THE LOGIC OF BUNDLES

JOHN HARDING AND TAEWON YANG

ABSTRACT. Since the work of Crown [5] in the 1970's, it has been known that the projections of a finite-dimensional vector bundle E form an orthomodular poset (OMP) $\mathcal{P}(E)$. This result lies in the intersection of a number of current topics, including the categorical quantum mechanics of Abramsky and Coecke [1], and the approach via decompositions of Harding [6]. Moreover, it provides a source of OMPs for the quantum logic program close to the Hilbert space setting, and admitting a version of tensor products, yet having important differences from the standard logics of Hilbert spaces.

It is our purpose here to initiate a basic investigation of the quantum logic program in the vector bundle setting. This includes observations on the structure of the OMPs obtained as $\mathcal{P}(E)$ for a vector bundle E, methods to obtain states on these OMPs, and automorphisms of these OMPs. Key theorems of quantum logic in the Hilbert setting, such as Gleason's theorem and Wigner's theorem, provide natural and quite challenging problems in the vector bundle setting.

1. INTRODUCTION

The quantum logic approach to quantum mechanics began with the work of Birkhoff and von Neumann [4] who argued that the projection operators of a Hilbert space should form the questions of a quantum system, and that these questions form a type of structure now known as an orthomodular lattice (abbrev. OML). Further impetus to the quantum logic program came with Mackey's argument [12] from basic physical principles that the questions of a quantum system should form a kind of structure known as an orthomodular poset (abbrev. OMP).

In the time period of Mackey's result, there was considerable interest in seeing if objects close to Hilbert spaces produced orthomodular structures of use in quantum mechanics (see [8] for a survey). Amemiya and Araki [3] showed that among the pre-Hilbert spaces, orthomodularity of the closed subspaces characterized Hilbert spaces. Perhaps motivated in part by this result, a substantial part of the quantum logic program turned from looking at examples near the Hilbert space setting to increasingly exotic examples. Difficulties with the existence of a tensor product for OMPs lead, in part, to consideration of wider classes such as orthoalgebras and effect algebras, where, under assumptions on state spaces, tensor products are obtained.

It is the purpose of this note to begin a more in-depth study of a class of OMPs lying close to the Hilbert space setting, the OMPs of projections of a finite-dimensional Hermitian vector bundle. This class of OMPs enjoys many properties of the finite-dimensional Hilbert space setting, including full state spaces, closure under a version of tensor product, and the existence of an intrinsic topology. Yet important differences exist, such as the structures being OMPs and not usually lattices. Moreover, these

OMPs obtained from vector bundles lie as interesting examples in the intersection of many current research themes. These include the categorical quantum mechanics of Abramsky and Coecke [1], Harding's orthomodular posets of decompositions [6], and Wilce's topological orthoalgebras [17].

This paper is organized in the following way. The second section provides some background information on vector bundles, quantum logic, topological OMPs, OMPs of decompositions, and categorical quantum mechanics. The third section provides a basic investigation of the quantum logic and categorical quantum mechanics programs in the setting of vector bundles. A more in-depth study of these OMPs of decomposition would entail proving quite difficult theorems, such as versions of Gleason's theorem and Wigner's theorem in the vector bundle setting.

2. Preliminaries

Definition 2.1. *An orthomodular poset (OMP) is a bounded poset P with bounds $0, 1$ and a unary operation $\perp$ such that*

(1) $\perp$ *is order inverting and period two;*
(2) $x \wedge x^\perp$ *exists and is* 0*, and* $x \vee x^\perp$ *exists and is* 1*;*
(3) *if* $x \leq y^\perp$*, then* $x \vee y$ *exists, and is written* $x \oplus y$*;*
(4) *if* $x \leq y^\perp$*, then* $x \oplus (x \oplus y)^\perp = y^\perp$*.*

An orthomodular poset that is a lattice is an orthomodular lattice (OML).

Examples of OMPs include Boolean algebras, the projection operators of a Hilbert space, and small pathological examples constructed using Greechie diagrams [9]. There are also general techniques to construct OMPs such as from the idempotents $Id(R)$ of a ring R with unit, from the complemented pairs $L^{(2)}$ of elements of a bounded modular lattice L, and from the direct product decompositions Fact X of any algebra, topological space, or relational structure X. See [6] for details.

Definition 2.2. *A state on an OMP P is a map $\mu : P \to [0, 1]$ such that*

(1) $\mu(0) = 0$ *and* $\mu(1) = 1$*;*
(2) $\mu(x^\perp) = 1 - \mu(x)$*;*
(3) $x \leq y^\perp \Rightarrow \mu(x \oplus y) = \mu(x) + \mu(y)$*.*

A σ-additive state is one that is countably additive rather than finitely additive in (3).

This definition of a state plays a key role in the passage from the Hilbert space treatment of quantum mechanics to the quantum logic treatment. In the Hilbert space treatment, one associates to a quantum system a Hilbert space $\mathcal{H}$. Observables of the system, such as position and momentum, are self-adjoint operators A of $\mathcal{H}$. Pure states of the system are given by unit vectors in $\mathcal{H}$, and time evolutions are given by families of unitary operators on $\mathcal{H}$.

