Varieties of Birkhoff Systems Part II

John Harding & Anna B. Romanowska

Order
A Journal on the Theory of Ordered Sets
and its Applications

ISSN 0167-8094

Order
DOI 10.1007/s11083-016-9392-1

Order
DOI 10.1007/s11083-016-9392-1

Varieties of Birkhoff Systems Part II

John Harding[1] · **Anna B. Romanowska**[2]

Received: 18 March 2015 / Accepted: 21 March 2016

Abstract This is the second part of a two-part paper on Birkhoff systems. A Birkhoff system is an algebra that has two binary operations $\cdot$ and $+$, with each being commutative, associative, and idempotent, and together satisfying $x \cdot (x + y) = x + (x \cdot y)$. The first part of this paper described the lattice of subvarieties of Birkhoff systems. This second part continues the investigation of subvarieties of Birkhoff systems. The 4-element subdirectly irreducible Birkhoff systems are described, and the varieties they generate are placed in the lattice of subvarieties. The poset of varieties generated by finite splitting bichains is described. Finally, a structure theorem is given for one of the five covers of the variety of distributive Birkhoff systems, the only cover that previously had no structure theorem. This structure theorem is used to complete results from the first part of this paper describing the lower part of the lattice of subvarieties of Birkhoff systems.

Keywords Birkhoff system · Variety · Splitting · Projective · Płonka sum · Lallement sum

1 Introduction

The first part of this paper [1] dealt with a description of the lattice $\mathcal{L}(\mathsf{BS})$ of subvarieties of Birkhoff systems. All notation and results from that paper will be carried over here. In particular, BS is the variety of Birkhoff systems, and DB, mDB, jDB are the varieties of

✉ John Harding
jharding@nmsu.edu

Anna B. Romanowska
A.Romanowska@mini.pw.edu.pl

1 Department of Mathematical Sciences, New Mexico State University, Las Cruces, NM 88003, USA

2 Faculty of Mathematics and Information Science, Warsaw University of Technology, 00-662 Warsaw, Poland

Published online: 08 April 2016

distributive Birkhoff systems, those that satisfy both distributive laws, and the varieties of meet-distributive and join-distributive Birkhoff systems.

It was shown in the first part that the poset $\mathcal{S}$ of finite, subdirectly irreducible, weakly projective bichains is embedded in $\mathcal{L}(\mathsf{BS})$. This poset $\mathcal{S}$ is infinite, and has many interesting properties. The second section of this paper describes properties of this poset $\mathcal{S}$. In the third section we describe all 4-element subdirectly irreducible Birkhoff systems, and place the varieties they generate in $\mathcal{L}(\mathsf{BS})$.

The fourth section gives a structure theorem for the largest subvariety $V(\mathrm{S}_{\mathbf{3_m}}, \mathrm{S}_{\mathbf{3_j}})$ of BS that does not contain either mDB or jDB. We show that each element of this variety is obtained from its subalgebras that are lattices using a generalization of Płonka sums known as Lallement sums [9]. This result is specialized to obtain a structure theorem for the variety $V(\mathbf{3_n})$ generated by the bichain $\mathbf{3_n}$. It follows that $V(\mathrm{S}_{\mathbf{3_m}}, \mathrm{S}_{\mathbf{3_j}})$ is the Mal'cev product [5] $\mathsf{L} \circ \mathsf{SL}$ of the varieties L of lattices and SL of semilattices within the variety BS.

In the fifth section, the results of the fourth are applied to describe the finite subdirectly irreducible algebras that belong to the variety $V(\mathbf{3_n})$. These are shown to be the bichains $\mathbf{2_l}$, $\mathbf{2_s}$, $\mathbf{3_d}$, $\mathbf{3_n}$, and one additional 4-element bichain we call $\mathbf{4_n}$. This allows us to give an equational axiomatization of the variety $V(\mathbf{3_n})$, and ultimately, to complete portions of the description of the lattice of subvarieties of BS left open in the first part of this paper [1].

The final section contains a list of open problems. The reader should consult the first part of this paper [1] for further background, and all notions not explicitly defined here.

2 The Poset $\mathcal{S}$

In this section, we consider the structure of the poset $\mathcal{S}$ of finite subdirectly irreducible, weakly projective bichains. We begin by recalling the definition from the first part of the paper [1, Def. 4.8].

Definition 2.1 Let $\mathcal{S}$ be the set of all finite, subdirectly irreducible, weakly projective bichains with meet order $1 <_{\cdot} \cdots <_{\cdot} n$ for some n, and partially ordered by setting $C \leq D$ if C is isomorphic to a subalgebra of D.

A recursive description of the finite subdirectly irreducible, weakly projective bichains was given in [2, Sc. 6]. In what follows, the greatest element of the join reduct of a bichain will be called its *join top* and the smallest element (or smallest two elements) of the join reduct its *join bottom*.

Theorem 2.2 *The set of all n-element weakly projective subdirectly irreducible bichains can be partitioned into the following groups:*

- $\mathcal{A}_n$ *is the set of all n-element bichains C with n as the join top element and such that $C \setminus \{n\}$ is subdirectly irreducible and weakly projective and not having $n-1$ as its join top.*
- $\mathcal{B}_n$ *is the set of all n-element bichains C with n as the join bottom and such that $C \setminus \{n\}$ is subdirectly irreducible and weakly projective and not having $n-1$ as its join bottom.*
- $\mathcal{C}_n$ *is the set of all n-element bichains C with $1 <_{+} n$ as the join bottom and such that $C \setminus \{1, n\}$ is subdirectly irreducible and weakly projective and not having $n-1$ as its join bottom.*

Using this recursive description, the bottom part of the poset $\mathcal{S}$ is constructed in Fig. 1. We also have the following result given in [2].

Corollary 2.3 ([2]) *The number, up to isomorphism, of subdirectly irreducible, weakly projective bichains with n elements is the* $(n+1)^{st}$ *Fibonacci number.*

We consider further properties of the poset $\mathcal{S}$.

Proposition 2.4 *Consider the groups of bichains* $\mathcal{A}_n$, $\mathcal{B}_n$ *and* $\mathcal{C}_n$ *of Theorem 2.2 in regards to the partial ordering of* $\mathcal{S}$.

(1) *Each member of* $\mathcal{A}_{n+1}$ *covers a member of* $\mathcal{B}_n$ *or* $\mathcal{C}_n$.
(2) *Each member of* $\mathcal{B}_{n+1}$ *covers a member of* $\mathcal{A}_n$ *or* $\mathcal{C}_n$.
(3) *Each member of* $\mathcal{C}_{n+1}$ *covers a member of* $\mathcal{B}_n$.

