So there are equations that hold in $\mathbb{M}$ that are not valid in all ortholattices. This gives the following.

Corollary 4. $\mathbb{M}$ *satisfies an equation not a consequence of the equations in* (1)–(7) *above. Thus this set of equations is not an equational base for the variety generated by* $\mathbb{M}$.

There are two immediate problems suggested. One is to exhibit a specific equation not a consequence of equations (1)–(7) and which holds in $\mathbb{M}$. The other is to extend the equations (1)–(7) to a family that does define the variety $\mathcal{V}(\mathbb{M})$. Concerning this, there is no a priori reason that there should exist a finite set of equations that defines this variety. In any case, we have not been able to find equational bases for the variety generated by $\mathbb{M}$ or for the various reducts discussed in Section 6. However we are able to produce a specific equation valid in $\mathbb{M}$ and not a consequence of these seven. That we do now.

Proposition 3. *The generalized distributive law holds in* $\mathbb{M}$, *but is not a consequence of equations* (1)–(7).

Here we have already noted that the generalized distributive law holds in $\mathbb{M}$, and that the generalized distributive law holds in a lattice if, and only if, the lattice is distributive. It is well known that there are ortholattices whose lattice reducts are not distributive,[1] so there are ortholattices that do not satisfy the generalized distributive law.

8. Concluding Remarks

Although we have listed here a number of properties about the equations satisfied by the algebra $\mathbb{M}$ of truth values of type-2 fuzzy sets, there remain a number of questions unanswered. We do not know whether or not a finite set of equations exist defining the variety $\mathcal{V}(\mathbb{M})$. Since $\mathcal{V}(\mathbb{M})$ is generated by a finite algebra, if its variety were congruence distributive, then a finite basis exists.[1] But we have no reason to believe this variety is congruence distributive. We conjecture that there is not a finite basis.

Concerning the two reducts of $\mathbb{M}$, we know that the variety $\mathcal{V}(([0,1]^{[0,1]}, \sqcap, \sqcup))$ is not congruence distributive. This follows as this variety is generated by the five-element "chain" of Fig. 4 when the constants are removed from this algebra, and is also generated by the four-element "chain" of Fig. 5. Since this five-element chain is subdirectly irreducible, the variety $\mathcal{V}(([0,1]^{[0,1]}, \sqcap, \sqcup))$ cannot be congruence distributive by Jónsson's Lemma.[1] It remains a possibility that with more operations, the variety $\mathcal{V}(\mathbb{M})$ is congruence distributive, but this seems unlikely.

It would be of interest to know whether the variety generated by $\mathbb{M}$ is generated by an algebra with fewer elements then the 12-element one depicted.

Finally, as a matter of basic curiosity, it would be of interest to explore various techniques used here in a more general setting. The very first step in defining the algebra $\mathbb{M}$ is as a "convolution" algebra of the unit interval with standard negation. Surely convolutions of other algebras can be considered, and may have some general theory. Another direction is provided by our study of the algebra $\mathbb{E}$. This algebra is naturally realized as the complex algebra of the unit interval with negation. This complex algebra can in turn be considered as a type of modal algebra on the Boolean algebra of all subsets of the real unit interval, with modal operators given by taking the upset and downset generated by a set. There may be a deeper connection here that is worth study.

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