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Varieties of Birkhoff Systems Part I

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Abstract A Birkhoff system is an algebra that has two binary operations $\cdot$ and $+$, with each being commutative, associative, and idempotent, and together satisfying $x \cdot (x + y) = x + (x \cdot y)$. Examples of Birkhoff systems include lattices, and quasilattices, with the latter being the regularization of the variety of lattices. A number of papers have explored the bottom part of the lattice of subvarieties of Birkhoff systems, in particular the role of meet and join distributive Birkhoff systems. Our purpose in this note is to further explore the lattice of subvarieties of Birkhoff systems. A primary tool is consideration of splittings and finite bichains, Birkhoff systems whose join and meet reducts are both chains. We produce an infinite family of subvarieties of Birkhoff systems generated by finite splitting bichains, and describe the poset of these subvarieties. Consideration of these splitting varieties also allows us to considerably extend knowledge of the lower part of the lattice of subvarieties of Birkhoff systems

Keywords Birkhoff system · Variety · Splitting · Projective · Quasilattice

1 Introduction

A *bisemilattice* is an algebra with two semilattice operations $\cdot$ and $+$ the first interpreted as a meet and the second as a join. In particular, both these operations are idempotent,

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commutative and associative. A *Birkhoff system* is a bisemilattice satisfying a weakened version of the absorption law for lattices known as *Birkhoff's equation*

$$x \cdot (x + y) = x + (x \cdot y). \tag{BS}$$

This equation was first introduced by Birkhoff in [2, Problem 7], asking for an investigation of algebras satisfying the lattice equations but with absorption weakened to this equation.

Obvious examples of Birkhoff systems are lattices and semilattices (considered as algebras with two equal semilattice operations). A less obvious example is the class of *quasilattices*, these are Birkhoff systems that satisfy $a + b = a \Rightarrow (a \cdot c) + (b \cdot c) = a \cdot c$ and its dual. Padmanabhan [15] showed that quasilattices form a variety, and that the variety of quasilattices is defined by the *regular equations* (equations having the same variables on each side) satisfied by lattices, or in other words, is the *regularization* of the variety of lattices.

The next varieties of Birkhoff systems to be extensively investigated were those that satisfy one, or both, distributive laws. These are the *meet-distributive*, *join-distributive*, and *distributive* Birkhoff systems. The distributive Birkhoff systems, ones that satisfy both distributive laws, form the regularization of the variety of distributive lattices [17].

In a series of papers, Romanowska [20–28], in conjunction with Dudek [5], Gierz [6], McKenzie [14], and Smith [29], considered the varieties of distributive and meet-distributive Birkhoff systems. Corresponding results for join-distributive Birkhoff systems follow by the duality between meet and join operations. Among many results were a characterization of the subvarieties of these varieties, a duality theorem for distributive Birkhoff systems, a structure theory for meet-distributive Birkhoff systems and ever better descriptions of some of the subdirectly irreducibles in these varieties.

A result of [20] involves a theme of primary importance in this paper. Each bisemilattice, and in particular each Birkhoff system, induces two partial orderings on its underlying set, one from the meet semilattice operation, and one from the join semilattice operation. A bisemilattice is called a *bichain* if each of these induced orderings is a chain. Every bichain is a Birkhoff system.

Romanowska [20] characterized all subdirectly irreducible bichains in the variety of meet-distributive Birkhoff systems. While this variety is generated by a 3-element bichain, there is no bound on the cardinality of the subdirectly irreducible bichains in it. This points to the fact that varieties of Birkhoff systems are not congruence distributive, and also to the difficulty inherent in investigations of these algebras.

Another variety of Birkhoff systems of interest here arose from an unexpected source — the truth-value algebra of type-2 fuzzy sets. In their investigation of this algebra, Harding, Walker and Walker [9] showed that the variety generated by its meet and join reduct is generated by a 4-element bichain. Attempts to find an equational basis for this variety lead to a characterization of the finite bichains that are weakly projective and subdirectly irreducible, hence splitting in the variety of Birkhoff systems [10].

It is the aim of this paper to continue the study of Birkhoff systems and their subvarieties. We are able to considerably expand knowledge of the lower part of the lattice of subvarieties, providing a number of covers for the varieties of distributive, meet-distributive, and join-distributive Birkhoff systems.

We also make use of results about splitting bichains to give an infinite family of subvarieties of Birkhoff systems, each generated by a finite, weakly projective, subdirectly irreducible bichain.

Finally, the overall structure of the lattice of subvarieties of Birkhoff systems is discussed, with several important classes of subvarieties identified. This is still far from a complete description of the lattice of subvarieties, and many question remain open. But it is surely a more illuminating picture than previously known.

This paper is arranged in the following manner. The second section contains some preliminaries on notation, and several general results from universal algebra. The third section provides a description of previously known varieties of Birkhoff systems and basic results about them. The fourth section contains results about splitting bichains and equations for the varieties they generate. The fifth section gives a detailed treatment of varieties generated by 3-element bichains and their place in the lattice of subvarieties. The sixth, and final, section deals with a number of other subvarieties of Birkhoff systems, and an overall view of the lattice of subvarieties of Birkhoff systems.

This paper is the first part of a two-part paper on varieties of Birkhoff system [8]. The second paper will include a description of the structure of the poset of varieties generated by splitting bichains, as well as a description of the 4-element subdirectly irreducible Birkhoff systems and the place of the varieties they generate in the lattice of subvarieties. We will also develop a structure theorem for the variety generated by a non-distributive 3-element bichain, and use this to extend our aim in this paper, of describing the bottom part of the lattice of subvarieties of Birkhoff systems.

2 Preliminaries

We give some general results, notations, and terminology about Birkhoff systems and bichains used in the paper. We also recall basics about Płonka sums and regular equations, and also about weakly projective algebras and splittings.

2.1 Birkhoff Systems and Bichains

As with any bisemilattice, a Birkhoff system induces two semilattice orderings on its underlying set. One called the meet order and denoted $\leq_\cdot$, the other called the join order and denoted $\leq_+$. These partial orders are defined as follows

$$x \leq_\cdot y \Leftrightarrow x \cdot y = x \quad \text{and} \quad x \leq_+ y \Leftrightarrow x + y = y.$$

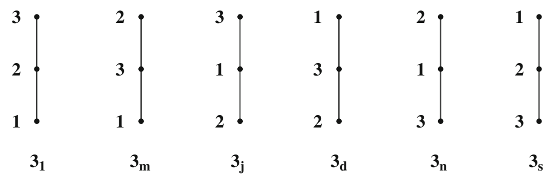
Finite bisemilattices may be illustrated by two pictures representing their two semilattice orderings. We usually draw the diagram of the meet semilattice on the left and that of the join semilattice on the right.

A bichain is a bisemilattice where both the meet and join semilattice are linear orders. In this paper, elements of a finite bichain with n elements will be denoted by $1, \ldots, n$, and the meet semilattice reducts will always be

$$1 <_\cdot 2 <_\cdot \ldots <_\cdot n.$$

When describing such a finite bichain, we will describe only its join semilattice order (from the smallest to the largest) and will often do so without using commas or $<$ symbols. So 231 will mean the bichain whose meet semilattice order is $1 <_\cdot 2 <_\cdot 3$ and whose join semilattice order is $2 <_+ 3 <_+ 1$. Then $\mathbf{n_l}$ will denote the bichain which is a lattice defined on the set $\{1, 2, \ldots, n\}$, and $\mathbf{n_s}$ will denote the semilattice defined on $\{1, 2, \ldots, n\}$. There are two 2-element bichains, $\mathbf{2_l}$ and $\mathbf{2_s}$. The six 3-element bichains will play an important role in this paper and are shown in Fig 1.

Fig. 1 The 3-element bichains



Definition 2.1 The dual of a bisemilattice $(S, \cdot, +)$ is the bisemilattice $(S, +, \cdot)$. The dual of a bisemilattice equation (ε) is the equation (ε^d) formed by interchanging $\cdot$ and $+$ in (ε).

Just as with lattices, a bisemilattice equation (ε) holds in a bisemilattice S if, and only if, the dual equation (ε^d) holds in the dual S^d of S. We next consider duality in the context of the six 3-element bichains of Fig. 1.

Proposition 2.2 *Each of the bichains $\mathbf{3_l}$, $\mathbf{3_d}$, $\mathbf{3_n}$, and $\mathbf{3_s}$ is isomorphic to its own dual, and each of $\mathbf{3_m}$ and $\mathbf{3_j}$ is isomorphic to the dual of the other.*

Proof We show the situation for $\mathbf{3_n}$, the others are similar. The bichain $\mathbf{3_n}$ has meet order $1 <. 2 <. 3$ and join order $3 <_+ 1 <_+ 2$. For clarity, let $\odot$ be the meet of the dual of $\mathbf{3_n}$, which is simply the join of $\mathbf{3_n}$, and $\oplus$ be the join of the dual of $\mathbf{3_n}$, which is the meet of $\mathbf{3_n}$. Since $3 + 1 = 1$ and $1 + 2 = 2$, we have $3 \odot 1 = 1$ and $1 \odot 2 = 2$. So the meet order of the dual of $\mathbf{3_n}$ is $2 <_\odot 1 <_\odot 3$. Since $3 \cdot 2 = 2$ and $2 \cdot 1 = 1$, we have $3 \oplus 2 = 2$ and $2 \oplus 1 = 1$, so the join order of the dual of $\mathbf{3_n}$ is $3 <_\oplus 2 <_\oplus 1$. It is then a simple matter to check that the dual of $\mathbf{3_n}$ is isomorphic to $\mathbf{3_n}$. □

We now collect several results about Birkhoff systems and bichains.