Proof Suppose C is an $(n+1)$-element bichain whose meet order is $1 <. \cdots <. n+1$. Lemma 6.5 of [2] shows that $C \in \mathcal{A}_{n+1}$ if and only if $n+1$ is the top of the join order of C and $C \setminus \{n+1\}$ belongs to $\mathcal{B}_n \cup \mathcal{C}_n$. Lemma 6.6 of [2] shows that $C \in \mathcal{B}_{n+1}$ if and only if $n+1$ is the bottom of the join order of C and $C \setminus \{n+1\}$ belongs to $\mathcal{A}_n \cup \mathcal{C}_n$. Lemma

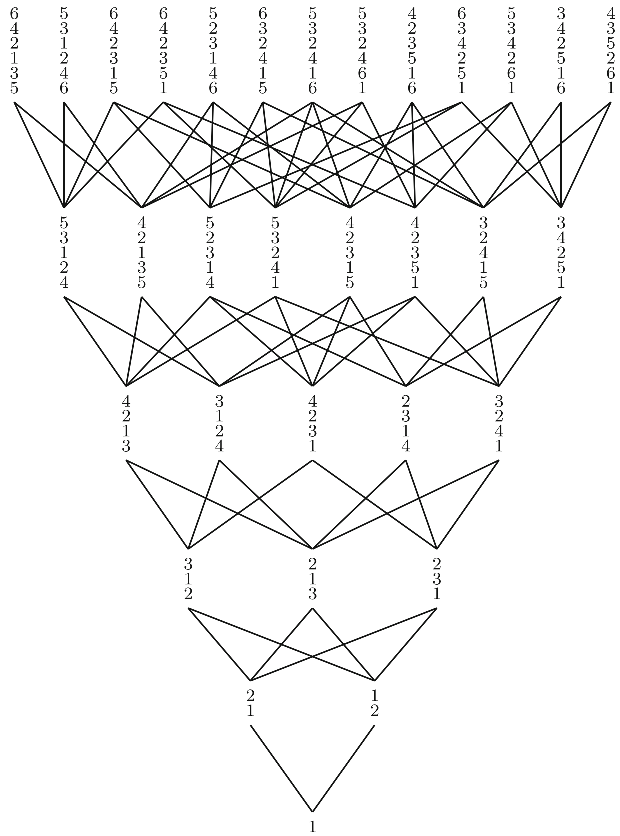


Fig. 1 The bottom portion of the poset $\mathcal{S}$ of finite, subdirectly irreducible, weakly projective bichains

6.7 of [2] shows that $C \in \mathcal{C}_{n+1}$ if and only if $1 <_+ n+1$ is the bottom of the join order of C and $C \setminus \{1, n+1\}$ is isomorphic to a member of $\mathcal{A}_{n-1} \cup \mathcal{C}_{n-1}$. The first two statements follow immediately.

For the third, suppose $C \in \mathcal{C}_{n+1}$. By Lemma 6.7 of [2], $C \setminus \{1, n+1\}$ is isomorphic to a member of $\mathcal{A}_{n-1} \cup \mathcal{C}_{n-1}$. But $C \setminus \{1\}$ is formed from $C \setminus \{1, 2\}$ by placing a new element $n+1$ on the top of its meet order and bottom of its join order. Hence, by Lemma 6.6 of [2], $C \setminus \{1\}$ is isomorphic to a member of $\mathcal{B}_n$. □

Remark 2.5 We refer to the members of $\mathcal{S}$ with n elements as the n^{th} level of $\mathcal{S}$. The result above shows that each member of the $(n+1)^{st}$ level covers a member of the n^{th} level, and Lemmas 6.5 and 6.6 of [2] show that each member of the n^{th} level is covered by a member of the $(n+1)^{st}$ level. We do not however know whether the only covers of a member of the n^{th} level belong to the $(n+1)^{st}$ level, i.e. if the ordering of the poset is determined by the containments between members of adjacent levels. In the portion of the poset depicted in Fig. 1 this is indeed the case.

We next introduce another partition of $\mathcal{S}$ that will help to describe some of the symmetries of $\mathcal{S}$. For this, we recall that there are four types of distributivity of bichains. A bichain may be (fully) distributive (i.e. both join- and meet-distributive), it may be only join-distributive, or only meet-distributive, or may be non-distributive.

Definition 2.6 The *distributivity type* of a bichain A depends on the kinds of distributivity of the 3-element subbichains of A which are not fully distributive. The type is t, if all such subbichains are isomorphic to $\mathbf{3}_\mathsf{t}$, for a unique $\mathsf{t} \in \{\mathsf{j}, \mathsf{m}, \mathsf{n}\}$, it is tu, if they are isomorphic to precisely two $\mathbf{3}_\mathsf{t}$ and $\mathbf{3}_\mathsf{u}$ of them (i.e. $\mathsf{t}, \mathsf{u} \in \{\mathsf{j}, \mathsf{m}, \mathsf{n}\}$), and finally it is jmn, if A contains 3-element subbichains of all three types.

In Fig. 1, the only distributive bichains of $\mathcal{S}$ are the two 2-element bichains of the second level. Each of the bichains of the third level has a singular type (j, n and m from left to right). All bichains of the fourth level have a double type (jn, jn, jm, mn, mn from left to right).

Definition 2.7 For each bichain $A \in \mathcal{S}$, the dual A^d of A is isomorphic to a unique member of $\mathcal{S}$ we call A^δ. This defines a map $\delta : \mathcal{S} \to \mathcal{S}$ we call the *duality map*.

Clearly δ is an automorphism of the poset $\mathcal{S}$ that is of period two. The following observation about the interaction of δ with distributivity types follows immediately from [1, Prop. 2.2].

Proposition 2.8 *Suppose A is a bichain in $\mathcal{S}$.*

(1) *A has j in its type if, and only if, A^δ has m in its type.*
(2) *A has m in its type if, and only if, A^δ has j in its type.*
(3) *A has n it its type if, and only if, A^δ has n in its type.*

So the duality map δ takes a bichain of type jn to one of mn, one of type jm to perhaps a different bichain of type jm, and one of type jmn to one of type jmn, and so forth. The axis of the symmetry δ goes through the trivial bichain of the first level, the bichain $\mathbf{3_n}$ on the third level, and the bichain B with the join-reduct 1324 in the middle of the fourth level.

Note that both $\mathbf{3_n}$ and B are self-dual. Other self-dual bichains are centered in the middle of Fig. 1 as well.

Lemma 2.9 *Let A be an n-element bichain whose meet order is given by $1 <_{.} \cdots <_{.} n$. Then the addition of one new element to A will change the distributivity type of A according to the following rules.*

(1) *The addition of a new element $n+1$ on the top of both the meet and join orders of A results in a possible addition of only* j *to the type of A.*
(2) *The addition of a new element $n+1$ on the top of the meet order and bottom of the join order of A results in a possible addition of only* n *to the type of A.*
(3) *First renumber elements of A as $2, 3, \ldots, n+1$ in place of $1, 2, \ldots, n$. The addition of a new element 1 to A on the bottom of both the meet and join orders results in a possible addition of only* m *to the type of A.*

Proof This is obvious from the definition of the type of a bichain, and the specific nature of the bichains $\mathbf{3_m}$, $\mathbf{3_j}$ and $\mathbf{3_n}$. □

Definition 2.10 For $n \geq 3$, define recursively bichains A_n and B_n by setting $A_3 = \mathbf{3_j}$ and $B_3 = \mathbf{3_n}$, and then for $n \geq 3$ defining

A_{n+1} is built from B_n by method (1) of Lemma 2.9,
B_{n+1} is built from A_n by method (2) of Lemma 2.9.