The quantum logic approach is built around several key theorems. The spectral theorem says that self-adjoint operators A correspond to σ-additive homomorphisms m_A from the Boolean algebra $\mathcal{B}(\mathbb{R})$ of Borel sets of the reals into the OML $\mathcal{P}(\mathcal{H})$ of projection operators of $\mathcal{H}$. Gleason's theorem says that the states of $\mathcal{H}$ correspond to

σ-additive states μ on the OML $\mathcal{P}(\mathcal{H})$. Born's correspondence rule says the probability of obtaining an outcome of the observable A in the Borel set B when the system is in state μ is given by $\mu(m_A(B))$.

$$\mathcal{B}(\mathbb{R}) \xrightarrow{\quad m_A \quad} \mathcal{P}(\mathcal{H}) \xrightarrow{\quad \mu \quad} [0,1].$$

Uhlhorn's version of Wigner's theorem [16] says that unitary operators on a real Hilbert space $\mathcal{H}$ correspond to automorphisms of the OML $\mathcal{P}(\mathcal{H})$, the complex case requiring also anti-unitary operators. This makes a bridge between the treatment of time evolutions in the Hilbert space approach, and that in the quantum logic approach.

We next consider the matter of compound systems. The following is a set of fairly minimal, physically motivated conditions, that would be required for a tensor product of OMPs. See [7, p. 790] for further discussion. In this definition we use $x \oplus y$ both to indicate that x and y are orthogonal and to represent their join.

Definition 2.3. *For* OMP*s A, B and C, we say $f : A \times B \to C$ is a bilinear map if for each $a, a_1, a_2 \in A$, and $b, b_1, b_2 \in B$*

(1) $f(a_1 \oplus a_2, b) = f(a_1, b) \oplus f(a_2, b)$;
(2) $f(a, b_1 \oplus b_2) = f(a, b_1) \oplus f(a, b_2)$;
(3) $f(1, 1) = 1$.

Definition 2.4. *A bilinear mapping $f : A \times B \to C$ is an* OMP *tensor product if for all states μ_A, μ_B on A, B, there is a state ω on C with $\omega(f(a, b)) = \mu_A(a)\mu_B(b)$.*

Compound systems in the Hilbert space approach are treated by taking the tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$ as the space for the compound system. The projection lattice $\mathcal{P}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ is then an OMP tensor product of $\mathcal{P}(\mathcal{H}_1)$ and $\mathcal{P}(\mathcal{H}_2)$. However, no such tensor product exists for general OMPs, a true problem for the quantum logic approach in the broad context. We turn next to the categorical quantum mechanics approach of Abramsky and Coecke [1] built specifically to treat such compound systems.

Definition 2.5. *A symmetric monoidal category is a category $\mathcal{C}$ equipped with a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and tensor unit I that satisfy certain coherence conditions (see* [11, p.162]*). A dagger symmetric monoidal category is also equipped with a contravariant functor $\dagger : \mathcal{C} \to \mathcal{C}$ that is the identity on objects, and is compatible in a natural way with the monoidal structure.*

Abramsky and Coecke [1] formulated a version of quantum mechanics in a certain type of dagger symmetric monoidal category called a strongly compact closed category. The idea was that quantum systems were the objects of the category, and processes on the systems were its morphisms. Forming compound systems was taken as the basic starting point, with the tensor product $A \otimes B$ of the category giving the object for the compound system.

In their original formulation, questions, and a Born rule, were developed in the setting of compact closed categories with biproducts. In this setting, and in somewhat more general ones, a link to the quantum logic program is established in [7].

Theorem 2.6. *If $\mathcal{C}$ is a dagger biproduct category, then for any object $A \in \mathcal{C}$, the collection of binary dagger biproduct decompositions of A forms an* OMP.

Another topic of importance here is that of topological OMPs. There had been a number of studies of topological OMLs, generalizing in an obvious way the notion topological groups and lattices, and having all basic operations continuous. However, these missed the primary example of topological structure on the OML $\mathcal{P}(\mathcal{H})$. For instance, considering subspaces of $\mathbb{R}^2$, there is a sequence of rays S_θ making angle θ with the x-axis T and converging in a natural sense to T. However, the sequence $S_\theta \wedge T$ is constantly the origin, and does not converge to $T \wedge T = T$. A more subtle notion of a topological OMP due to Wilce [17] does capture this key example, and will be the notion of interest here.

Definition 2.7. *Let P be an* OMP *and $O = \{(x, y) \in P^2 \,|\, x \perp y\}$. We say that P, equipped with a Hausdorff topology τ, is a topological* OMP *if the following hold.*

1. *The set O is closed.*
2. *The operation $\oplus : O \to P$ is continuous.*
3. *The orthocomplementation $\perp : P \to P$ is continuous.*

For any Hilbert space $\mathcal{H}$, the strong and weak operator topologies on $\mathcal{P}(\mathcal{H})$ agree, and with this topology $\mathcal{P}(\mathcal{H})$ is a topological OMP. In fact, this topological OMP is compact, a fact of some importance.

We turn next to our final topic of this preliminary section, some basics on vector bundles. For further details, the reader should consult [2, 10].

Definition 2.8. *Let E and X be topological spaces, $\pi : E \to X$ be continuous, and suppose that each fiber $\pi^{-1}\{x\}$ is equipped with the structure of a real vector space. This data is called a real vector bundle over X if for each $x \in X$, there is a natural number k, a neighborhood U of x, and a homeomorphism*

$$\varphi : U \times \mathbb{R}^k \to \pi^{-1}[U]$$

such that $\varphi : \{y\} \times \mathbb{R}^k \to \pi^{-1}\{y\}$ is a vector space isomorphism for all $y \in U$.