Proposition 2.3 *Every subset of a bichain is a subalgebra.*

Proof If x, y are elements of a bichain, then $x \cdot y$ is equal to either x or y, and $x + y$ is equal to either x or y. □

Corollary 2.4 *Every variety that is generated by a family of bichains is locally finite.*

Proof Proposition 2.3 shows that any class $\mathcal{K}$ of bichains is what is known as uniformly locally finite, meaning there is a finite upper bound on the cardinality of any n-generated subalgebra of a member of $\mathcal{K}$. In our case, this upper bound is n itself. It follows from [1, Theorem 3.7] that the variety generated by $\mathcal{K}$ is locally finite. □

Proposition 2.5 *Every bichain satisfies Birkhoff's equation (BS), hence is a Birkhoff system.*

Proof Suppose A is a bichain and $x, y \in A$. Then by Proposition 2.3 $\{x, y\}$ is a subalgebra of the bichain, hence itself is a bichain. Up to isomorphism, there are exactly three bichains having at most two elements, the trivial 1-element bichain, the 2-element lattice, and the 2-element semilattice. All satisfy Birkhoff's equation. □

Birkhoff systems might not have distributive congruence lattices. This is seen by noting that semilattices do not have distributive congruence lattices, and that each semilattice can naturally be considered as a Birkhoff system with two equal binary operations. However, the congruence lattices of semilattices are known to satisfy the following quasi-equation known as *meet-semidistributivity* or (SD$\wedge$). This condition, and its dual join-semidistributivity (SD$\vee$), are given below.

$$x \wedge y = x \wedge z \quad \Rightarrow \quad x \wedge (y \vee z) = x \wedge y; \tag{SD$\wedge$}$$

$$x \vee y = x \vee z \quad \Rightarrow \quad x \vee (y \wedge z) = x \vee y. \tag{SD$\vee$}$$

Proposition 2.6 *The congruence lattice of every Birkhoff system, is meet-semidistributive. Therefore the lattice of subvarieties of Birkhoff systems is join-semidistributive.*

Proof The congruence lattice of a bisemilattice is a sublattice of the congruence lattice of each of its semilattice reducts. In fact, it is the intersection of the congruence lattices of its semilattice reducts. As a sublattice of a meet-semidistributive lattice is meet-semidistributive, the result follows.

From well known results in universal algebra [3, Cor. 14.10], the lattice of subvarieties of the variety of Birkhoff systems is dually isomorphic to the lattice of fully invariant congruences on the free Birkhoff system on countably many generators. This lattice of fully invariant congruences is a sublattice of a meet-semidistributive lattice, hence is meet-semidistributive. Thus its dual is join-semidistributive. □

Finally, to conclude our discussion of basics of Birkhoff systems and bichains, to aid readability, we use several common conventions when writing expressions and equations. The multiplication symbol $\cdot$ will often be omitted and implied by juxtaposition. Also, multiplication is understood to have precedence over addition. Thus $x + yz$ represents the expression $x + (y \cdot z)$.

2.2 Płonka Sums and Regular Equations

In [16], Płonka introduced a technique for constructing algebras that has since become known by the name of *Płonka sums*. Since their introduction, Płonka sums have received a considerable amount of attention. Here we give a more current approach to the topic as found in [19] and [30, Ch. 4].

In what follows we consider only algebras of so-called *plural* type, that means without symbols of nullary operations and with at least one symbol of at least binary operation. Let $\mathcal{K}$ be a category of algebras of a given plural type τ with homomorphisms as morphisms. And consider a (join) semilattice S as a small category with elements as objects and one morphism $a \to b$ in the case $a \leq b$.

Definition 2.7 Let $F : S \to \mathcal{K}$ be a (covariant) functor from the category S to the category $\mathcal{K}$. The functor F assigns to each morphism $s \to t$ of S a homomorphism $\varphi_{s,t} : A_s \to A_t$ of $\mathcal{K}$. The *Płonka sum of algebras* A_s (over the semilattice S by the functor F) is the algebra defined on the disjoint sum of the sets A_s with operations of type τ given as follows. The n-ary operation ω on the Płonka sum is defined by setting

$$\omega(a_1, \ldots, a_n) = \omega(\varphi_{s_1,t}(a_1), \ldots, \varphi_{s_n,t}(a_n)),$$

where $a_1 \in A_{s_1}, \ldots, a_n \in A_{s_n}$ and $t = s_1 \vee \cdots \vee s_n$.

Płonka sums provide a link between algebraic properties of a variety and syntactic properties. The key notion is that of a regular equation.

Definition 2.8 An equation $p = q$ is *regular* if the same variables appear on each side. A variety is said to be *regular* if all equations valid in it are regular.

Theorem 2.9 (Płonka, [16]) *The equations preserved by Płonka sums (over non-trivial semilattices) are exactly the regular equations valid in each component. Therefore, a variety with no constant operations is regular if, and only if, it is closed under the formation of Płonka sums.*

If V is a regular variety and W is a variety containing V, then the equations valid in W are a subset of those valid in V. Therefore every variety that contains a regular variety is regular. Further, for each variety V, there is a smallest regular variety containing V called the *regularization* of V and denoted $\widetilde{\mathsf{V}}$. Clearly the regularization of V is the variety defined by all regular equations that are valid in V. If V is a *strongly irregular variety*, i.e. it satisfies an irregular equation $x \circ y = x$ for some binary term $x \circ y$ in which y appears, then the regularization $\widetilde{\mathsf{V}}$ consists precisely of Płonka sums of algebras in V [18].

Theorem 2.10 (Dudek, Graczyńska [4]) *If V is a strongly irregular variety, then the lattice $\mathcal{L}(\widetilde{\mathsf{V}})$ of subvarieties of its regularization $\widetilde{\mathsf{V}}$ is isomorphic to the direct product $\mathcal{L}(\mathsf{V}) \times 2$ of the lattice of subvarieties of V and the 2-element lattice 2*

In this result it is shown that every subvariety of $\widetilde{\mathsf{V}}$ is either a subvariety W of V or the regularization $\widetilde{\mathsf{W}}$ of a subvariety of V, and that these are always distinct. Further, denoting the elements of the 2-element lattice 2 as 0 and 1, the required isomorphism from $\mathcal{L}(\mathsf{V}) \times 2$ to $\mathcal{L}(\widetilde{\mathsf{V}})$ takes an element $(\mathsf{W}, 0)$ to W and $(\mathsf{W}, 1)$ to $\widetilde{\mathsf{W}}$.

2.3 Splitting Varieties

The notion of a splitting pair of subvarieties was introduced by McKenzie in [13]. For a more complete account of the material discussed here, we suggest for example [11].

Definition 2.11 A pair (u, w) of elements of a complete lattice L is called a *splitting pair* or briefly a *splitting* of L, if L is the disjoint union $(u] \cup [w)$ of the set of elements that are underneath of u and the set of elements that are above of w.

Our interest is in pairs of subvarieties U and W of a variety V that split the lattice of subvarieties of V. Such a pair (U, W) is called a *splitting pair of subvarieties* of V. A discussion of the following well known results due to McKenzie is in [11, p. 22].

Proposition 2.12 *Let (U, W) be a splitting pair of subvarieties of V. Then there is a subdirectly irreducible algebra S in V that generates W. The variety U is the largest subvariety of V that does not contain S. It is defined by the equations satisfied in V and one additional equation.*

Definition 2.13 The subdirectly irreducible algebra S in Proposition 2.12 is called a *splitting algebra* in V, the variety U is called the *splitting variety* of S, and the additional equation defining the splitting variety of S is called the *splitting equation* for S.

We turn next to a source of splitting algebras.

Definition 2.14 An algebra P in a variety V is *weakly projective in* V if for any algebra $A \in \mathsf{V}$ and any homomorphism $f : A \to P$ onto P there is a subalgebra B of A such that the restriction $f|_B : B \to P$ is an isomorphism.

The term weakly projective is used rather than the categorical notion of projective since epimorphisms in a variety need not coincide with onto homomorphisms. In fact, we do not know if all epimorphisms are onto in the variety of Birkhoff systems.

Definition 2.15 For a variety V and an algebra S in V, define V_S to be the class of all algebras in V that do not contain a subalgebra that is isomorphic to S.

$$\mathsf{V}_S = \{A \in \mathsf{V} \mid S \not\leq A\}.$$

Using $V(S)$ for the variety generated by an algebra S, we have the following key result. Its proof may be found by combining [11, Lemma 2.10] and [11, Theorem 2.11].

Proposition 2.16 *Let S be an algebra that is subdirectly irreducible and weakly projective in a variety V. Then S is a splitting algebra in V and $(\mathsf{V}_S, V(S))$ is a splitting pair of subvarieties of V.*

One further result will be most useful.

Proposition 2.17 *Suppose S and T are subdirectly irreducible and weakly projective in a variety V. Then the following are equivalent.*

(1) *S is isomorphic to a subalgebra of T.*
(2) $V(S) \subseteq V(T)$.

Thus if S and T are finite, then $V(S) = V(T)$ if, and only if, S is isomorphic to T.

Proof The implication (1) $\Rightarrow$ (2) is trivial. To show (2) $\Rightarrow$ (1), suppose that $V(S) \subseteq V(T)$. Then S is in the variety generated by T. This means that S is a homomorphic image of a subalgebra of a power of T, say $B \leq T^X$ and $f : B \to S$ is an onto homomorphism. As S is weakly projective, S is isomorphic to a subalgebra of B, hence to a subalgebra of T^X. Since S is subdirectly irreducible, it follows that S is isomorphic to a subalgebra of T. □

3 Some Varieties of Birkhoff Systems

Let BS be the variety of Birkhoff systems, and $\mathcal{L}(\mathsf{BS})$ be the lattice of subvarieties of BS. Here we summarize a number of basic results about $\mathcal{L}(\mathsf{BS})$, many of which were previously known. As notation, for algebras $A_1, \ldots, A_n$, we denote the variety generated by these

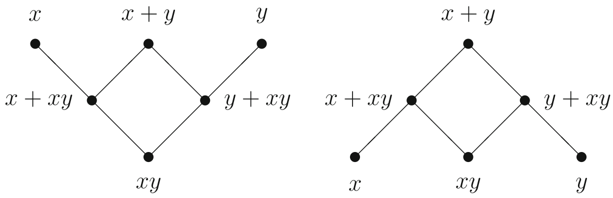


Fig. 2 The free Birkhoff system on two generators

algebras by $V(A_1, \ldots, A_n)$. If a variety is defined by the axioms of Birkhoff systems and equations $\alpha_1, \ldots, \alpha_n$, then it is denoted by $V(\alpha_1, \ldots, \alpha_n)$.