Then set $\mathcal{JN} = \{A_n, B_n \mid n \geq 4\}$.

In Fig. 1, $\mathcal{JN}$ consists of two left-most bichains from each level.

Proposition 2.11 *For $n \geq 3$, the join orders of A_n and B_n are:*

$$A_n: \quad n-1 <_+ n-3 <_+ n-5 <_+ \cdots <_+ n-4 <_+ n-2 <_+ n.$$
$$B_n: \quad n <_+ n-2 <_+ n-4 <_+ \cdots <_+ n-5 <_+ n-3 <_+ n-1.$$

Further, each A_n belongs to $\mathcal{A}_n$ and each B_n belongs to $\mathcal{B}_n$.

Proof That the join orders of A_n and B_n have the indicated forms is a simple induction based on their definitions. It is easily seen that each is subdirectly irreducible with their minimal congruence generated by collapsing the pair $(1, 2)$. Also, it is easily seen from the forms of the join orders, decreasing, then increasing, that none can contain a bichain isomorphic to $\mathbf{3_d}$. Thus each A_n and B_n is a weakly projective bichain, hence a member of $\mathcal{S}$. From the definitions of $\mathcal{A}_n$ and $\mathcal{B}_n$ it is clear that A_n belongs to $\mathcal{A}_n$ and B_n to $\mathcal{B}_n$ for $n \geq 3$. □

Proposition 2.12 *The bichains in $\mathcal{JN}$ form a subposet of $\mathcal{S}$ whose members all have distributivity type* jn.

Proof This follows by a simple induction using Definition 2.10 and Lemma 2.9 once noting that A_4 and B_4 have distributivity type jn. □

Definition 2.13 Let $\mathcal{MN}$ be the set of bichains of $\mathcal{S}$ consisting of images of members of $\mathcal{JN}$ under the duality map δ.

In Fig. 1, members of $\mathcal{MN}$ are the two rightmost members of each level. The following is immediate from the Definition 2.13 and Proposition 2.8.

Corollary 2.14 *The bichains in* $\mathcal{MN}$ *form a subposet of* $\mathcal{S}$ *whose members all have distributivity type* mn.

We have previously described the distributivity types of members of the first three levels. We next describe the distributivity types of all members of higher levels.

Proposition 2.15 *For* $n \geq 4$, *each member of the* n^{th} *level of* $\mathcal{S}$ *that is not a member of* $\mathcal{JN}$ *or* $\mathcal{MN}$ *has distributivity type* jmn.

Proof The proof is by induction on the level n. For $n = 4$ this is valid because the only member of the 4^{th} level not belonging to either $\mathcal{JN}$ or $\mathcal{MN}$ is the bichain B in the middle, and it has type jmn.

Suppose our result holds for the n^{th} level, and that C is a bichain in $\mathcal{S}$ with $n+1$ elements. If C contains a bichain isomorphic to one with type jmn, then C also has type jmn. So we may assume that C only covers members of the n^{th} level belonging to either $\mathcal{JN}$ or to $\mathcal{MN}$. We assume that C only covers members belonging to $\mathcal{JN}$, the argument in the other case follows using the duality map δ. Since C covers only members of the n^{th} level belonging to $\mathcal{JN}$, then C covers only the bichains A_n and B_n of the n^{th} level. We consider several cases.

Suppose $C \in \mathcal{A}_{n+1}$. Proposition 2.4 (1), with the assumption that C covers only A_n or B_n, provides that C covers B_n, and is built from B_n by adding a new element to the top of both the join and meet order of B_n. Thus $C = A_{n+1}$ and therefore belongs to $\mathcal{JN}$.

Suppose $C \in \mathcal{B}_{n+1}$. Proposition 2.4 (2), with the assumption that C covers only A_n or B_n, provides that C covers A_n, and is built from A_n by adding a new element to the top of the meet order and bottom of the join order of B_n. Thus $C = B_{n+1}$ and therefore belongs to $\mathcal{JN}$.

Suppose $C \in \mathcal{C}_{n+1}$. Proposition 2.4 (3), with the assumption that C covers only A_n or B_n, provides that C covers B_n, and is built from B_n by adding a new element to the bottom of the meet and join order of B_n. Then C clearly has a subalgebra isomorphic to $\mathbf{3_m}$. Thus the type of C is jmn. □

Theorem 2.16 *The set of all bichains of* $\mathcal{S}$ *in levels 4 or higher, can be partitioned into three disjoint groups:*

(1) $\mathcal{JN}$ *of distribuivity type* jn*;*
(2) $\mathcal{MN}$ *of distributivity type* mn*;*
(3) $\mathcal{JNM}$ *of distributivity type* jmn.

Proof This follows directly from Proposition 2.12, Corollary 2.14, and Proposition 2.15. □

Corollary 2.17 *The lattice* $\mathcal{L}(\mathsf{BCh})$ *of subvarieties of* BCh *contains as subposets the three posets* $\mathcal{JN}$, $\mathcal{MN}$ *and* $\mathcal{JMN}$. *Each member of* $\mathcal{JN}$ *contains the variety* $V(\mathbf{3_j}, \mathbf{3_n})$ *and is contained in the splitting variety of* $\mathbf{3_m}$*; each member of* $\mathcal{MN}$ *contains the variety*

$V(\mathbf{3_m}, \mathbf{3_n})$ and is contained in the splitting variety of $\mathbf{3_j}$; and each member of $\mathcal{JMN}$ contains the variety $V(\mathbf{3_j}, \mathbf{3_m}, \mathbf{3_n})$.

3 4-element Subdirectly Irreducibles

Up to isomorphism, there are two 2-element subdirectly irreducible members of BS, both of which are bichains, and four 3-element subdirectly irreducible members of BS, all of which are bichains. These facts are easily checked by hand. Using Sage in conjunction with a universal algebra package from Peter Jipsen, we obtained the following.

Proposition 3.1 *Up to isomorphism, there are 16 4-element subdirectly irreducible members of* BS*. Of these, 12 are bichains, the five weakly projective bichains of Fig. 1 and the seven bichains given in Fig. 2. The other four 4-element subdirectly irreducibles are described in Fig. 3.*

Figure 3 shows the meet and join orders of two of the four 4-element subdirectly irreducible members of BS that are not bichains. The other two subdirectly irreducibles that are not bichains are the duals of the two shown, and thus have Boolean lattices for their meet orders and chains for their join orders.