In general, the value of k in this definition depends one the point x. However, for each k, the set $\{x \in X \mid \dim(E_x) = k\}$ for each $k \in \mathbb{N}$ is clopen. So if the base space X is connected, then the dimensions of fibers remain constant. If all fibers have the same dimension $k \geq 0$, the bundle is called k-dimensional. An important example of a k-dimensional vector bundle is the trivial bundle $\pi : X \times \mathbb{R}^k \to X$ where $\mathbb{R}^k$ is given the usual topology and π is the projection onto the first coordinate.

Definition 2.9. *Let $\pi : E \to X$ be a bundle. We say $S \subseteq E$ is a subbundle of E if the restriction $\pi|_S : S \to X$ is a vector bundle and each of its fibers is a subspace of the corresponding fiber of E. The set of all subbundles of E is denoted by $Sub(E)$.*

Definition 2.10. *For a bundle $\pi : E \to X$ let $E \oplus E = \{(u, v) \in E \times E \,|\, \pi(u) = \pi(v)\}$. A Hermitian metric $\langle \cdot, \cdot \rangle$ on E is a continuous map*

$$E \oplus E \to \mathbb{R}$$

which is a positive-definite inner product on each fiber of E. A bundle that is equipped with a Hermitian metric is called a Hermitian bundle.

While there are examples of bundles that do not admit a Hermitian metric, it is known that every bundle over a compact Hausdorff space X does admit a Hermitian metric. Clearly the trivial bundle $\pi : X \times \mathbb{R}^k \to X$ has a Hermitian metric given by the usual inner product in each fiber. When we speak of the trivial bundle, we shall mean it to be equipped with this metric.

Definition 2.11. *Let $\pi : E \to X$ be a bundle. A continuous map $s : X \to E$ is called a global section if $\pi s = id_X$. A global section s of a Hermitian bundle is called normalized if $s(x)$ is a unit vector for each $x \in X$.*

Every bundle has a global section, namely the zero section $s : X \to E$ defined by $x \rightsquigarrow 0$ for each $x \in X$. Also, every Hermitian bundle that has an everywhere non-zero section s has a normalized global section obtained by normalizing the vector $s(x)$ for each $x \in X$. However, there are bundles that do not have everywhere non-zero global sections, such as the familiar Möbius band [10].

Definition 2.12. *An orthonormal basis for a Hermitian bundle over X is a set $\{s_1, \ldots, s_k\}$ of global sections such that for each $x \in X$ the set $\{s_1(x), \ldots, s_k(x)\}$ is an orthonormal base of the fiber over x.*

For a trivial bundle $X \times \mathbb{R}^k$, we let $\tilde{e}_1, \ldots, \tilde{e}_k$ be the obvious orthonormal basis and call this the standard basis of the trivial bundle. There are examples of Hermitian bundles that have no basis. Indeed, it is known that a bundle has a basis if, and only if, it is equivalent to a trivial bundle. Also known is that any bundle over a compact Hausdorff space that is contractible to a point is equivalent to a trivial one.

Definition 2.13. *Suppose s is an everywhere non-zero global section of a bundle $\pi : E \to X$. Then the subbundle*

$$\bigcup \{[s(x)] \mid x \in X\},$$

where $[s(x)]$ is the subspace of the fiber E_x generated by $s(x)$ for each $x \in X$, is called the subbundle generated by the global section s and is denoted by $[s]$.

While each everywhere non-zero global section gives rise to a one-dimensional subbundle, the notions are not equivalent. The Möbius band is itself a one-dimensional bundle, yet has no everywhere non-zero global section. Moreover, it is possible that a bundle may not have any one-dimensional subbundles, for example the tangent bundle over the unit sphere [13].

Definition 2.14. *A bundle map between bundles $\pi : E \to X$ and $\pi' : F \to X$ is a map $\varphi : E \to F$ that is continuous, fiberwise linear, and satisfies $\pi = \pi' \circ \varphi$. The category of bundles over X and their bundle maps is denoted by $\mathscr{E}(X)$. We let $\mathscr{H}(X)$ be the category of Hermitian bundles and the bundle maps between them.*

In the presence of a basis for a bundle, one can work with bundle maps via matrices, much as one works with linear transformations of vector spaces. Here, we let $\mathcal{M}_{k\times k}(\mathbb{R})$ be the set of $k \times k$ real matrices with usual topology.

Proposition 2.15. *Let E be a vector bundle over X with a basis $\{s_1 \cdots, s_k\}$. For any bundle map $\varphi : E \to E$, there is the continuous map $\hat{\varphi} : X \to \mathcal{M}_{k\times k}(\mathbb{R})$ with $\hat{\varphi}(x)$ being the matrix for the linear map $\varphi_x : E_x \to E_x$ in the basis $\{s_1(x), \cdots, s_k(x)\}$.*

Constructions in the vector space setting, such as direct sum $V \oplus W$, duals V^*, function spaces $\mathrm{Hom}(V, W)$, and tensor products $V \otimes W$, are transferred to the vector bundle setting by performing them fiberwise, and topologizing them appropriately. There are general results that state that these constructions enjoy properties of those in vector spaces [10]. We consider one such construction in detail, the others are similar.

Definition 2.16. *For vector bundles $\pi : E \to X$ and $\pi' : F \to X$, let $E \oplus F$ be the subspace of the product space $E \times F$ consisting of those (a, b) with $\pi(a) = \pi'(b)$. Then the map $\pi \oplus \pi' : E \oplus F \to X$ defined in the obvious way is a bundle over X called the Whitney sum.*

General considerations show that the Whitney sum of bundles $E \oplus F$ gives a biproduct in the category $\mathscr{E}(X)$, and with an obvious lifting of inner products, a biproduct also in $\mathscr{H}(X)$. Similarly, tensor products $\otimes$ provide symmetric monoidal structure for both categories. Fiberwise considerations also provide the following.