3.1 The Bottom of the Lattice $\mathcal{L}$(BS)

Determining the bottom of the lattice $\mathcal{L}$(BS) is accomplished by understanding the structure of the free BS on two generators. This is a simple matter of applying Birkhoff's equation (BS), giving a 6-element bisemilattice whose two semilattice reducts are shown in Fig. 2 [5].

Using $x + xy = x(x + y)$ and $y + xy = y(x + y)$ provided by Birkhoff's equation (BS), this description of the free Birkhoff system on two generators makes a convenient method to check equations in two variables for BS. In particular, we obtain the following two equations valid in all Birkhoff systems that have frequent use.

$$xy(x + y) = xy, \tag{BS1}$$

$$x + y + xy = x + y. \tag{BS2}$$

Note that the free Birkhoff system on three generators is infinite since it has the free lattice on three generators as a quotient and this is known to be infinite. It seems to be an open problem whether the free word problem for Birkhoff systems is decidable.

Definition 3.1 Let Triv be the variety of all 1-element Birkhoff systems, DL be the variety of distributive lattices, SL be the variety of semilattices (considered as bisemilattices with two equal operations), and L be the variety of lattices.

Clearly Triv is the smallest subvariety of BS, and is generated by the 1-element bichain. The variety DL is generated by the two-element lattice $\mathbf{2_l}$, and SL is generated by the 2-element semilattice $\mathbf{2_s}$. It is well known that DL and SL have no non-trivial subvarieties, so they are atoms in the lattice $\mathcal{L}$(BS).

Proposition 3.2 *Every subvariety of* BS *that is neither a variety of lattices nor the variety* SL *of semilattices contains the join of the varieties* DL *and* SL.

Proof Let A be a Birkhoff system. If A is not a lattice, then there are $x, y \in A$ such that x and $x + xy$ are different. The description of the free Birkhoff system on two generators in Fig. 2 shows $x(x+xy) = x+xy$ and $x+x+xy = x+xy$. So x and $x+xy$ are a 2-element subalgebra of A isomorphic to the 2-element semilattice $\mathbf{2_s}$. If A is not a semilattice, then there are x and y such that xy and $x+y$ are distinct. Figure 2 again shows that xy and $x+y$ are a two-element subalgebra of A isomorphic to the two-element lattice $\mathbf{2_l}$. It follows that each subvariety V of BS that is not a variety of lattices or semilattices contains DL and SL, and hence contains their join. □

Corollary 3.3 *A subvariety of* BS *is regular (see Definition 2.8) if, and only if, it is not a variety of lattices. Thus* SL *is the smallest regular subvariety of* BS.

Proof Each variety of lattices satisfies $x(x + y) = x$, and this equation is not regular. So no variety of lattices is regular. The variety SL of semilattices is defined by the equations defining Birkhoff systems and the equation $xy = x + y$, all of which are regular, and it follows that SL is a regular variety. Proposition 3.2 shows that every subvariety of BS that is not a variety of lattices contains the regular variety SL, hence is regular. □

3.2 Quasilattices

Definition 3.4 The variety Q of quasilattice is the subvariety of BS defined by the equations

$$(x + y)z + yz = (x + y)z, \tag{mQ}$$

$$(xy + z)(y + z) = xy + z. \tag{jQ}$$

Let us note that this definition is equivalent to the definition given in the Introduction. Since the equations defining quasilattices are regular, quasilattices form a regular variety. In fact, much more is true.

Theorem 3.5 *The variety* Q *of quasilattices is the regularization* $\widetilde{\mathsf{L}}$ *of the variety* L *of lattices, and the subvarieties of* Q *are the subvarieties* V *of lattices and their regularizations* $\widetilde{\mathsf{V}}$.

The first part of this theorem was proved by Padmanabhan [15], generalizing an earlier result of Płonka [17] concerning the regularization of the variety of distributive lattices. For the second part, the variety L of lattices is strongly irregular since it satisfies absorption $x \cdot (x + y) = x$, so Theorem 2.10 applies. This shows that the subvarieties of Q are the subvarieties of lattices and their regularizations.

Theorem 2.10, and the discussion surrounding this theorem, has a number of other consequences. The lattice $\mathcal{L}(\mathsf{Q})$ of subvarieties of Q is isomorphic to the direct product $\mathcal{L}(\mathsf{L}) \times 2$ of the lattice of subvarieties of lattices and the 2-element lattice. So for a subvariety V of lattices, its regularization $\widetilde{\mathsf{V}}$ covers V. Further, since SL is the regularization of the trivial variety of lattices, $\widetilde{\mathsf{V}}$ is the join of the varieties V and SL. As shown in [18], the members of $\widetilde{\mathsf{V}}$ are all algebras that are Płonka sums of members of V, and by Theorem 2.9 the equations valid in $\widetilde{\mathsf{V}}$ are exactly the regular equations valid in V.

Definition 3.6 The variety DB of distributive Birkhoff systems is the subvariety of BS that satisfies the meet and join-distributive equations.

$$x(y + z) = xy + xz, \tag{mD}$$

$$x + yz = (x + y)(x + z). \tag{jD}$$

Corollary 3.7 [17] *The variety* DB *is the regularization of the variety* DL *of distributive lattices. Its members are the Płonka sums of distributive lattices. Further,* DB *is the join of the varieties* DL *and* SL.

We have seen that DB is generated by the 2-element lattice $\mathbf{2_l}$ and the 2-element semilattice $\mathbf{2_s}$. It is also generated by the 3-element bichain $\mathbf{3_d}$ of Fig. 1. This bichain $\mathbf{3_d}$ can be realized as the Płonka sum of the 2-element lattice 23 and the 1-element lattice 1. Finally, we remark that the free Birkhoff system on 2-generators satisfies (mD) and (jD) hence belongs to DB. It follows that every 2-generated Birkhoff system is distributive.

3.3 Distributive Bisemilattices

Definition 3.8 The variety mDB of meet-distributive bisemilattices is defined by the biseilattice equations and (mD), and the variety jDB of join-distributive bisemilattices is defined by the bisemilattice equations and (jD).

Each of (mD) and (jD) implies Birkhoff's equation (BS), so mDB and jDB are subvarieties of BS. The algebra $\mathbf{3_j}$ of Fig. 1 satisfies (jD) but not (mD), and the algebra $\mathbf{3_m}$ of Fig. 1 satisfies (mD) but not (jD). Thus varieties mDB and jDB are distinct. So in the context of bisemilattices, meet and join-distributivity do not imply one another. The varieties mDB and jDB have been extensively studied. The following was established in [14].

Theorem 3.9 *Each of the varieties* mDB *and* jDB *covers* DB.

Each of the varieties mDB and jDB contains an infinite number of subdirectly irreducible algebras (see [20, 22, 28]). However, since these varieties cover DB, each algebra belonging to them and not to DB generates its variety. In particular, the variety mDB is generated by the bichain $\mathbf{3_m}$ of Fig. 1, and the variety jDB is generated by the bichain $\mathbf{3_j}$ of Fig. 1.

Definition 3.10 A bisemilattice is called *non-distributive* if it satisfies neither of the two distributive laws (mD) nor (jD).

Our discussion of varieties thus far has placed many of the varieties generated by small bichains. The trivial variety is generated by the 1-element bichain. The two 2-element bichains generate the varieties DL and SL. The six 3-element bichains are shown in Fig. 1. Of these, $\mathbf{3_l}$ and $\mathbf{3_s}$ again generate DL and SL. We have seen that $\mathbf{3_d}$ generates DB, that $\mathbf{3_m}$ generates mDB, and that $\mathbf{3_j}$ generates jDB.

The structure of algebras in these varieties is well understood. We have already seen that distributive bisemilattices are Płonka sums of distributive lattices. The structure of meet-distributive bisemilattices was investigated in [29]. It was shown there that each such algebra is a quotient of certain special bisemilattice called a *bisemilattice of subsemilattices*, defined as follows. For a meet semilattice T, let $\mathbf{S}_f(T)$ be the set of all non-empty finite subsemilattices of T. Define two binary operations on the set $\mathbf{S}_f(T)$ by

$$U \cdot W = \{uw \mid u \in U,\ w \in W\} \quad \text{and} \quad U + W = U \vee W.$$

Then the algebra $(\mathbf{S}_f(T), +, \cdot)$ of finite non-empty subsemilattices of T is a meet-distributive bisemilattice and the following theorem holds.

Theorem 3.11 [29] *Each meet-distributive bisemilattice* $(B, +, \cdot)$ *is a quotient of the bisemilattice* $(\mathbf{S}_f(B), +, \cdot)$ *of the finite non-empty subsemilattices of its meet-semilattice reduct* $(B, \cdot)$.

What has remained a mystery is the variety $V(\mathbf{3_n})$ generated by the non-distributive 3-element bichain. This variety, and the structure of its algebras, will be investigated in the sequel to this paper [8].

3.4 The Variety Generated by Bichains

Definition 3.12 Let BCh be the variety generated by all bichains.

Proposition 2.5 shows that each bichain is a Birkhoff system, so BCh is a subvariety of BS. We next show that these varieties are different.