Proposition 3.2 *The bichain $\mathbf{4_n}$ of Fig. 2 generates the variety $V(\mathbf{3_n})$, the bichains M_1 and M_2 generate the variety* mDB *and the bichains J_1 and J_2 generate the variety* jDB*. The varieties $V(E)$ and $V(F)$ both properly contain $V(\mathbf{3_j}, \mathbf{3_m}, \mathbf{3_n})$ and are incomparable to one another.*

Proof Clearly $\mathbf{4_n}$ has a subalgebra isomorphic to $\mathbf{3_n}$, so $V(\mathbf{3_n}) \subseteq V(\mathbf{4_n})$. From earlier results, the bichain $\mathbf{3_d}$ belongs to $V(\mathbf{3_n})$, hence so also does the product $\mathbf{3_n} \times \mathbf{3_d}$. Removing the element $(3, 1)$ from this product produces a subalgebra T of it. Then there is a congruence ϕ of T that collapses the following pairs of elements: $(3, 3)$ and $(3, 2)$, and $(1, 1)$ and $(2, 1)$. One can check that the quotient T/ϕ is isomorphic to $\mathbf{4_n}$. It follows that $V(\mathbf{4_n}) \subseteq V(\mathbf{3_n})$, hence $\mathbf{4_n}$ generates this variety.

It is easily checked that M_1 and M_2 satisfy (mD) and contain $\mathbf{3_m}$, hence generate the variety mDB. Similarly J_1 and J_2 satisfy (jD) and contain $\mathbf{3_j}$, hence generate the variety jDB. Both E and F have all of $\mathbf{3_m}$, $\mathbf{3_j}$ and $\mathbf{3_n}$ as subalgebras. Thus both $V(E)$ and $V(F)$ contain the variety $V(\mathbf{3_m}, \mathbf{3_j}, \mathbf{3_n})$. To see this containment is strict, consider the equation (χ) below.

$$xz(u + y)(xy + u + z) = xz(u + y)(zy + u + x). \tag{χ}$$

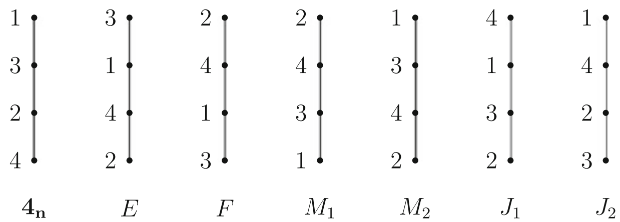


Fig. 2 The 4-element subdirectly irreducible bichains with meet order $1 \lessdot 2 \lessdot 3 \lessdot 4$ that are not weakly projective

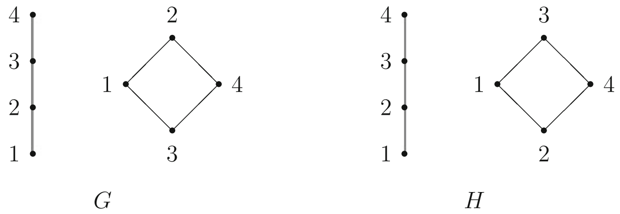


Fig. 3 Two 4-element subdirectly irreducibles that are not bichains. The other two are their duals

One checks that (χ) is valid in E, hence valid in all 3-element bichains, and fails in F. Since E and F are isomorphic to the duals of one another, it follows that (χ^d) holds in F and fails in E. These results imply that both $V(E)$ and $V(F)$ properly contain $V(\mathbf{3_j}, \mathbf{3_m}, \mathbf{3_n})$ and are incomparable to one another. $\square$

Proposition 3.3 *Consider the subdirectly irreducible algebra G shown in Fig. 3. Then G and its dual G^d generate varieties that properly contain $V(\mathbf{3_n})$, do not contain either $V(\mathbf{3_m})$ or $V(\mathbf{3_j})$, are incomparable to one another, and are not subvarieties of* BCh.

Proof One easily checks that $\mathbf{3_n}$ is isomorphic to a subalgebra of G, and that $\mathbf{3_m}$ and $\mathbf{3_j}$ are not. So $V(G)$ contains $V(\mathbf{3_n})$, and since $\mathbf{3_m}$ and $\mathbf{3_j}$ are splitting, $V(G)$ does not contain either $V(\mathbf{3_m})$ or $V(\mathbf{3_j})$. Since $\mathbf{3_n}$ is isomorphic to its own dual, and $\mathbf{3_m}$ and $\mathbf{3_j}$ are isomorphic to the duals of one another, these same statements hold for the variety $V(G^d)$. To see that $V(G)$ is not contained in BCh, note that the equation (BCh4) from the first part of the paper [1] that is valid in all bichains fails in G by setting $x = 1$, $y = 3$, and $z = 4$. The same reasoning with the dual of (BCh4) shows $V(G^d)$ is not contained in BCh. Finally, that $V(G)$ and $V(G^d)$ are incomparable follows from considering the following equation.

$$y(x+y)(y+z) = y(y+xz). \tag{κ}$$

This equation, given by the Universal Algebra Calculator, holds in G but not in G^d. So its dual equation holds in G^d but not in G. $\square$

Proposition 3.4 *Consider the subdirectly irreducible algebra H of Fig. 3. Then H and its dual H^d generate varieties that properly contain $V(\mathbf{3_m}, \mathbf{3_j})$, do not contain $V(\mathbf{3_n})$, are incomparable to one another, and are not contained in* BCh.

Proof One easily checks that $\mathbf{3_m}$ and $\mathbf{3_j}$ are isomorphic to subalgebras of H, and that $\mathbf{3_n}$ is not. It follows that $V(H)$ contains $V(\mathbf{3_m}, \mathbf{3_j})$, and as $\mathbf{3_n}$ is splitting, that $V(H)$ does not contain $V(\mathbf{3_n})$. Since $\mathbf{3_n}$ is isomorphic to its own dual and $\mathbf{3_m}$ and $\mathbf{3_j}$ are isomorphic to the duals of one another, the same comments hold for the variety $V(H^d)$. To see that $V(H)$ is not contained in BCh, and therefore properly contains $V(\mathbf{3_m}, \mathbf{3_j})$, consider the equation (BCh5) from [1]. This equation holds in all bichains, but fails in H with $x = 1$, $y = 2$, and $z = 4$. Similar comments hold for $V(H^d)$ using the dual of (BCh5). Finally, to see that $V(H)$ and $V(H^d)$ are incomparable, consider the following equation.

$$z(x+y)(y+z) = z(y+xz). \tag{λ}$$

This equation, given by the Universal Algebra Calculator, holds in H but not in H^d. So its dual equation holds in H^d but not in H. $\square$

Using Sage in conjunction with a universal algebra package there are, up to isomorphism, 125 subdirectly irreducible 5-element algebras in BS. We have not explored these further.

4 A Structure Theorem

In this section we give a structure theorem for the variety $V(\mathrm{S}_{\mathbf{3_m}}, \mathrm{S}_{\mathbf{3_j}})$, and in particular for its subvariety $V(\mathbf{3_n})$, by showing that each algebra A in this variety is built in a manner somewhat like a Płonka sum from a family of lattices that are subalgebras of A. We begin with several definitions and lemmas.

Definition 4.1 Let A be a Birkhoff system and $u, v \in A$. If $u \leq_{\cdot} v$, let $[u, v]_{\cdot}$ be the interval in the meet semilattice reduct of A, and if $u \leq_{+} v$, let $[u, v]_{+}$ be the interval in the join semilattice reduct of A.