Proposition 2.17. *The category $\mathscr{H}(X)$ of real Hermitian vector bundles is a strongly compact closed categories with biproducts in the sense of Abramsky and Coecke* [1].

Several remarks are in order. The Hermitian metric $\langle \cdot, \cdot \rangle$ on a real bundle E is a bilinear mapping from $E \oplus E$ to the tensor unit I, and this lifts to a bundle map $\epsilon_E : E \otimes E \to I$. This serves as the counit, and its adjoint the unit $\eta_E : I \to E \otimes E$, of a compact closed structure on $\mathscr{H}(X)$ where each object is self dual. Verifying the necessary commutativity of diagrams is fiberwise and follows as in [1].

The result also holds for complex Hermitian bundles, but uses conjugate bundles exactly as they are used in showing that finite-dimensional complex Hilbert spaces are strongly compact closed. Finally, we remark that modest conditions on X, such as paracompactness, ensure a rich supply of Hermitian bundles over X. But this is not needed in the statement of the theorem.

3. OMPs from Hermitian vector bundles

Here we consider OMPs constructed from Hermitian vector bundles, and their situation in the quantum logic program. The first construction of an OMP from vector bundles was in Crown [5] and proceeded via Baer *-semigroups, but equivalent constructions are available in more accessible ways that we now describe.

Definition 3.1. *For a Hermitian vector bundle E, the endomorphisms of E form a ring $End(E)$ with involution $\dagger$ where addition is fiberwise addition, involution is the fiberwise adjoint, and multiplication is composition. Self-adjoint idempotents of this ring are called projections of E, and the collection of all such is $\mathcal{P}(E)$.*

For a trivial bundle E over X with basis $s_1, \ldots, s_k$, this endomorphism ring can be realized as the ring of matrices $\mathcal{M}_{k\times k}(C(X))$ having continuous functions

$f : X \to \mathbb{R}$ as entries. Addition, multiplication and adjoint in this ring are done componentwise. Constructing an OMP from an involutive ring is a well-known process.

Proposition 3.2. *For R a ring with involution $*$ and $Id^*(R) = \{e \,|\, e = e^2 = e^*\}$, there is a partial ordering $\leq$ on $Id^*(R)$ given by $e \leq f \Leftrightarrow ef = e = fe$. With this partial ordering and orthocomplementation $e^\perp = 1 - e$, $Id^*(R)$ forms an* OMP.

Corollary 3.3. *The projections $\mathcal{P}(E)$ of a Hermitian bundle E form an* OMP.

This is an obvious extension of the construction of an OML from the projections of a Hilbert space. The projections of a Hilbert space correspond to closed subspaces, and in the finite-dimensional case, simply to subspaces. An analogous result is true also for bundles. For each projection p of a Hermitian bundle E, it is known that the image of p is a subbundle of E (a non-trivial fact that uses idempotence [2]), and for each subbundle M of E there is a unique projection p^M of E whose image is M. Further, the image of the orthocomplement $1 - p$ of p is the subbundle we denote $M^\perp$ of E given by

$$M^\perp = \bigcup \{M_x^\perp \,|\, x \in X\},$$

where M_x is the fiber of M over x and $M_x^\perp$ is the orthogonal subspace of M_x in the fiber E_x. This immediately gives the following.

Proposition 3.4. *For a Hermitian bundle E, the collection $Sub(E)$ of subbundles of E forms an* OMP *under set inclusion with the orthocomplementation $M \rightsquigarrow M^\perp$. Further, this* OMP *is isomorphic to the* OMP *$\mathcal{P}(E)$.*

The above proposition enables us to identify subbundles with their corresponding projections. We will often use $Sub(E)$ and $\mathcal{P}(E)$ interchangeably. This construction of $Sub(E)$ and $\mathcal{P}(E)$ can also be realized as the OMP of †-biproduct decompositions of an object E in the category of Hermitian vector bundles over X as described in [7]. So the vector bundle setting provides an interesting case to study an example of the categorical quantum mechanics program that differs from the base example of finite-dimensional vector spaces, yet has much more interesting structure than some toy models such as the category REL of sets and relations.

Proposition 3.5. *The* OMP *$Sub(E)$ of a Hermitian bundle E need not be a lattice.*

Proof. Let E be the trivial 3-dimensional bundle over the the real unit interval $[0, 1]$. We will define four subbundles A, B, C and D of E by giving their fibers for each time t in $[0, 1]$. Let the fiber A_0 of A at time $t = 0$ be the x-axis in $\mathbb{R}^3$, let this rotate around the y-axis at a constant rate until it becomes the z-axis at time $t = 1/2$, and then let it stay at the z-axis until time $t = 1$. Let the fiber of B at time t be the $x - z$ plane for all times t from 0 to 1/2. At time $t = 1/2$ let this plane rotate around the z-axis until it becomes the $y - z$ plane at time $t = 1$. Let C be the z-axis at all times t and let D be the $x - z$ plane at all times t.