Proposition 3.13 *The following equation is satisfied in the variety* BCh *generated by all bichains but not in the variety* BS.

$$(yx + x + z)xz = (yz + x + z)xz. \tag{BCh1}$$

Proof First note that an equation with three variables is satisfied in a bichain if, and only if, it is satisfied in each of its k-element subalgebras for $k \leq 3$. So it is enough to check that the equation (BCh1) is valid in the 3-element chains $\mathbf{3_m}$, $\mathbf{3_j}$ and $\mathbf{3_n}$, since each of them contains a 2-element lattice and a 2-element semilattice as subalgebras. We can show even more. Note that the meet-distributive law implies that both sides of (BCh1) are equal to $xyz + xz$. Similarly, the join-distributive law implies that both sides are equal to $(x + z)(x + y + z)xz$. Hence each of the distributive laws implies (BCh1). To check that $\mathbf{3_n}$ satisfies (BCh1), it is enough to substitute all triples of $1, 2, 3$ for x, y, z. Of course, many cases, such as $x = 1$ or $z = 1$, can be dealt with trivially.

To see that not all Birkhoff systems satisfy (BCh1), consider the bisemilattice with the join reduct $4 <_+ 3 <_+ 2 <_+ 1$ and the meet reduct a Boolean lattice with $3 <_\cdot 1, 4 <_\cdot 2$. It is a Birkhoff system since any pair of elements belongs to a 3-element subalgebra satisfying at least one of distributive laws. To see that it does not satisfy (BCh1), substitute 4 for x, 1 for y and 2 for z. □

Corollary 3.14 *The variety* BCh *is strictly contained in* BS.

By changing the role of join and meet in the equation (BCh1), one obtains a dual equation which is also valid in BCh but not in BS. Note that the example in the proof and its dual version are both subdirectly irreducible Birkhoff systems not belonging to the variety BCh. Let us also mention four more equations valid in all bichains but not in all Birkhoff systems:

$$x(xy + xz) = xy + xz, \tag{BCh2}$$

$$x(x + y)(xz + y) = x(x + y)(xz + y + z), \tag{BCh3}$$

$$xy + yz(x + z) = (xy + z)(xy + yz). \tag{BCh4}$$

$$z(y + z)(xy + z) = z + yz(x + z). \tag{BCh5}$$

To see that these equations hold in all bichains, it is enough to check that they hold in $\mathbf{3_m}$, $\mathbf{3_j}$, and $\mathbf{3_n}$ as in the proof of Proposition 3.13. This is not difficult to do by hand as many cases can be eliminated. For instance, (BCh2) surely holds if x is either 1 or 3, and since the equation is obviously true in DL and SL we may assume that all of x, y, z are distinct. Symmetry then leaves the single case of $x = 2, y = 1, z = 3$.

Finding Birkhoff systems where these equations do not hold is made easy by Prover9/Mace4 [12]. One asks the program to prove an equation such as (BCh2) from the equations defining Birkhoff systems, and runs Mace4 to find a counterexample. For each equation, a counterexample with either 4 or 5 elements is produced. The real difficulty in all this is finding the equations. There is no magic secret for this.

4 Splitting Pairs in the Lattice $\mathcal{L}(\mathsf{BCh})$

We begin by recalling several results established in [10].

Theorem 4.1 *A finite bichain is weakly projective in the variety* BS *if, and only if, it does not contain a subalgebra isomorphic to* $\mathbf{3_d}$.

In establishing this result, more was obtained. Suppose that C is an n-element bichain that does not contain a subalgebra isomorphic to $\mathbf{3_d}$. Let φ be the canonical homomorphism from the free Birkhoff system $F_{\mathsf{BS}}(n)$ on n generators onto C. In [10] an algorithm is given to produce for each element $c \in C$, a bisemilattice term $p_c(x_1, \ldots, x_n)$ so that $\{p_c(x_1, \ldots, x_n) \mid c \in C\}$ is a subalgebra of $F_{\mathsf{BS}}(n)$ with the restriction of φ to this subalgebra an isomorphism. Then, the general theory of splittings provides the following. Here, the *monolith* of a subdirectly irreducible algebra is its least non-trivial congruence. This will be a congruence $\theta(a, b)$ generated by collapsing a certain pair of elements.

Proposition 4.2 *Suppose that C is an n-element weakly projective, subdirectly irreducible bichain with monolith $\theta(a, b)$. Then the splitting variety* $\mathsf{BS}_C = \{A \in \mathsf{BS} \mid C \not\leq A\}$ *is defined by the splitting equation*

$$p_a(x_1, \ldots, x_n) = p_b(x_1, \ldots, x_n). \qquad (\mathrm{S}_C)$$

The smallest subdirectly irreducible, weakly projective bichains are the 2-element lattice $\mathbf{2}_\mathrm{l}$ and the 2-element semilattice $\mathbf{2}_\mathrm{s}$. It is illustrative, and informative, to consider matters in these two simple cases.

Proposition 4.3 *The splitting variety* $\mathsf{BS}_{\mathbf{2}_\mathrm{l}}$ *of* $\mathbf{2_l}$ *is the variety* SL *of semilattices, and the splitting equation* $(\mathrm{S}_{\mathbf{2}_\mathrm{l}})$ *is* $xy = x + y$.

Proof It is easy to use Birkhoff's equation to verify that $\{xy, x + y\}$ is a subalgebra of $F_{\mathsf{BS}}(2)$ that is isomorphic to $\mathbf{2_l}$. Then the splitting equation is $xy = x + y$, and with the equations true of Birkhoff systems, this defines the variety of semilattices. ☐

Proposition 4.4 *The splitting variety* $\mathsf{BS}_{\mathbf{2}_\mathrm{s}}$ *of* $\mathbf{2_s}$ *is the variety* L *of lattices, and its splitting equation* $(\mathrm{S}_{\mathbf{2}_\mathrm{s}})$ *is absorption,* $x + xy = x$.

Proof The algorithm to produce a subalgebra of $F_{\mathsf{BS}}(2)$ isomorphic to $\mathbf{2_s}$ produces $\{x + xy, x\}$. Then the splitting equation is the absorption law $x + xy = x$, and with the equations true of Birkhoff systems, this defines the variety of lattices. ☐

In the following, we often use equations that are provably equivalent over BS to splitting equations. This aids readability.

Proposition 4.5 *Each of the bichains* $\mathbf{3_m}$, $\mathbf{3_j}$ *and* $\mathbf{3_n}$ *from Fig. 1 is subdirectly irreducible and weakly projective. Their splitting equations are the following.*

$$(z + xyz)(z + yz + xyz) = z + xyz, \tag{S_{3_m}}$$

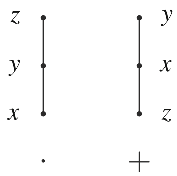
$$z(x + y + z) + z(y + z)(x + y + z) = z(x + y + z), \tag{$\mathrm{S}_{\mathbf{3}_\mathrm{j}}$}$$

$$(z + xyz)(z + yz + xyz) = z + yz + xyz. \tag{$\mathrm{S}_{\mathbf{3}_\mathrm{n}}$}$$

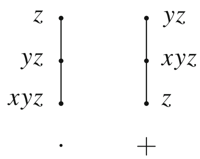
These equations define the varieties $\mathsf{BS}_{\mathbf{3}_\mathrm{m}}$, $\mathsf{BS}_{\mathbf{3}_\mathrm{j}}$ *and* $\mathsf{BS}_{\mathbf{3}_\mathrm{n}}$*, respectively.*

Proof We give the proof for the $\mathbf{3_n}$. This is also found in [9, Prop. 7], but we include it here for the convenience of the reader. The verification for the other two algebras is similar.

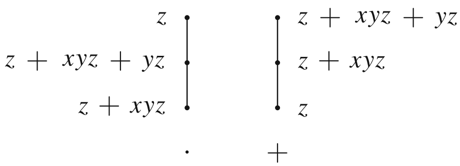
The bichain $\mathbf{3_n}$ is subdirectly irreducible with its minimal congruence being the one collapsing 1 and 2. We apply the algorithm of [10]. Let φ be the canonical homomorphism from $F_{\mathsf{BS}}(3)$ to $\mathbf{3_n}$ mapping x to 1, y to 2, and z to 3. The aim is to produce a subalgebra of $F_{\mathsf{BS}}(3)$ that is isomorphic to $\mathbf{3_n}$ and on which φ restricts to an isomorphism. Below is a copy of $\mathbf{3_n}$ of Fig. 1 with x, y, z in place of 1,2,3 respectively.



The elements shown above do not form a subalgebra of $F_{\mathsf{BS}}(3)$ since they are not closed under joins or meets. In particular, the joins and meets of these elements are not as shown in this diagram. As the first step of the algorithm, we repair meets as shown below. When doing so, we list elements for the join order so as to retain the pattern of an algebra isomorphic to $\mathbf{3_n}$.



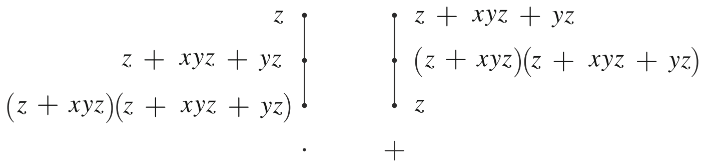
Now meets are corrected, but joins are a problem. We repair joins, and update the listing of elements of the meet order to reflect these changes. Of course, this may ruin our previously repaired meets.



Before we continue to again repair meets, we note that Birkhoff's equation $a(a+b) = a + ab$ gives the following.

$$\begin{aligned} z(z+xyz+yz) &= z+z(yz+xyz) \\ &= z+zyz(yz+x) \\ &= z+yz(yz+x) \\ &= z+yz+xyz \end{aligned}$$

So in fixing meets again, we may leave intact the top two elements of the meet order to obtain the following.