Our key notions now follow.

Definition 4.2 Let A be a Birkhoff system. We say that a subset $S \subseteq A$ is a *sublattice* of A if S is a subalgebra of A that is a lattice. We say that S is a *convex sublattice* of A if S is a sublattice of A and is convex in each semilattice reduct of A.

Our setting of interest will be the variety $V(\mathrm{S}_{\mathbf{3_m}}, \mathrm{S}_{\mathbf{3_j}})$ defined by the splitting equations of the bichains $\mathbf{3_m}$ and $\mathbf{3_j}$. Thus a Birkhoff system belongs to $V(\mathrm{S}_{\mathbf{3_m}}, \mathrm{S}_{\mathbf{3_j}})$ if, and only if, it contains no subalgebra isomorphic to either $\mathbf{3_m}$ or $\mathbf{3_j}$. See [1] for a complete account.

Lemma 4.3 *Assume that A belongs to the variety $V(\mathrm{S}_{3_m}, \mathrm{S}_{3_j})$. Let $a, b \in A$. If $a \leq_{\cdot} b$ and $a \leq_{+} b$, then the intervals $[a, b]_{\cdot}$ and $[a, b]_{+}$ coincide and form a convex sublattice of A.*

Proof We first show that each element x in the interval $[a, b]_{\cdot}$ belongs to $[a, b]_{+}$. Symmetry then shows the other inclusion. If $x = a$ or $x = b$ or $a = b$, then there is nothing to show. Suppose that $a <_{\cdot} x <_{\cdot} b$. Since $a + b = b$, the following inequalities hold:

$$x \leq_{+} a + x \leq_{+} a + b + x = b + x. \tag{4.1}$$

Also, $a <_{\cdot} x <_{\cdot} b$ gives $ax = a$ and $xb = x$. Using Birkhoff's equation (BS), we then have $a(a + x) = a + ax = a$, $x(x + a) = x + xa = x + a$, $x(x + b) = x + xb = x$, and $b(b + x) = b + xb = b + x$. This gives the following inequalities:

$$a \leq_{\cdot} a + x \leq_{\cdot} x \leq_{\cdot} b + x \leq_{\cdot} b. \tag{4.2}$$

Suppose $a \not<_{+} x$. Then $x <_{+} a + x$. Also, as we have $a <_{+} b$, we cannot have $b <_{+} x$, so $x <_{+} b + x$. The strictness of these inequalities in conjunction with (4.2) provides $a + x <_{\cdot} x <_{\cdot} b + x$. Then (4.1) provides $x <_{+} a + x <_{+} b + x$. This gives a subalgebra of A that is isomorphic to $\mathbf{3_j}$, contrary to our assumption. Hence $a <_{+} x$.

Suppose $x \not<_{+} b$. Then $b <_{+} b + x$. Also, as $a <_{+} b$, we cannot have $x <_{+} a$, so $a <_{+} a + x$. The strictness of these inequalities in conjunction with (4.2) gives $a <_{\cdot} b + x <_{\cdot} b$. Then (4.1) provides $a <_{+} b <_{+} b + x$. This gives a subalgebra of A that is isomorphic to $\mathbf{3_m}$, contrary to our assumption. Hence $x <_{+} b$.

We have shown that if $a \leq_{\cdot} b$ and $a \leq_{+} b$, then the intervals $[a, b]_{\cdot}$ and $[a, b]_{+}$ are equal. Now we will show that the partial orderings $\leq_{\cdot}$ and $\leq_{+}$ agree on these equal intervals.

Suppose that $x, y \in [a, b]_\cdot$ and $x \leq_\cdot y$. Then $x, y \in [a, b]_+$. So we have $a \leq_\cdot y$ and $a \leq_+ y$. Therefore, by the first part of the proof, applied to a, y, the intervals $[a, y]_\cdot$ and $[a, y]_+$ are equal. Then since $x \leq_\cdot y$ we have $x \in [a, y]_\cdot$. Hence $x \in [a, y]_+$, giving $x \leq_+ y$. A symmetrical argument shows that $x \leq_+ y$ implies $x \leq_\cdot y$. So the partial orderings $\leq_\cdot$ and $\leq_+$ agree on the two equal intervals. It follows that these intervals form a convex sublattice of A. □

Definition 4.4 For a Birkhoff system A, define a binary relation θ on A by setting $a\,\theta\,b$ if a and b generate a sublattice of A.

While this relation is defined on any Birkhoff system, it is for those A in $V(\mathrm{S}\mathbf{3_m}, \mathrm{S}\mathbf{3_j})$ that it enjoys good properties.

Theorem 4.5 *If $A \in V(\mathrm{S}3_m, \mathrm{S}3_j)$, then θ is a bisemilattice congruence of A, the equivalence classes of θ are convex sublattices, and the quotient A/θ is a semilattice.*

Proof On any Birkhoff system, θ is reflexive and symmetric. To show that it is transitive, suppose that $a\,\theta\,b$ and $b\,\theta\,c$. This means that $\{a, b\}$ and $\{b, c\}$ generate sublattices of A. Then, writing $x \leq y$ to mean that both $x \leq_\cdot y$ and $x \leq_+ y$ hold, we have the following:

$$ab \leq a, b \leq a + b, \tag{4.3}$$

$$bc \leq b, c \leq b + c. \tag{4.4}$$

Next, we note that the following holds for any elements of any Birkhoff system [2, Lemma 2.7]. The proof is a simple application of Birkhoff's equation (BS).

$$abc \leq a + b + c. \tag{4.5}$$

This equation, together with Lemma 4.3, implies that the intervals $[abc, a + b + c]_\cdot$ and $[abc, a+b+c]_+$ coincide and form a convex sublattice of A. We will show that a, c belong to this sublattice, giving $a\,\theta\,c$.

We first establish the following:

$$ab + abc \leq_\cdot ab \leq_\cdot b, \tag{4.6}$$

$$ab \leq_+ ab + abc \leq_+ b. \tag{4.7}$$

It is a simple consequence of (BS) that $x + xy \leq_\cdot x$. Applying this with $x = ab$ and $y = bc$ yields the first inequality in (4.6). The second inequality in (4.6) is trivial, as is the first inequality in (4.7). For the second inequality in (4.7), note (BS) implies $x + xy \leq_+ x + y$, and this yields $ab + abc \leq_+ ab + bc$. Since (4.3) and (4.4) give $ab, bc \leq_+ b$, the second inequality in (4.7) follows.

Since A has no subalgebra isomorphic to $\mathbf{3_j}$, (4.6) and (4.7) imply that the elements $ab + abc$, ab, and b cannot all be distinct. It is a simple matter to see that this implies $ab + abc = ab$. It is clear that $abc \leq_\cdot ab \leq_\cdot b$. Further, the equality $ab + abc = ab$, along with (4.3), give $abc \leq_+ ab + abc = ab \leq_+ b$. We therefore have

$$abc \leq ab \leq b. \tag{4.8}$$

Dually, using the assumption that A does not contain a subalgebra isomorphic to $\mathbf{3_m}$, we obtain

$$b \leq a + b \leq a + b + c. \tag{4.9}$$

This and (4.3) give

$$abc \leq ab \leq a, b \leq a + b \leq a + b + c. \tag{4.10}$$

It follows that a belongs to the interval subalgebra $[abc, a + b + c]$, and by symmetry, that c also belongs to this interval subalgebra. Thus a, c generate a sublattice of A, showing $a \, \theta \, c$.