Then A_t is contained in B_t and D_t for all times t, and C_t is contained in B_t and D_t for all times t. Thus A and C are lower bounds of B and D. It is clear that A and C are incomparable, and that B and D are incomparable. For B and D to have a greatest lower bound, it would have to be contained in $B \cap D$. But at time $t > 1/2$

$B_t \cap D_t$ is one-dimensional. Since $[0,1]$ is connected, each subbundle has constant dimension, so any non-trivial lower bound of B and D is one-dimensional. Clearly then there is no greatest lower bound of B and D. □

We continue to investigate basic structural properties of these OMPs $Sub(E)$. We recall that a block of an OMP is a maximal Boolean subalgebra. Some basics can easily be determined about blocks of $Sub(E)$, but matters can be quite delicate.

Proposition 3.6. *Suppose E is a k-dimensional Hermitian bundle over a connected compact Hausdorff space X.*

(1) *Each block of $Sub(E)$ has at most k atoms.*
(2) *If X is contractible, then each block of $Sub(E)$ has exactly k atoms.*
(3) *The blocks of even trivial bundles can have fewer than k atoms.*

Proof. (1) If subbundles S and T of E are orthogonal, then for each x, their fibers S_x and T_x are orthogonal subspaces of the k-dimensional space E_x. So there can be at most k pairwise orthogonal non-zero subbundles of E. (2) Any bundle over a compact contractible space is trivial, hence has an everywhere non-zero global section. It follows that an atom of a block, which in general must be an atom of the OMP, is a one-dimensional subbundle. Its orthogonal subbundle has dimension $k-1$ and a simple induction applies. (3) Every bundle can be embedded as a subbundle into a trivial bundle, and there are bundles that have no one-dimensional subbundles. □

In the absence of connectedness or compactness, one has difficulties establishing the existence of atoms. When X is compact and connected, we can bound the size of blocks in $Sub(E)$. But some blocks can be of different sizes than others, and the situation can involve highly complex questions about bundles. It is only in the case of bundles over contractible spaces where matters are simple.

Definition 3.7. *A subset of an OMP P is compatible if it is contained in a block of P. We say P is regular if every pairwise compatible subset of P is compatible.*

Proposition 3.8. *For any ring R, the OMP $Id\,(R)$ of its idempotents is regular, and for any $*$-ring R, the OMP $Id^*(R)$ of its projections is regular.*

Proof. It is known that elements a, b of an OMP are compatible iff there are pairwise orthogonal elements x, y, z with $a = x \oplus y$ and $b = y \oplus z$, and in this case the meet of a, b is y. It is also known that elements x, y of $Id\,(R)$ are orthogonal iff $xy = yx = 0$ and in this case $x \oplus y = x + y$. It follows from a simple calculation that if elements $a, b \in Id\,(R)$ are compatible, then they commute and their meet is ab. Conversely, if a, b commute, then $a - ab, ab, b - ab$ is a pairwise orthogonal set showing that a, b are compatible. It is well known [14] that an OMP is regular if $\{a, b, c\}$ being compatible implies $\{a, b \wedge c\}$ is compatible, and it follows that $Id\,(R)$ is regular. The argument for a $*$-ring is the same, one has only to check various elements are self-adjoint. □

Corollary 3.9. *For a Hermitian bundle E, the OMP $Sub(E)$ is regular.*

We consider one final structural property of the OML $\mathcal{P}(\mathcal{H})$ that persists only in a weakened form even in the setting of quite special bundles.

Suppose $\mathcal{H}$ is a Hilbert space of dimension at least 3. Then for any atoms p, q of $\mathcal{P}(\mathcal{H})$, there is an atom r that is orthogonal to both p and q. To see this, it is enough to show this for $\mathcal{H}$ being $\mathbb{R}^3$ since any two atoms of $\mathcal{P}(\mathcal{H})$ belong to an interval $[0, s]$ that is isomorphic to $\mathcal{P}(\mathbb{R}^3)$. If $p = q$, we then take any atom beneath their orthocomplement. On the other hand, if $p \neq q$, then p corresponds to a 1-dimensional subspace spanned by a vector u, and q to a one dimensional subspace spanned by v. Let $w = u \times v$ the usual cross product in $\mathbb{R}^3$. Then the subspace spanned by w is an atom r that is orthogonal to both p, q.

This argument does not carry through even for trivial bundles over contractible spaces. Suppose E is a 3-dimensional Hermitian bundle over a contractible space X. Given two atoms p, q of $Sub(E)$, there are everywhere non-zero global sections s, t so that for each $x \in X$, the the span $[s(x)]$ is the fiber p_x and the span $[t(x)]$ is q_x. We can compute the cross product $s \times t$ of these global sections componentwise, and the result is a global section. However $s \times t$ need not be everywhere non-zero, so will not determine a 1-dimensional subbundle. With effort, one can however show the following, whose technical proof involves a suitable choice of an everywhere non-zero global section whose cross product does behave well with both s and t [18].

Proposition 3.10. *For E the trivial 3-dimensional bundle over the space $[0, 1]$, for any two atoms p, q of $Sub(E)$, there are atoms r, s with $p \perp r \perp s \perp q$.*

In terms of Greechie diagrams, any two atoms of $\mathcal{P}(\mathbb{R}^3)$ can be connected by at most two blocks, and any two atoms of $Sub([0, 1] \times \mathbb{R}^3)$ can be connected by at most three blocks. We next turn our attention to the matter of topological OMPs in the sense of Definition 2.7. The first result is a simple adaptation of the usual proof that the endomorphisms of a finite-dimensional vector space form a C^*-algebra.

Proposition 3.11. *For a vector bundle E over a compact space X, the endomorphisms $End(E)$ form a C^*-algebra under the norm*

$$\|\varphi\| = \sup\{\|\varphi_x\| \mid x \in X\},$$

where $\|\varphi_x\|$ denotes the operator norm of the fiber map φ_x on E_x for each $x \in X$.