By Birkhoff's equation, $(z+xyz)(z+xyz+yz) = z+xyz+yz(z+xyz)$. So the join of the bottom two elements of the join order are correct since z will be absorbed when added to this element. To see the join of the top two elements of the join order are correct, we again use Birkhoff's identity.

$$\begin{aligned} &z+xyz+yz+(z+xyz+yz)(z+xyz) \\ =\ &(z+xyz+yz)(z+xyz+yz+z+xyz) \\ =\ &(z+xyz+yz)(z+xyz+yz) \\ =\ &z+xyz+yz \end{aligned}$$

So after the last round of repairing meets, joins are also correct.

We then have that $\{z, z+xyz+yz, (z+xyz)(z+xyz+yz)\}$ is a subalgebra of $F_{\mathsf{BS}}(3)$, and it is seen that the canonical homomorphism φ restricts to an isomorphism from this subalgebra onto $\mathbf{3_n}$. The splitting equation of $\mathbf{3_n}$ in BS is obtained by equating the terms corresponding to the elements of this algebra that generate its monolith, hence is the equation $(z+xyz)(z+yz+xyz) = z+yz+xyz$. □

An algebra P that is weakly projective in a variety V is weakly projective in any subvariety U of V that contains P. If P is subdirectly irreducible, we can consider the splitting variety V_P of P in V, and also the splitting variety U_P of P in U. Clearly $\mathsf{U}_P \subseteq \mathsf{V}_P$, and the containment can be strict. The splitting equations for P in U and V can differ. The splitting equation of P in U, together with the equations defining U, defines the variety U_P.

Proposition 4.6 [10] *The splitting equation defining* $\mathsf{BCh}_{\mathbf{3_n}}$ *is*

$$x(y+z)(xy+xz) = x(y+z)+(xy+xz). \qquad (\mathrm{S}'_{\mathbf{3_n}})$$

We note that the equation $(\mathrm{S}'_{\mathbf{3_n}})$ has a very nice form. It is $p \cdot q = p + q$ for $p = q$ the meet-distributive law (mD). Further discussion of what we call *doubled* lattice equations is given in Proposition 6.10.

Corollary 4.7 *The subvarieties of* BCh *that are varieties of lattices are the trivial variety* Triv *and the variety* DL *of distributive lattices.*

Proof Let V be a subvariety of BCh that is a variety of lattices. Then V does not contain the bichain $\mathbf{3}_\mathrm{n}$, so V satisfies the splitting equation $x(y+z)(xy+xz) = x(y+z)+(xy+xz)$. As mentioned above, this equation is of the form $pq = p + q$ where $p = x(y+z)$ and $q = xy+xz$. If a lattice satisfies $pq = p+q$, then it also satisfies $p = q$. So V is contained in the variety of distributive lattices. □

Definition 4.8 Let $\mathcal{S}$ be the set of all finite, subdirectly irreducible, weakly projective bichains with meet order $1 <. \cdots <. n$ for some n, and partially ordered by setting $C \leq D$ if C is isomorphic to a subalgebra of D.

Theorem 4.9 *The lattice* $\mathcal{L}(\mathsf{BCh})$ *of subvarieties of* BCh *contains a subposet isomorphic to the poset* $\mathcal{S}$ *of finite, subdirectly irreducible, weakly projective bichains.*

Proof We show that the map $C \mapsto V(C)$ from $\mathcal{S}$ to $\mathcal{L}(\mathsf{BCh})$ is an order-embedding. Suppose that $C, D \in \mathcal{S}$, so in particular are subdirectly irreducible and weakly projective. Then Proposition 2.17 provides that C is isomorphic to a subalgebra of D if, and only if, $V(C) \subseteq V(D)$. But C being isomorphic to a subalgebra of D by definition means that $C \leq D$ in the partial ordering of $\mathcal{S}$. This shows that $C \mapsto V(C)$ is an order-embedding. □

Corollary 4.10 *The variety* BCh *is not generated by any finite set of finite bichains.*

Proof Let $\mathcal{F}$ be a finite set of finite bichains. There is an upper bound n on the size of algebras in $\mathcal{F}$. Results of [10] show there is a subdirectly irreducible, weakly projective bichain S with $n+1$ elements. Surely $S \not\leq A$ for any $A \in \mathcal{F}$. This means that each $A \in \mathcal{F}$ belongs to the splitting variety BS_S. Hence the variety $V(\mathcal{F})$ generated by $\mathcal{F}$ is contained in BS_S. As S does not belong to BS_S and S is a bichain, BS_S does not contain BCh. So $V(\mathcal{F})$ is not equal to BCh. □

Let us note that an equation with n variables is satisfied in a bichain if and only if it is satisfied in each of its k-element subalgebras for $k \leq n$.

Corollary 4.11 *For each* $n \geq 2$*, there is a subvariety of* BCh *that is not defined by any set of equations involving only* n *variables.*

Proof Define an equivalence relation $\equiv_n$ on the collection of all finite bichains by setting $C \equiv_n C'$ if C and C' have, up to isomorphism, the same n-element sub-bichains. Note that there are only finitely many equivalence classes of $\equiv_n$. If two bichains have, up to isomorphism, the same n-element sub-bichains, then they satisfy the same equations in n variables. So members of the same equivalence class of $\equiv_n$ satisfy the same equations in n variables.

By Theorem 4.9, there are infinitely many non-isomorphic finite, subdirectly irreducible weakly projective bichains. So one of these equivalence classes must contain two non-isomorphic finite, subdirectly irreducible, weakly projective bichains S and T. By Proposition 2.17, $V(S) = V(T)$ implies that S is isomorphic to T. Consequently $V(S)$ and $V(T)$ must be different. Yet S and T satisfy exactly the same equations in n variables. It follows that $V(S)$ and $V(T)$ cannot both be defined solely by equations in n variables. □

The finite, subdirectly irreducible, weakly projective bichains play an important role in the description of the lattice $\mathcal{L}(\mathsf{BS})$ of Birkhoff systems. A recursive description of these algebras was given in [10, Sc. 6]. In the second part of this paper we will continue the investigation of these algebras, and in particular, give a description of the poset $\mathcal{S}$.

5 Varieties Generated by Three Element Bichains

The bottom part of the lattice $\mathcal{L}(\mathsf{BCh})$ of subvarieties of the variety BCh consists of varieties satisfying one, or both, distributive laws. A great deal is known about these varieties (cp. Section 3.3), both their place in the lattice of subvarieties, and the structure of the algebras belonging to them.

Theorem 5.1 *The variety* DB *of distributive bisemilattices has exactly four subvarieties,* Triv, DL, SL *and* DB. *Each subvariety of* BCh *either contains* DB *or is contained in it, and both* mDB *and* jDB *cover* DB.

Proof That DB has exactly the four given subvarieties was established in [14], as was the fact that mDB and jDB cover DB. Proposition 3.2 and Corollary 3.7 show that every subvariety of BS that is not a variety of lattices is either contained in DB or contains DB, and Corollary 4.7 shows that the only varieties of lattices that are subvarieties of BCh are contained in DB. □

Figure 3 shows a subposet of the lattice $\mathcal{L}(\mathsf{BCh})$ consisting of the variety $\mathsf{mDB} \vee \mathsf{jDB}$ and some of its subvarieties. Each of these varieties is generated by a set of bichains, and this is indicated as well.

We next show that this is in fact a complete picture of the lattice of subvarieties of $\mathsf{mDB} \vee \mathsf{jDB}$.

Proposition 5.2 *The lattice of subvarieties of* $\mathsf{mDB} \vee \mathsf{jDB}$ *is exactly as shown in Fig. 3.*

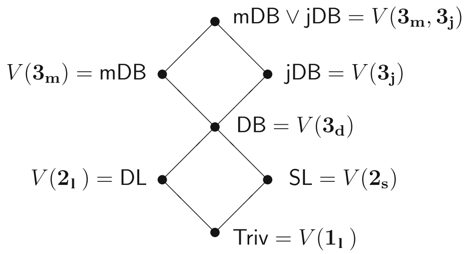


Fig. 3 The bottom of the lattice $\mathcal{L}(\mathsf{BCh})$

Proof We will show that if V is a subvariety of $\mathsf{mDB} \vee \mathsf{jDB}$ that does not contain mDB, then V is contained in jDB. By symmetry it follows that if V does not contain jDB, then V is contained in mDB. If V is a proper subvariety of $\mathsf{mDB} \vee \mathsf{jDB}$, then V cannot contain both mDB and jDB. Without loss of generality, assume that V does not contain mDB. Then we will have that V is contained in jDB. It then follows from Theorem 5.1 that V is either equal to jDB, or is one of the four subvarieties of DB. This will show that the subvarieties of $\mathsf{mDB} \vee \mathsf{jDB}$ are exactly those in Fig. 3, and the result follows.

Suppose that V is a subvariety of $\mathsf{mDB} \vee \mathsf{jDB}$. Then V satisfies every equation valid in both $\mathbf{3_m}$ and $\mathbf{3_j}$. In particular, V satisfies the following equation known as the *symmetric distributive law*, which can be easily seen to hold in both $\mathbf{3_m}$ and $\mathbf{3_j}$.

$$xy + xz + yz = (x + y)(x + z)(y + z). \tag{SD}$$

If, in addition, V does not contain mDB, then it is contained in the splitting variety of $\mathbf{3_m}$. Then by Proposition 4.5 the variety V satisfies the splitting equation ($\mathrm{S}_{\mathbf{3}_\mathrm{m}}$). Prover9 provides a 114 line proof that the two equations (SD) and ($\mathrm{S}_{\mathbf{3}_\mathrm{m}}$), together with the equations defining BS, imply the join distributive law. Thus V is contained in jDB. □

We continue to look at the bottom of the lattice $\mathcal{L}(\mathsf{BCh})$. So far, we have placed the varieties generated by all 1-element and 2-element bichains, and have also placed five of the six 3-element bichains of Fig. 1. The final 3-element bichain, the non-distributive one $\mathbf{3_n}$, also plays a key role. The following extends, and provides an alternative proof of a result of [14].