To show that θ is compatible with both bisemilattice operations, assume that $a, a', b, b' \in A$ and that both $a \, \theta \, a'$ and $b \, \theta \, b'$. Then a, a' generate a sublattice of A. So by Lemma 4.3, they lie in the convex sublattice $[aa', a + a']$ of A. Let $x = aa'$ and $x' = a + a'$. Similarly, let $y = bb'$ and $y' = b + b'$. We then have

$$x \leq a, a' \leq x', \tag{4.11}$$

$$y \leq b, b' \leq y'. \tag{4.12}$$

Note that Birkhoff's equation (BS) is easily seen to imply $pq \leq_\cdot p+q$ and $pq \leq_+ p+q$, hence $pq \leq p + q$ for any $p, q \in A$. Thus

$$xy \leq x + y, \tag{4.13}$$

$$x'y' \leq x' + y'. \tag{4.14}$$

Also, since $\cdot$ preserves $\leq_\cdot$ in each coordinate, and $+$ preserves $\leq_+$ in each coordinate, (4.11) and (4.12) with (4.13) and (4.14) give

$$xy \leq_\cdot ab, a'b' \leq_\cdot x'y' \leq_\cdot x' + y', \tag{4.15}$$

$$xy \leq_+ x + y \leq_+ a + b, a' + b' \leq_+ x' + y'. \tag{4.16}$$

Thus $xy, x' + y'$ form a 2-element sublattice of A. By Lemma 4.3, these elements generate a convex sublattice of A. Since $\leq_\cdot$ and $\leq_+$ agree on this sublattice, (4.15) says that ab and $a'b'$ belong to this convex sublattice, and (4.16) says that $a + b$ and $a' + b'$ belong to it. Then $ab \, \theta \, a'b'$ and $a + b \, \theta \, a' + b'$. This shows θ is compatible with the operations, whence it is a bisemilattice congruence.

To see that each equivalence class of A is a lattice, note that as the basic operations are idempotent, each equivalence class is a subalgebra of A. In particular, each equivalence class is a Birkhoff system in its own right. Suppose that a and b belong to an equivalence class of A. By the definition of θ, we have that a and b generate a (convex) sublattice of A. Thus $a(a + b) = a = a + ab$. So absorption holds in this subalgebra, showing that this subalgebra is a lattice.

It remains to show that A/θ is a semilattice. Suppose that $a, b \in A$. We have noted that (BS) imply $(ab) \, \theta \, (a+b)$. Therefore $(a/\theta) \cdot (b/\theta) = (ab)/\theta = (a+b)/\theta = (a/\theta) + (b/\theta)$. Therefore A/θ is a semilattice. □

Definition 4.6 We say that a Birkhoff system A is a *semilattice sum of lattices*, if there is a congruence relation θ on A such that the congruence classes a/θ are sublattices of A and the quotient A/θ is a semilattice. If we denote the quotient A/θ by S and the corresponding congruence classes a/θ by A_s, then the semilattice sum is denoted by $\bigsqcup_{s \in S} A_s$.

Proposition 4.7 *In a semilattice sum $\bigsqcup_{s \in S} A_s$, the summands A_s are necessarily convex sublattices of A, and the congruence θ is unique.*

Proof Let θ be a congruence of A such that its congruence classes are sublattices of A and the quotient A/θ is a semilattice. We first show that the congruence classes of θ are convex sublattices. Suppose $x\,\theta\,y$ and that $x \leq_\cdot a \leq_\cdot y$. Since x, y belong to a sublattice of A, then $x = x + xy$ and $y = y + xy$. Then as A/θ is a semilattice, $x/\theta = x/\theta \cdot y/\theta = y/\theta$. Therefore $a/\theta = a/\theta \cdot y/\theta = a\theta \cdot x/\theta = x/\theta$. So a belongs to the same equivalence class as x, y, so x/θ is convex with respect to the meet order. A similar argument shows that it is convex with respect to the join order as well.

We have seen that $x\,\theta\,y$ implies that x, y belong to a convex sublattice of A. We now show the converse. Suppose that $a, b \in A$ are such that $[a, b]$ is a convex sublattice of A and that $x, y \in [a, b]$. The argument above shows that $x/\theta = x/\theta \cdot b/\theta = x/\theta \cdot a/\theta = a/\theta$, and similarly that $y/\theta = a/\theta$. Thus $x\,\theta\,y$. So the congruence θ is uniquely determined. □

Observe that a Płonka sum of lattices is a special case of a semilattice sum of lattices. However, unlike the situation with Płonka sums, where the Płonka sum of lattices is known to always be a Birkhoff system, it is not the case that an algebra A with a congruence θ having the above properties must be a Birkhoff system. A simple example is given by the 3-element bisemilattice a, b, c, ordered by $a >_\cdot b <_\cdot c$ and $a <_+ c >_+ b$.

Corollary 4.8 *A Birkhoff system A belongs to the variety $V(\mathrm{S}_{\mathbf{3}_m}, \mathrm{S}_{\mathbf{3}_j})$ if, and only if, it is a semilattice sum of lattices.*

Proof By Theorem 4.5, each member of $V(\mathrm{S}_{\mathbf{3}_\mathbf{m}}, \mathrm{S}_{\mathbf{3}_\mathbf{j}})$ is a semilattice sum of its convex sublattices.

Suppose that A is a semilattice sum $\bigsqcup_{s\in S} A_s$ of lattices A_s over a semilattice S. We show that A cannot contain subalgebras isomorphic to $\mathbf{3_j}$ or $\mathbf{3_m}$. Suppose to the contrary that A contains a subalgebra C isomorphic to $\mathbf{3_j}$. Let us identify C with $\mathbf{3_j}$. Since the summands of A are convex, neither of the 2-element sublattices $\{2, 3\}$ and $\{1, 3\}$ of C can be contained in one summand. Since the subalgebra $\{1, 2\}$ of C is a semilattice, it cannot be contained in a summand. So all three elements of C belong to different summands of A. Hence there are $s >_\cdot t >_\cdot u$ in S such that $3 \in A_s,\ 2 \in A_t,\ 1 \in A_u$. But $2 <_+ 3$, providing a contradiction to $s <_+ t$. One obtains a similar contradiction if one starts with $\mathbf{3_m}$. □

Since $V(\mathbf{3_n})$ is a subvariety of $V(\mathrm{S}_{\mathbf{3}_\mathbf{m}}, \mathrm{S}_{\mathbf{3}_\mathbf{j}})$, the result of Corollary 4.8 can be applied to this setting. We note that [1, Prop.6.5] shows that the intersection of BCh with the variety L of lattices is the variety DL of distributive lattices. The following is immediate.