In particular, for a vector bundle E over a compact space X, we have $End(E)$ is a topological $*$-ring in the obvious sense of being a $*$-ring with a topology making the basic operations continuous.

Proposition 3.12. *If R is a topological $*$-ring, then the projections $P = Id^*(R)$ are a topological* OMP *in the sense of Wilce.*

Proof. Multiplication $\cdot : R^2 \to R$ is continuous. The set O is the intersection of the closed sets $\{(e, f) \mid ef = 0\}$ and $\{(e, f) \mid fe = 0\}$ with the square P^2 of our OMP $Id^*(R)$, hence is a closed subset of P^2. The operations of orthocomplementation $\perp$ and orthogonal sum are given from continuous operations on R, $e^\perp = 1 - e$ and $e \oplus f = e + f$, hence are continuous on P. □

Corollary 3.13. *For E a Hermitian bundle over a compact space X, there is a complete metric space topology on $Sub(E)$ making this a topological* OMP *in the sense* *of Wilce. Further, if X is connected, and E is k-dimensional, then the set of all m-dimensional subbundles of E is a clopen subspace of $Sub(E)$.*

Proof. Let $R = End(E)$. Since $Sub(E) \simeq \mathcal{P}(E)$ and $\mathcal{P}(E)$ is the OMP $Id^*(R)$, the metric topology on R gives a metric topology on $\mathcal{P}(E)$ that transfers to $Sub(E)$. The metric on R is complete, and the limit of a sequence of self-adjoint idempotents is a self-adjoint idempotent since adjoint and multiplication are continuous. So this metric on $Id^*(R)$ is complete.

For a projection p, we have $p = p^M$ where M is the image of p. Then for each $x \in X$ the trace of p_x is the dimension of the fiber M_x. It is easily seen that the map $\mathrm{Tr}(p)$ that assigns to each $x \in X$ the trace of p_x is continuous and takes values in $\{0, \ldots, k\}$. So if X is connected, then it must be a constant function m for some natural number $0 \leq m \leq k$. Therefore, the trace map Tr may be viewed as a map from $\mathcal{P}(E)$ to $\{0, \ldots, k\}$, and as such is continuous. Thus each $\mathrm{Tr}^{-1}\{m\}$ is a clopen subspace of $\mathcal{P}(E)$ corresponding to the set of all m-dimensional subbundles. □

We see yet another difference between OMPs $Sub(E)$ from a Hermitian bundle over a compact space X and OMPs from the subspaces of a finite-dimensional Hilbert space. While both have complete metric space topologies, the topologies on ones from finite-dimensional Hilbert spaces are compact.

Proposition 3.14. *The topological* OMP *$Sub(E)$ need not be compact.*

Proof. Consider the three dimensional trivial bundle $E = [0, 1] \times \mathbb{R}^3$. We produce a sequence that has no convergent subsequence. For each natural number n, consider the everywhere non-zero normalized global section $u_n : [0, 1] \to \mathbb{R}^3$ defined as follows.

$$u_n(x) = (\frac{x^n}{\sqrt{1 + x^{2n}}}, \frac{1}{\sqrt{1 + x^{2n}}}, 0)$$

Then let $[u_n]$ be the 1-dimensional subbundle determined by u_n. Using the standard base $\{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3\}$ for the bundle E, notice that the ij-component of the 3×3 matrix representing the bundle map $p^{[u_n]}$, when $i = 1$ and $j = 1$, is the function

$$f_n(x) = \frac{x^{2n}}{1 + x^{2n}}$$

The sequence $\{f_n\}_{n \in \mathbb{N}}$ in $C([0, 1])$ does not contain any convergent subsequence since any such would converge to a function taking value $\frac{1}{2}$ at 1 and value 0 otherwise. It follows that the sequence $p^{[u_n]}$ contains no convergent subsequence. □

We next turn our attention to states on these OMPs built from Hermitian bundles. First, we discuss notation surrounding states on the OMP of projections of $\mathbb{R}^n$.

Definition 3.15. *For a unit vector v in a finite-dimensional inner product space V, define a state $\mu^v : \mathcal{P}(V) \to [0, 1]$ by setting*

$$\mu^v(p) = \|p(v)\|^2$$

In the following, we consider a probability measure on a topological space X, meaning a probability measure on the Baire σ-algebra of X.

Proposition 3.16. *Let E be a Hermitian bundle over a compact Hausdorff space X. Then for each probability measure m on X, and each global section f of E that is non-zero and normalized on the support of m, there is a continuous state $\mu^{f,m}$ on $\mathcal{P}(E)$ given by*

$$\mu^{f,m}(p) = \int_X \mu^{f(x)}(p_x)dm$$

Proof. The correspondence $x \rightsquigarrow \|p(f(x))\|^2$ defines a nonnegative, continuous, and integrable map with $\|p(f(x))\| \leq 1$, showing that $\mu^{f,m}$ is well-defined. Clearly, $\mu^{f,m}(0) = 0$. Let S be the support of the measure m. As f is normalized on S, then $\mu^{f,m}(id_E) = 1$. Also, for any projection p, on the support S we have $\|(1-p)(f(x))\|^2 = \|f(x)\|^2 - \|p(f(x))\|^2 = 1 - \|p(f(x))\|^2$ showing that $\mu^{f,m}(p^\perp) = 1 - \mu^{f,m}(p)$. If $p \perp q$, as $\|(p+q)(f(x))\| = \|p(f(x))\|^2 + \|q(f(x))\|^2$, it follows that $\mu^{f,m}(p \oplus q) = \mu^{f,m}(p) + \mu^{f,m}(q)$. Finally, if p_n is convergent to p in $\mathcal{P}(E)$, then the convergence $\mu^{f,m}(p_n) \to \mu^{f,m}(p)$ follows from the the inequality $\|p_n(f(x)) - p(f(x))\| \leq \|p_n - p\|$ for all $x \in X$ and $m(X) = 1$. □