Theorem 5.3 *The variety* DB *has three covers in the lattice* $\mathcal{L}(\mathsf{BCh})$*, the varieties* mDB*,* jDB *and* $V(\mathbf{3_n})$*. Further, each subvariety of* BCh *that properly contains* DB *contains at least one of these covers.*

Proof Proposition 4.5 shows that each of $\mathbf{3_m}$, $\mathbf{3_j}$ and $\mathbf{3_n}$ is subdirectly irreducible and weakly projective. It then follows from Proposition 2.17 that there are no containments among the varieties mDB, jDB, and $V(\mathbf{3_n})$. Suppose V is a subvariety of BCh that does not contain any of mDB, jDB, or $V(\mathbf{3}_\mathrm{n})$. Then V satisfies the splitting equations ($\mathrm{S}_{\mathbf{3}_\mathrm{m}}$), ($\mathrm{S}_{\mathbf{3}_\mathrm{j}}$), and ($\mathrm{S}_{\mathbf{3}_\mathrm{n}}$) given in Proposition 4.5. Also, as V is a subvariety of BCh, it satisfies the equation (BCh3). Prover9 shows that together the equations ($\mathrm{S}_{\mathbf{3}_\mathrm{m}}$), ($\mathrm{S}_{\mathbf{3}_\mathrm{j}}$), ($\mathrm{S}_{\mathbf{3}_\mathrm{n}}$) and (BCh3) imply the meet-distributive law and the join-distributive law. Hence V is contained in DB. □

We next consider the subvarieties of BCh that are generated by bichains having at most 3 elements. The aim is to show that the lattice of Fig. 4 is a sublattice of the lattice of subvarieties of the largest such variety $V(\mathbf{3_m}, \mathbf{3_j}, \mathbf{3_n})$. In the sequel to this paper [8], we will further establish that $V(\mathbf{3_m}, \mathbf{3_n}) \wedge V(\mathbf{3_j}, \mathbf{3_n})$ is equal to $V(\mathbf{3_n})$, and obtain other results about this portion of the lattice of subvarieties.

Definition 5.4 Let B be the 4-element bichain whose join reduct is given by $1 <_+ 3 <_+ 2 <_+ 4$.

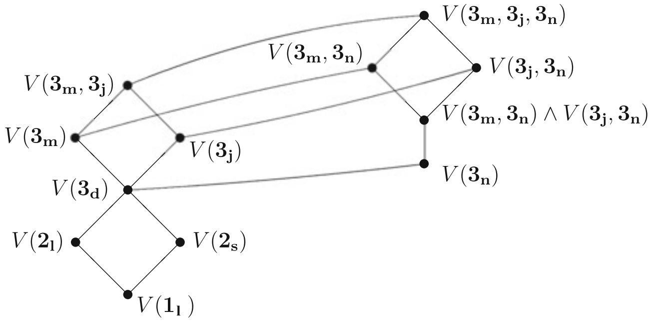


Fig. 4 Varieties generated by bichains of size ≤ 3

Lemma 5.5 *The variety* $V(\mathbf{3_m}, \mathbf{3_j})$ *is contained in the variety* $V(B)$. *The varieties* $V(\mathbf{3_n})$ *and* $V(B)$ *are incomparable, and the varieties* $V(\mathbf{3_m}, \mathbf{3_j})$ *and* $V(\mathbf{3_n})$ *are incomparable.*

Proof We have seen in Proposition 4.5 that each of $\mathbf{3_m}$, $\mathbf{3_j}$, and $\mathbf{3_n}$ is subdirectly irreducible and weakly projective. By Theorem 4.1, the same is true of the bichain B. It is clear that both of $\mathbf{3_m}$ and $\mathbf{3_j}$ are isomorphic to subalgebras of B, so $V(\mathbf{3_m}, \mathbf{3_j}) = V(\mathbf{3_m}) \vee V(\mathbf{3_j})$ is contained in $V(B)$. Similarly, it is clear that $\mathbf{3_n}$ is not isomorphic to a subalgebra of B, and surely B is not isomorphic to a subalgebra of $\mathbf{3_n}$. Hence Proposition 2.17 gives that $V(\mathbf{3_n})$ is incomparable to $V(B)$. Finally, by Theorem 5.3, neither $V(\mathbf{3_m})$ nor $V(\mathbf{3_j})$ is contained in $V(\mathbf{3_n})$. We then have that $V(\mathbf{3_m}, \mathbf{3_j})$ is not contained in $V(\mathbf{3_n})$, and $V(\mathbf{3_n})$ cannot be contained in $V(\mathbf{3_m}, \mathbf{3_j})$ since the latter is contained in $V(B)$. □

Lemma 5.6 *The varieties in Fig. 4 that are contained in* $V(\mathbf{3_m}, \mathbf{3_j})$ *are distinct, and none is equal to any of the varieties in this figure that contain* $V(\mathbf{3_n})$. *Further,* $V(\mathbf{3_n})$ *is distinct from* $V(\mathbf{3_m}, \mathbf{3_j}, \mathbf{3_n})$.

Proof Earlier results have shown that each of the listed subvarieties of $V(\mathbf{3_m}, \mathbf{3_j})$ in Fig. 4 are distinct. By Lemma 5.5, $V(\mathbf{3_m}, \mathbf{3_j})$ and $V(\mathbf{3_n})$ are incomparable, so none of the listed varieties containing $V(\mathbf{3_n})$ can be equal to any of the varieties contained in $V(\mathbf{3_m}, \mathbf{3_j})$. Also, the incomparability of $V(\mathbf{3_n})$ and $V(\mathbf{3_m}, \mathbf{3_j})$ implies that $V(\mathbf{3_m}, \mathbf{3_j}, \mathbf{3_n})$ cannot be equal to $V(\mathbf{3_n})$. □

Lemma 5.7 *Neither of the varieties* $V(\mathbf{3_m}, \mathbf{3_n})$ *nor* $V(\mathbf{3_j}, \mathbf{3_n})$ *is equal to* $V(\mathbf{3_m}, \mathbf{3_j}, \mathbf{3_n})$. *Moreover, the varieties* $V(\mathbf{3_m}, \mathbf{3_n})$ *and* $V(\mathbf{3_j}, \mathbf{3_n})$ *are incomparable.*

Proof Suppose one of $V(\mathbf{3_m}, \mathbf{3_n})$ or $V(\mathbf{3_j}, \mathbf{3_n})$ is equal to $V(\mathbf{3_m}, \mathbf{3_j}, \mathbf{3_n})$. Then by symmetry both are equal to $V(\mathbf{3_m}, \mathbf{3_j}, \mathbf{3_n})$. Proposition 2.6 shows that $\mathcal{L}(\mathsf{BS})$ satisfies join-semidistributivity (SD$\vee$). Then as $V(\mathbf{3_m}, \mathbf{3_n}) = V(\mathbf{3_m}) \vee V(\mathbf{3_n})$ and $V(\mathbf{3_j}, \mathbf{3_n}) = V(\mathbf{3_j}) \vee V(\mathbf{3_n})$, it follows from join-semidistributivity that x

$$V(\mathbf{3_n}) \vee (V(\mathbf{3_m}) \wedge V(\mathbf{3_j})) = V(\mathbf{3_m}, \mathbf{3_j}, \mathbf{3_n}).$$

Since $V(\mathbf{3_m}) \wedge V(\mathbf{3_j}) = V(\mathbf{3_d})$, and $V(\mathbf{3_d})$ is contained in $V(\mathbf{3_n})$, it follows that $V(\mathbf{3_n}) = V(\mathbf{3_m}, \mathbf{3_j}, \mathbf{3_n})$, a contradiciton. Thus neither of $V(\mathbf{3_m}, \mathbf{3_n})$ nor $V(\mathbf{3_j}, \mathbf{3_n})$ is equal to $V(\mathbf{3_m}, \mathbf{3_j}, \mathbf{3_n})$. This additionally implies that $V(\mathbf{3_m}, \mathbf{3_n})$ and $V(\mathbf{3_j}, \mathbf{3_n})$ are incomparable, as if one of them contained the other, the larger would be $V(\mathbf{3_m}, \mathbf{3_j}, \mathbf{3_n})$. □

Theorem 5.8 *The lattice of Fig. 4, or its quotient collapsing* $V(\mathbf{3_n})$ *and* $V(\mathbf{3_m}, \mathbf{3_n}) \wedge V(\mathbf{3_j}, \mathbf{3_n})$*, is a sublattice of* $\mathcal{L}(\mathsf{BCh})$.

Proof Lemmas 5.6 and 5.7 show that the varieties in Fig. 4 are distinct, except possibly that $V(\mathbf{3_n})$ is equal to $V(\mathbf{3_m}, \mathbf{3_n}) \wedge V(\mathbf{3_j}, \mathbf{3_n})$. Indeed, Lemma 5.6 shows that all varieties contained in $V(\mathbf{3_m}, \mathbf{3_j})$ are distinct and distinct from all varieties containing $V(\mathbf{3_n})$. Lemma 5.7 shows that $V(\mathbf{3_m}, \mathbf{3_n})$ and $V(\mathbf{3_j}, \mathbf{3_n})$ are incomparable, and thus they, and their meet, $V(\mathbf{3_m}, \mathbf{3_n}) \wedge V(\mathbf{3_j}, \mathbf{3_n})$, and join $V(\mathbf{3_m}, \mathbf{3_j}, \mathbf{3_n})$ are also distinct.

The containments indicated in Fig. 4 obviously hold. No other containments can hold, except possibly that $V(\mathbf{3_m}, \mathbf{3_n}) \wedge V(\mathbf{3_j}, \mathbf{3_n})$ is contained in $V(\mathbf{3_n})$. Indeed, Proposition 5.2 shows that no other containments hold among subvarieties of $V(\mathbf{3_m}, \mathbf{3_j})$, and Lemma 5.6 shows that no variety containing $V(\mathbf{3_n})$ is contained in any variety contained in $V(\mathbf{3_m}, \mathbf{3_j})$. Finally, any additional containments among varieties containing $V(\mathbf{3_n})$ would contradict the incomparability of $V(\mathbf{3_m}, \mathbf{3_n})$ and $V(\mathbf{3_j}, \mathbf{3_n})$. Thus, the poset of Fig. 4, or its quotient collapsing the two indicated varieties, is order-isomorphic to a subposet of $\mathcal{L}(\mathsf{BCh})$.