Corollary 4.9 *Each member of the variety $V(\mathbf{3_n})$ is a semilattice sum of distributive lattices.*

Classes of general algebras built in a similar way as the semilattice sums of lattices were considered by Mal'cev (see [3–5]). We will adapt his definition to the case of Birkhoff systems.

Definition 4.10 Let V and W be two varieties of Birkhoff systems. Then the *Mal'cev product* $\mathsf{V} \circ \mathsf{W}$ of V and W consists of Birkhoff systems A with a congruence φ such that the quotient A/φ is in W, and each congruence class a/φ of A is in V.

It follows by a theorem of Mal'cev concerning general Mal'cev products that the Mal'cev product $\mathsf{V} \circ \mathsf{W}$ is a quasivariety.

Corollary 4.11 *The class of Birkhoff systems that are semilattice sums of lattices is the Mal'cev product* $\mathsf{L} \circ \mathsf{SL}$ *of the varieties* L *of lattices and* SL *of semilattices within the class of Birkhoff systems.*

Corollary 4.12 *The following three classes of Birkhoff systems are equal: the variety* $V(\mathrm{S}_{\mathbf{3}_m}, \mathrm{S}_{\mathbf{3}_j})$*, the class of Birkhoff systems that are semilattice sums of lattices, and the quasivariety* $\mathsf{L} \circ \mathsf{SL}$.

It is natural to ask how to reconstruct the bisemilattice structure of a semilattice sum $\bigsqcup_{s\in S} A_s$ from its lattice summands A_s and the quotient semilattice S. Such a construction exists and was introduced for general algebras in [8, §6.2], under the name of a Lallement sum, as a generalization of a Płonka sum (see also [6]). The primary idea is to relax the requirement of functoriality in the definition of Płonka sums. In the case of bisemilattices this construction can be described as follows.

Definition 4.13 Let $(S, *)$ be a semilattice with partial ordering $\leq_*$. For each $s \in S$, let a lattice A_s be given, and two extensions $(E_s^{\cdot}, \cdot)$ of $(A_s, \cdot)$, and $(E_s^{+}, +)$ of $(A_s, +)$. For each pair $t \leq_* s$ of S, let two homomorphisms be given:

$$\varphi_{s,t}^{\cdot} : (A_s, \cdot) \to (E_t^{\cdot}, \cdot)$$

and

$$\varphi_{s,t}^{+} : (A_s, +) \to (E_t^{+}, +),$$

such that the following three conditions are satisfied for $\dagger \in \{\cdot, +\}$:

(1) $\varphi_{s,s}^{\dagger}$ is the embedding of A_s into $E_s^{\dagger}$;
(2) $\varphi_{s,s*t}^{\dagger}(A_s) \dagger \varphi_{t,s*t}^{\dagger}(A_t) \subseteq A_{s*t}$;
(3) for each $u \leq_* s * t$ in S and $a \in A_s,\ b \in A_t$

$$\varphi_{s*t,u}^{\dagger}(\varphi_{s,s*t}^{\dagger}(a) \dagger \varphi_{t,s*t}^{\dagger}(b)) = \varphi_{s,u}^{\dagger}(a) \dagger \varphi_{t,u}^{\dagger}(b).$$

Here, for two sets A and B, $A \dagger B = \{a \dagger b \mid a \in A, b \in B\}$.

With this data provided, a bisemilattice structure on the disjoint sum A of all A_s is given by defining the operations $\cdot$ and $+$ for $a \in A_s$ and $b \in A_t$ as follows:

$$a \cdot b = \varphi_{s,s*t}^{\cdot}(a) \cdot \varphi_{t,s*t}^{\cdot}(b),$$

and

$$a + b = \varphi_{s,s*t}^{+}(a) + \varphi_{t,s*t}^{+}(b).$$

Then all A_s are subalgebras of the bisemilattice A and S is its quotient. The semilattice sum A of A_s is said to be the *semilattice sum of* A_s *by the mappings* $\varphi_{s,t}^{\dagger}$. If additionally, for each $t \in S$, one has that $E_t^{\dagger} = \{\varphi_{s,t}^{\dagger}(a) \mid s \geq_* t,\ a \in A_s\}$, and all E_s are certain canonical extensions of A_s (see [8, §6.1]), then this semilattice sum is called a *Lallement sum*.

By [8, Th. 624] each semilattice sum of lattices can be reconstructed as a Lallement sum of these lattices. The usefulness of Lallement sums depends on properties of the available extensions $E_s^{\dagger}$. In particular, a nice situation appears if for each $s \in S$, both extensions $E_s^{\cdot}$ and E_s^{+} coincide with the summand A_s. Such Lallement sum is called *strict*.

Example 4.14 The subdirectly irreducible Birkhoff system G of Fig. 3, and its dual G^d, are strict Lallement sums of 2-element lattices. Specifically, G is the Lallement sum of A_s

being $3 < 4$ and A_t being $1 < 2$ where $s * t = t$, the map $\varphi_{s,t}$ takes both 3, 4 to 2 and $\psi_{s,t}$ takes 3 to 1 and 4 to 2. The minimal congruence of G collapses $\{1, 2\}$.

We next consider a special case of strict Lallement sums of lattices.

Definition 4.15 A Birkhoff system A is a semilattice sum $\bigsqcup A_s$ of bounded lattices if A has a congruence θ such that A/θ is a semilattice S, and the congruence classes A_s of θ are bounded lattices.

For A a semilattice sum $\bigsqcup A_s$ of bounded lattices, where 0_s and 1_s are the bounds of A_s, define for $a \in A$ and $t \leq_* s$ in S the maps

$$\varphi_{s,t} : A_s \to A_t; a \mapsto a \cdot 1_t \quad \text{and} \quad \psi_{s,t} : A_s \to A_t; a \mapsto a + 0_t.$$

Note that $t \leq_* s$ in S means $s \cdot t = t$ and $s + t = t$, so these maps are well defined. It is easily seen that each $\varphi_{s,t}$ is a homomorphism of the meet reducts of A_s and A_t, and that $\psi_{s,t}$ is a homomorphism of the join reducts of A_s and A_t. It is also not difficult to see that they satisfy the requirements of a strict Lallement sum. This gives the following theorem, a corollary of Theorem 624 of [8].

Theorem 4.16 *Let A be a Birkhoff system. Then A is a semilattice sum $\bigsqcup_{s\in S} A_s$ of bounded lattices A_s over a semilattice S if, and only if, it is a strict Lallement sum of the lattices A_s over the semilattice S given by the homomorphisms $\varphi_{s,t}$ and $\psi_{s,t}$ described above.*

The following is then immediate from Corollaries 4.8 and 4.9.

Corollary 4.17 *Each finite algebra in the variety $V(\mathrm{S}_{\mathbf{3}_m}, \mathrm{S}_{\mathbf{3}_j})$ is a strict Lallement sum of lattices, and each finite algebra in the variety $V(\mathbf{3_n})$ is a strict Lallement sum of distributive lattices.*

Problem 4.18 Is each semilattice sum of lattices embeddable into a semilattice sum of bounded lattices?