The set of states of any OMP is convex, meaning that if μ_1 and μ_2 are states, then $\lambda\mu_1 + (1-\lambda)\mu_2$ is also a state, for any $0 \leq \lambda \leq 1$. Further, it is known [14] that this convex set is compact. Its extreme points, that is, states that cannot be expressed as a non-trivial convex combination, are called pure states. For $\mathcal{P}(\mathcal{H})$, these are the states μ^v given by unit vectors.

Proposition 3.17. *If the state $\mu^{f,m}$ is pure, then m is a point charge.*

Proof. For any A with $m(A) > 0$ we obtain a state $\mu^{f,m,A}(p) = \frac{1}{m(A)}\int_A \mu^{f(x)}(p_x)\,dm$ If m is other than a point charge, there is A so that both A and its complement A' have positive measure. Then $\mu^{f,m} = m(A)\mu^{f,m,A} + m(A')\mu^{f,m,A'}$. □

It was shown in [18] that for the trivial bundle E over $[0,1]$, each $\mu^{f,m}$ with m a point charge is pure. It was also shown that each state on this $\mathcal{P}(E)$ that attains value 1 on some atom is continuous. However, unlike the Hilbert space setting, there are states on the trivial bundle E that attain value 1 on an atom and are not pure [18].

Definition 3.18. *A density operator on $\mathbb{R}^n$ is a positive self-adjoint operator that has trace 1. A density map on a Hermitian bundle E over a space X is a bundle endomorphism $\rho \in End(E)$ such that ρ_x is a density operator on E_x for each $x \in X$.*

Proposition 3.16 has an obvious extension.

Proposition 3.19. *Let E be a Hermitian bundle over a compact Hausdorff space X. Let m be a probability measure on X and ρ be a density map on E. Then there is a continuous state $\mu^{\rho,m}$ on $\mathcal{P}(E)$ given by*

$$\mu^{\rho,m}(p) = \int_X \mathrm{Tr}(\rho\, p)_x\, dm$$

Gleason's theorem [9] provides that for a Hilbert space $\mathcal{H}$ of dimension more than 2, that the density operators on $\mathcal{H}$ correspond to states on $\mathcal{P}(\mathcal{H})$. We ask the following questions about states on OMPs built from bundles.

Problem 1. *Suppose E is a Hermitian bundle over a compact connected space X. Characterize the states on $\mathcal{P}(E)$. Are all states continuous? Do all states arise from density maps? What are the pure states?*

In the previous section we noted that a version of Wigner's theorem characterizes automorphisms of the OMP $\mathcal{P}(\mathcal{H})$ as those maps built in an obvious way from a unitary endomorphism of the real Hilbert space $\mathcal{H}$. An endomorphism U of a Hermitian bundle E over X is called unitary if it is invertible and its adjoint $U^\dagger$ is its inverse, or equivalently, if each U_x is unitary. The following is easily established.

Proposition 3.20. *Suppose E is a Hermitian bundle over X and U is a unitary endomorphism of E. Then the map $\hat{U} : \mathcal{P}(E) \to \mathcal{P}(E)$ defined by $\hat{U}(p) = UpU^\dagger$ is an automorphism of the* OMP *$\mathcal{P}(E)$. Further, the automorphisms $\hat{U}$ arising from unitary maps form a subgroup of the automorphism group of $\mathcal{P}(E)$.*

We come to the second of our problems.

Problem 2. *For a Hermitian bundle E, describe its automorphism group* AUT *$\mathcal{P}(E)$. Does every automorphism of $\mathcal{P}(E)$ arise from a unitary bundle map of E?*

Remark 3.21. For a Hermitian bundle $\pi : E \to X$ over a compact connected space X, it appears there is another way to form automorphisms of $\mathcal{P}(E)$. Given a homeomorphism α of X to itself, we define a mapping $\hat{\alpha} : E \to E$ in the following way. For a element (v, x) in the fiber over x choose a neighborhood U of x for which the restrictions $E|U$ and $E|\alpha U$ are trivial. Then consider the trivializing maps

$$\varphi_U : U \times \mathbb{R}^n \to \pi^{-1}[U]$$
$$\varphi_{\alpha U} : \alpha U \times \mathbb{R}^n \to \pi^{-1}[\alpha U]$$

We then define $\hat{\alpha}$ by requiring that it send (v, x) to the element $\varphi_{\alpha U} \circ (\alpha \times id) \circ \varphi_U^{-1}$ applied to (v, x). Note that $\hat{\alpha}$ is not a bundle map, but for any projection p, the map $\hat{\alpha}^{-1} \circ p \circ \hat{\alpha}$ is a bundle map, and is seen to be a projection. It is also easily seen that $\Phi_\alpha = \hat{\alpha}^{-1} \circ (\cdot) \circ \hat{\alpha}$ is an automorphism of $\mathcal{P}(E)$. However, Φ_α may be induced by a unitary bundle map, we do not know.