That each of the joins in Fig. 4 is a join in $\mathcal{L}(\mathsf{BCh})$ follows from earlier results on subvarieties of $V(\mathbf{3_m}, \mathbf{3_j})$, from obvious containments, or from the equality $V(S) \vee V(T) = V(S \cup T)$, which holds for sets of bichains S and T.

It remains to show that meets in Fig. 4 agree with those in $\mathcal{L}(\mathsf{BCh})$. For two varieties that are contained in $V(\mathbf{3_m}, \mathbf{3_j})$ this was given in Proposition 5.2. For two varieties containing $V(\mathbf{3_n})$ the only case that does not involve comparable varieties is the meet of $V(\mathbf{3_m}, \mathbf{3_n})$ and $V(\mathbf{3_j}, \mathbf{3_n})$ and their meet in this figure is by definition their meet in $\mathcal{L}(\mathsf{BCh})$. Finally, that any meet involving a variety containing $V(\mathbf{3_n})$ and a variety contained in $V(\mathbf{3_m}, \mathbf{3_j})$ is as shown in the Figure follows from the fact established in Proposition 5.2 that the only subvarieties of $V(\mathbf{3_m}, \mathbf{3_j})$ are the ones shown in the Figure. □

Several additional facts can be gleaned from Fig. 4.

Proposition 5.9 *The variety* $V(\mathbf{3_m}, \mathbf{3_n})$ *covers* $V(\mathbf{3_m})$*, and* $V(\mathbf{3_j}, \mathbf{3_n})$ *covers* $V(\mathbf{3_j})$.

Proof We have seen in Proposition 4.5 that each of $\mathbf{3_m}$, $\mathbf{3_j}$, and $\mathbf{3_n}$ is subdirectly irreducible and weakly projective. Proposition 4.5 also gives the splitting equations ($S_{\mathbf{3_m}}$), ($S_{\mathbf{3_j}}$), and ($S_{\mathbf{3_n}}$). Using Prover9, one shows that together the equations ($S_{\mathbf{3_j}}$), ($S_{\mathbf{3_n}}$), and the equation (BCh3) earlier seen to be valid in all bichains, imply the meet-distributive law (mD) that defines $V(\mathbf{3_m})$. So any subvariety of BCh that does not contain $V(\mathbf{3_j})$ or $V(\mathbf{3_n})$ is contained in $V(\mathbf{3_m})$.

Suppose V is a variety that is strictly contained in $V(\mathbf{3_m}, \mathbf{3_n})$ and contains $V(\mathbf{3_m})$. Since V is strictly contained in $V(\mathbf{3_m}, \mathbf{3_n})$ and contains $V(\mathbf{3_m})$, it cannot contain $V(\mathbf{3_n})$. Also V cannot contain $V(\mathbf{3_j})$ since V is contained in $V(\mathbf{3_m}, \mathbf{3_n})$ and Proposition 5.7 shows that $V(\mathbf{3_m}, \mathbf{3_n})$ does not contain $V(\mathbf{3_j})$. So the result of the paragraph above shows that V is contained in $V(\mathbf{3_m})$, hence is equal to $V(\mathbf{3_m})$. It follows that $V(\mathbf{3_m}, \mathbf{3_n})$ covers $V(\mathbf{3_m})$. The argument that $V(\mathbf{3_j}, \mathbf{3_n})$ covers $V(\mathbf{3_j})$ is dual. □

As a final comment, let us note that knowing if every subvariety of BCh is generated by the bichains it contains would be of the utmost value in determining the subvarieties of BCh.

6 The Lattice of Subvarieties of Birkhoff Systems

In this section we provide a description of portions of the lattice $\mathcal{L}(\mathsf{BS})$ of subvarieties of BS. Our results are summarized in Fig. 5. We note that this figure is not intended to show

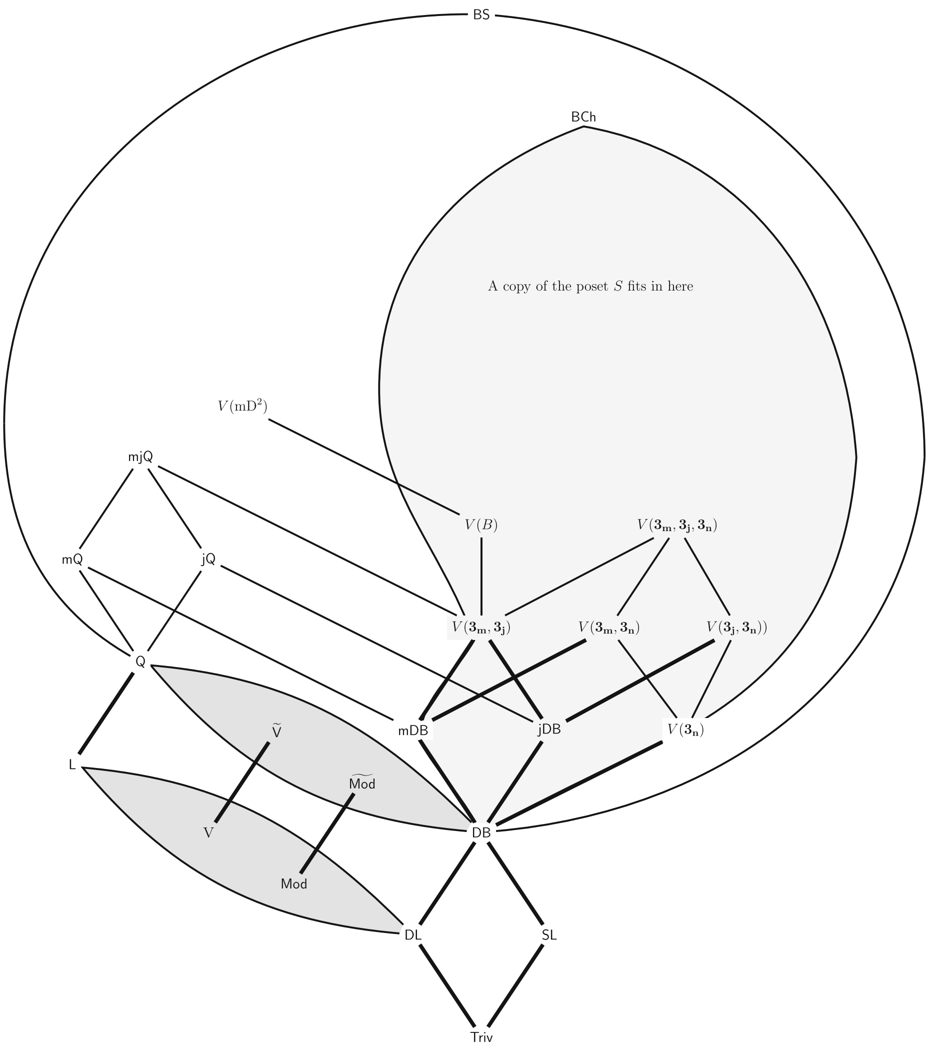


Fig. 5 Large scale view of the lattice of subvarieties of BS. Thick lines represent covers, thin lines indicate proper containment. It does not indicate joins and meets.

joins and meets, only containments. All of the containments shown are proper. Coverings are shown by thick lines. A number of further results, such as those involving certain joins and meets, are also developed in this section but are not indicated in the figure. We begin by reviewing results already obtained.

Theorem 6.1 *The lattice of subvarieties of the variety* Q *of quasilattices is isomorphic to the product of the two-element lattice and the lattice of subvarieties of the variety* L *of lattices. In particular, for each variety* V *of lattices, its regularization* $\tilde{\mathsf{V}}$ *covers* V.

Proof This follows from Padmanabahn's result [15] stated in Theorem 3.5 that Q is the regularization of L, and the result of Dudek and Graczyńska given in Theorem 2.10. See the discussion in Section 3. □

This shows that the portion of Fig. 5 beneath the variety Q is complete and correct, with covers as indicated. Our results in Section 5, in particular Proposition 5.2, Theorems 5.3 and 5.8, and Proposition 5.9 show that the portion of Fig. 5 beneath $V(\mathbf{3_m}, \mathbf{3_j}, \mathbf{3_n})$ is correct with respect to proper containments and covers. Additionally, these results show the portion beneath $V(\mathbf{3_m}, \mathbf{3_j})$ is complete, and that all joins and meets beneath $V(\mathbf{3_m}, \mathbf{3_j}, \mathbf{3_n})$ are as indicated, with the possible exception of $V(\mathbf{3_m}, \mathbf{3_n}) \wedge V(\mathbf{3_j}, \mathbf{3_n})$ (which will be established in [8]). These results then show that all meets of varieties beneath either Q or $V(\mathbf{3_m}, \mathbf{3_j}, \mathbf{3_n})$ are as shown. No attempt has been made to show joins of varieties beneath Q with varieties beneath $V(\mathbf{3_m}, \mathbf{3_j}, \mathbf{3_n})$.

Definition 6.2 Using the equations (mQ) and (jQ) of Definition 3.4 make the following definitions.

1. mQ is the subvariety of BS defined by (mQ),
2. jQ is the subvariety of BS defined by (jQ),
3. mjQ is the join of the varieties mQ and jQ.

We call mQ the variety of *meet quasilattices* and jQ the variety of *join quasilattices*. We note that the meet of mQ and jQ is by definition the variety Q of quasilattices.