Besides questions about states and automorphisms, there are questions involving the relationship between the two. For each state μ on $\mathcal{P}(\mathcal{H})$ that attains value 1 on some one-dimensional subspace $[u]$ spanned by a vector u, a basic fact is that the value $\mu([v])$ for any vector v depends only on the angle between u and v. So such a state μ is invariant under any rotation, i.e. action of a unitary map, that leaves u fixed. This is an essential point in Piron's discussion of states [15].

Problem 3. *Suppose μ is a state on $\mathcal{P}(E)$ that has a value 1 on some projection q onto a one-dimensional subbundle of E. If U is a unitary bundle map on E where $\hat{U}$ leaves q fixed, is $\mu(\hat{U}(p)) = \mu(p)$ for all projections p of E?*

In Definition 2.4, we gave a definition of a tensor product of OMPs P and Q. This involved the existence of a bilinear map from $P \times Q$ into the tensor product $P \otimes Q$ that had certain properties, especially with respect to the lifting of states on P and Q to a state on $P \otimes Q$. For bundles E and F over X, we have a bundle tensor product $E \otimes F$ constructed from E and F. Also, for projections $p \in \mathcal{P}(E)$ and $q \in \mathcal{P}(F)$, there is the natural projection on $E \otimes F$, written $p \otimes q$, namely, the usual tensor product of the bundle maps p and q. The following is straightforward.

Proposition 3.22. *For Hermitian bundles E and F over a compact space X, the map $\Phi : \mathcal{P}(E) \times \mathcal{P}(F) \to \mathcal{P}(E \otimes F)$ defined by $\Phi(p,q) = p \otimes q$ is a bilinear* OMP *map.*

It is not difficult to verify that if states μ_E and μ_F on $\mathcal{P}(E)$ and $\mathcal{P}(F)$ are given by density maps on E and F respectively, then there is a state ω on $\mathcal{P}(E \otimes F)$ given by a density map with $\omega(p \otimes q) = \mu_E(p)\mu_F(q)$ for all projections $p \in \mathcal{P}(E)$ and $q \in \mathcal{P}(F)$. However, to fully answer the following problem, we would need to know that all states on $\mathcal{P}(E)$ and $\mathcal{P}(F)$ arise from density operators, or a way to extend such states that do not.

Problem 4. *Is $\Phi : \mathcal{P}(E) \times \mathcal{P}(F) \to \mathcal{P}(E \otimes F)$ an* OMP *tensor product?*

We note that this problem could be bypassed by extending the notion of a quantum logic to allow for a specified set of states, and to then specify this set of states for $P(E)$ to be those given by density maps. It remains of interest to know whether such measures are necessary.

References

[1] S. Abramsky and B. Coecke, *A categorical semantics of quantum protocols*, Proceedings of the 19th Annual IEEE Symposium on Logic in Computer Science, 415-425, IEEE Comput. Soc. Press, Los Alamitos, CA, 2004.
[2] M. Atiyah, *K-theory*, Lecture Notes Series by W. A. Benjamin, Inc., 1967.
[3] I. Amemiya and H. Araki, *A remark on Pirons paper*, Publ. Res. Inst. Math. Sci., Sect A 12 (1966), 423-427.
[4] G. Birkhoff and J. von Neumann, *The logic of quantum mechanics.* Ann. of Math. (2) 37 (1936), no. 4, 823-843.
[5] G. Crown, *On some orthomodular posets of vector bundles.* J. Natur. Sci. and Math. 15 (1975), no. 1-2, 11-25.
[6] J. Harding, *Decompositions in quantum logic*, Trans. Amer. Math. Soc. 348 (1996), no. 5, 1839-1862.
[7] J. Harding, *A link between quantum logic and categorical quantum mechanics*, Internat. J. Theoret. Phys. 48 (2009), no. 3, 769-802.
[8] S. S. Holland Jr., *The current interest in orthomodular lattices*, Trends in Lattice Theory (Sympos., U.S. Naval Academy, Annapolis, Md., 1966), Van Nostrand Reinhold, New York, 1970, pp. 41126.
[9] G. Kalmbach, *Orthomodular lattices*, London Mathematical Society Monographs, 18. Academic Press, Inc. London, 1983.
[10] M. Karoubi, *K-theory*, Reprint of the 1978 edition, Classics in Mathematics, Springer-Verlag, Berlin, 2008.
[11] S. Mac Lane, *Categories for the working mathematician*, second ed., Springer, New York, 1998.
[12] G. W. Mackey, *The mathematical foundations of quantum mechanics*, A lecture-note volume by W. A. Benjamin, Inc., New York-Amsterdam, 1963.

[13] J. Milnor and J. Stasheff, *Characteristic classes*, Annals of Mathematics Studies, No. 76, Princeton University Press, Princeton, N. J., 1974.

[14] P. Pták and S. Pulmannová, *Orthomodular structures as quantum logics*, Fundamental Theories of Physics, 44. Kluwer Academic Publishers Group, Dordrecht, 1991.

[15] C. Piron, *Foundations of quantum physics*, Mathematical Physics Monograph Series 19, Benjamin-Cummings, 1976.

[16] U. Uhlhorn, *Representation of symmetry transformations in quantum mechanics*, Arkiv Fysik 23 (1962) 307-340.

[17] A. Wilce, *Compact orthoalgebras*, Proc. Amer. Math. Soc. 133 (2005), no. 10, 2911-2920.

[18] T. Yang, *The Logic of Bundles*, Ph.D. Thesis, New Mexico State University, 2013.