Theorem 6.3 *For the splitting varieties* $\mathsf{BS}_{\mathbf{3_j}}$, $\mathsf{BS}_{\mathbf{3_m}}$, $\mathsf{BS}_{\mathbf{3_n}}$ *of the bichains* $\mathbf{3_m}, \mathbf{3_j}, \mathbf{3_n}$, *we have* $\mathsf{mQ} = \mathsf{BS}_{\mathbf{3_j}} \wedge \mathsf{BS}_{\mathbf{3_n}}$, $\mathsf{jQ} = \mathsf{BS}_{\mathbf{3_m}} \wedge \mathsf{BS}_{\mathbf{3_n}}$, *and* $\mathsf{Q} = \mathsf{BS}_{\mathbf{3_j}} \wedge \mathsf{BS}_{\mathbf{3_m}} \wedge \mathsf{BS}_{\mathbf{3_n}}$.

Proof The splitting equations defining the varieties $\mathsf{BS}_{\mathbf{3_j}}$, $\mathsf{BS}_{\mathbf{3_m}}$, and $\mathsf{BS}_{\mathbf{3_n}}$ are the equations ($S_{\mathbf{3_j}}$), ($S_{\mathbf{3_m}}$), and ($S_{\mathbf{3_n}}$) given in Proposition 4.5. Using Prover9, there is a 131 line proof of (mQ) from ($S_{\mathbf{3_j}}$), ($S_{\mathbf{3_n}}$), and the equations defining BS. This shows that $\mathsf{BS}_{\mathbf{3_j}} \wedge \mathsf{BS}_{\mathbf{3_n}} \subseteq \mathsf{mQ}$. It is easily seen that neither $\mathbf{3_j}$ nor $\mathbf{3_n}$ satisfies (mQ), hence do not belong to mQ. This shows that mQ is contained in both splitting varieties $\mathsf{BS}_{\mathbf{3_j}}$ and $\mathsf{BS}_{\mathbf{3_n}}$, hence establishes the first statement. The second is dual. The third then follows from the first two as Q is by definition the meet of the varieties mQ and jQ. □

Corollary 6.4 *There are five covers of* DB *in* BS, *the varieties* mDB, jDB, $V(\mathbf{3_n})$, *and the regularizations of the varieties* $V(M_5)$ *and* $V(N_5)$ *generated by the two non-distributive 5-element lattices. Further, each subvariety of* BS *that properly contains* DB *contains one of these covers.*

Proof We have already noted that each of mDB, jDB, and $V(\mathbf{3_n})$ cover DB. The results above describing the lattice of subvarieties of Q, along with standard results about varieties of lattices, show that the regularizations of $V(M_5)$ and $V(N_5)$ are covers of DB, and that any subvariety of Q that properly contains DB contains one of these two covers. In particular, each of the indicated varieties is indeed a cover of DB.

Suppose V is a subvariety of BS that properly contains DB. If V does not contain any of mDB, jDB, or $V(\mathbf{3_n})$, then V is contained in the splitting variety of each of these varieties, hence V is contained in $\mathsf{BS}_{\mathbf{3_j}} \wedge \mathsf{BS}_{\mathbf{3_m}} \wedge \mathsf{BS}_{\mathbf{3_n}}$. By Proposition 6.3, V is contained in Q. It is well known [11] that any variety of lattices that properly contains DL contains one of its two covers $V(M_5)$ or $V(N_5)$. Since V is contained in Q and properly contains the regular variety DB, it follows from the discussion after Theorem 3.5 that V contains the regularization of $V(M_5)$ or $V(N_5)$. □

Proposition 6.5 *The following describe the meets of the indicated varieties:* mQ ∧ BCh = mDB, jQ ∧ BCh = jDB, *and* Q ∧ BCh = DB.

Proof It is easily seen that (mD) implies (mQ) and (jD) implies (jQ). So mDB ⊆ mQ, jDB ⊆ jQ, and DB ⊆ Q. Prover9 gives a 96 line proof that the equations defining BS, the equations (BCh2) and (BCh3) valid in all bichains, and the equation (mQ), imply the equation (mD). This shows mQ ∧ BCh ⊆ mDB, hence equality. The second statement is dual. The third follows since Q = mQ ∧ jQ and BD = mDB ∧ jDB. □

We note that we do not know whether the meet of mjQ and BCh is the variety $V(\mathbf{3_m}, \mathbf{3_j})$.

Proposition 6.6 *The following describe the meets of the indicated varieties:* $\mathsf{BCh}_{\mathbf{3_j}} \wedge \mathsf{BCh}_{\mathbf{3_n}} = \mathsf{mDB}$ *and* $\mathsf{BCh}_{\mathbf{3_m}} \wedge \mathsf{BCh}_{\mathbf{3_n}} = \mathsf{jDB}$.

Proof Since $\mathsf{BCh}_{\mathbf{3_j}}$ and $\mathsf{BCh}_{\mathbf{3_n}}$ are the splitting varieties of $\mathbf{3_j}$ and $\mathbf{3_n}$ in the variety BCh, and neither $\mathbf{3_j}$ nor $\mathbf{3_n}$ belongs to mDB, we have $\mathsf{BCh}_{\mathbf{3_j}} \wedge \mathsf{BCh}_{\mathbf{3_n}} \supseteq \mathsf{mDB}$. For the other containment, Prover9 gives a 369 line proof that the equations defining BS, the splitting equations ($S_{\mathbf{3_j}}$) and ($S_{\mathbf{3_n}}$) for $\mathbf{3_j}$ and $\mathbf{3_n}$ in BS, and equations (BCh2) and (BCh3) valid in all bichains, imply the equation (mD). The second statement is dual. □

Corollary 6.7 *Every subvariety of* BCh *that is not contained in either* mDB *or* jDB *contains either* $V(\mathbf{3_n})$ *or* $V(\mathbf{3_m}, \mathbf{3_j})$.

Proof Suppose V is a subvariety of BCh and that V does not contain $V(\mathbf{3_n})$ or $V(\mathbf{3_m}, \mathbf{3_j})$. Since V does not contain $V(\mathbf{3_n})$, we have $\mathsf{V} \subseteq \mathsf{BCh}_{\mathbf{3_n}}$. Since V does not contain $V(\mathbf{3_m}, \mathbf{3_j})$ we have that either V does not contain $V(\mathbf{3_m})$ or V does not contain $V(\mathbf{3_j})$. In the first case $\mathsf{V} \subseteq \mathsf{BCh}_{\mathbf{3_m}}$, and in the second $\mathsf{V} \subseteq \mathsf{BCh}_{\mathbf{3_j}}$. Then Proposition 6.6 gives that V is contained in either mDB or jDB. □

We next turn our attention to matters related to the 4-element bichain B given in Definition 5.4. The variety $V(B)$ was studied in [9], where it was shown to be the variety generated by the meet and join reduct of the truth value algebra of type-2 fuzzy sets.

Proposition 6.8 *The variety* $V(B)$ *properly contains* $V(\mathbf{3}_\mathrm{m}, \mathbf{3}_\mathrm{j})$.

Proof Lemma 5.5 shows that $V(\mathbf{3_m}, \mathbf{3_j})$ is contained in $V(B)$. The result then follows as B being subdirectly irreducible and weakly projective implies there is a largest variety not containing it. However, it is perhaps useful to have a more direct argument. Consider the following equation:

$$(y+u)(xy+x+z) = (y+u)(yz+x+z). \qquad (\varrho)$$

It is easily checked that this equation holds in both $\mathbf{3_m}$ and $\mathbf{3_j}$, hence holds in $V(\mathbf{3_m}, \mathbf{3_j})$. Alternately, a direct proof of (ϱ) from either the meet-distributive law, or join-distributive law, is easily obtained. But (ϱ) does not hold in B as is seen by substituting 1 for x, 2 for y, 3 for z and 4 for u. □

We note that (ϱ) fails also in $\mathbf{3_n}$.

Definition 6.9 For a bisemilattice equation (ε) $p = q$, we define its *doubling* to be the equation (ε^2) $p \cdot q = p + q$. We then let $V(\varepsilon^2)$ be the subvariety of BS defined by (ε^2).

The notion of the doubling of an equation arose in consideration of the variety $V(B)$ in [9]. There it was conjectured that $V(B)$ is the splitting variety of $\mathbf{3_n}$ in BCh, and it was shown that this splitting variety of $\mathbf{3_n}$ in BCh is defined by the doubled version of the meet distributive law (mD) (see Proposition 4.6). This points to a path to create a many interesting varieties of Birkhoff systems as described in the following result.

Proposition 6.10 *Suppose* (ε) *is an equation for bisemilattices and* V *is the variety defined by the lattice equations and* (ε). *Then* V *is covered by its regularization* $\tilde{\mathsf{V}}$, *and* $\tilde{\mathsf{V}}$ *is contained in* $V(\varepsilon^2)$.

Proof That V covered by $\tilde{\mathsf{V}}$ is part of Theorem 6.1. It is easily seen that if (ε) is valid in a Birkhoff system, then the idempotence of the operations $\cdot$ and $+$ give that (ε^2) is also valid in this Birkhoff system. For lattices the converse is true as well, but we don't need this fact. Then (ε^2) is valid in V and, as this equation is regular, it is valid in $\tilde{\mathsf{V}}$. Thus $\tilde{\mathsf{V}}$ is contained in $V(\varepsilon^2)$. □

The variety $V(\text{mD}^2)$ obtained from the doubling of the meet distributive law appears in Fig. 5. No others have been placed here because we do not understand their theory so well. In particular, we do not know if an equation (β) being a consequence of an equation (α) for lattices implies (β^2) is a consequence of (α^2) for BS.

Remark 6.11 For the meet and join-distributive laws (mD) and (jD), we attempted to prove that (mD^2) and (jD^2) were equivalent in BS using Prover9. Using a 2013 MacBook, the run was terminated after approximately 5 days. Later, Thomas Hillebrand used Waldmeister to show that these statements are equivalent in BS. This seems an interesting area to test the performance of equational theorem provers.

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