5 The Variety $V(\mathbf{3_n})$

In this section we give an equational aximoatization of the variety $V(\mathbf{3_n})$. Key to this will be a description of the finite subdirectly irreducibles in this variety obtained using the results of the previous section on semilattice sums of lattices.

Definition 5.1 Let us consider the following equation:

$$x + z(x + y) = x + yz(x + y) \tag{N}$$

This equation, and its dual, (N^d), play a key role.

Proposition 5.2 *Both (N) and (N^d) are valid in the bichain $\mathbf{3_n}$.*

Proof Label $\mathbf{3_n}$ as before, with join order $3 <_+ 1 <_+ 2$. We will show that (N) holds. Since $\mathbf{3_n}$ is self-dual it follows that (N^d) will hold as well. Note that (N) holds in any bichain in which x is the join top since then both sides are x, and it holds in any bichain in which x is the join bottom, since it then becomes $zy = yzy$. So it holds in any bichain with 1 or 2 elements. To see it holds in $\mathbf{3_n}$, we only consider when x, y, z are distinct, and when x is neither the join top nor join bottom, so when $x = 1$. It clearly holds when $yz = z$, so we may assume $y <. z$. Since x, y, z are distinct and $x = 1$, this leaves $y = 2$ and $z = 3$. Then we have $x + z(x + y) = 1 + 3(1 + 2) = 1 + 2 = 2$ and $x + yz(x + y) = 1 + 2(1 + 2) = 1 + 2 = 2$. □

Proposition 5.3 *In any lattice, (N) is equivalent to the distributive law, and so is (N^d)*

Proof Since $\mathbf{3_n}$ satisfies (N), so does the 2-element distributive lattice. Hence all distributive lattices satisfy (N). Therefore the equations for distributive lattices imply (N). For the converse, assume L is a lattice that satisfies (N). To see that L is distributive, it is enough to show that it doesn't have a sublattice isomorphic to either of the two 5-element non-distributive lattices. For M_3 let x, y, z be the three middle elements to get a failure of (N). For the pentagon, let $x < z$ be the two elements in one side and y the element on the other to get a failure of (N). Then $x + z(x + y) = x + z$ and is equal to 1 in M_3 and to z in the pentagon, and $x + yz(x + y) = x + yz = x$. □

Proposition 5.4 *We have the following.*

(1) *(N) implies the splitting equation ($S_{\mathbf{3_m}}$) for $\mathbf{3_m}$.*
(2) *(N^d) implies the splitting equation ($S_{\mathbf{3_j}}$) for $\mathbf{3_j}$.*
(3) *$\mathbf{3_j}$ satisfies (N).*
(4) *$\mathbf{3_m}$ satisfies (N^d).*

Proof (1) The computer program Prover9 provides a fairly short proof of ($S_{\mathbf{3_m}}$) from (N).

(2) This follows from duality.

(3) The join order of $\mathbf{3_j}$ is $2 <_+ 1 < +3$. We have seen that (N) is valid in any 2-element bichain, so to verify that (N) holds in $\mathbf{3_j}$ it is enough to check all instances where x, y, z are distinct. It clearly holds when x is either the join top or the join bottom, so we assume it is the join middle $x = 1$. It also holds when $yz = z$. So we are left with just the case $x = 1, y = 2, z = 3$. The equation then becomes $x + y = x + y(x + y)$ and this is valid.

(4) This follows from (3) by duality. □

Lemma 5.5 *The equation (N) is valid in the bisemilattice G, but is not valid in G^d. The equation (N^d) is valid in the bisemilattice G^d, but is not valid in G.*

Proof We show that the first statement is valid. The second follows by a dual argument. First note that the sets $\{1, 2, 4\}$ and $\{2, 3, 4\}$ form distributive subalgebras of G, and $\{1, 2, 3\}$ is isomorphic to the non-distributive $\mathbf{3_n}$. So it is sufficient to show that the elements 1, 3 and 4 satisfy (N). If $x = 3$, then both sides of (N) are equal to yz. If $x = 1$, then both sides are 2 or both sides are 1. And finally, if $x = 4$, then both sides are always equal to 2. Hence G satisfies (N). Now substituting 3 for x, 4 for y and 1 for z, one obtains $x(z + xy) = 1$ and $x(y + z + xy) = 2$. This shows that (N^d) is not satisfied in G. □

We next turn to the matter of applying these axiomatics to help describe the finite subdirectly irreducible algebras in $V(\mathbf{3_n})$. A key step is Corollary 4.9, that states that each algebra

A in this variety is a semilattice sum of distributive lattices. We introduce some further terminology for discussing these semilattice sums.

Definition 5.6 For any Birkhoff system A, the largest quotient A/θ of A that is a semilattice is called the *semilattice replica* of A.

Thus A being a semilattice sum of lattices means that for θ, the congruence producing its semilattice replica, the congruence classes of θ are lattices. For $S = A/\theta$ its semilattice replica, we denote the identical operations of $+$ and $\cdot$ on S by $*$. We treat S as a meet semilattice with order $\leq_*$ given by $s \leq_* t$ iff $s * t = s$. The following definition and proposition are adapted to the setting of BS from a more general notion. (See [8, p. 73] and [7].)

Definition 5.7 For a Birkhoff system A, we call a subset B of A a *sink* if for each $a \in A$ and $b \in B$, we have $a + b \in B$ and $a \cdot b \in B$.

Sinks are automatically subalgebras. If A is a semilattice S, then sinks are also called *downsests* or *ideals* of S. The principal downset generated by $s \in S$ is denoted $\downarrow s$. The result below follows from the fact that lattices have no nontrivial sinks.

Proposition 5.8 [7] *If a Birkhoff system A is a semilattice sum of lattices $\bigsqcup_{s \in S} A_s$, then a subset B of A is a sink of A if, and only if, $B = \bigsqcup_{s \in D} A_s$ for a downset D of S.*

It is easily seen that a congruence on a sink in any algebra can be extended in an obvious way to a congruence of the whole algebra. The following then follows from the proposition above.

Corollary 5.9 *Suppose A is a semilattice sum of lattices $\bigsqcup_{s \in S} A_s$ and D is a downset of S. Then for each congruence ϕ on $\bigsqcup_{s \in D} A_s$, the relation on A which agrees with ϕ on $\bigsqcup_{s \in D} A_s$, and is equal to the equality relation otherwise, is a congruence on A.*

The following will be key in our investigation of finite subdirectly irreducible algebras in $V(\mathbf{3_n})$.

Corollary 5.10 *If A is subdirectly irreducible and a semilattice sum of lattices $\bigsqcup_{s \in S} A_s$, then for any downset D of S, the sink $\bigsqcup_{s \in D} A_s$ either has one element or is subdirectly irreducible.*

The other facet of our investigation will involve small configurations of points possibly arising in semilattice sums. This arises frequently in conjunction with the following easily proved observation.

Proposition 5.11 *Let A be a semilattice sum $\bigsqcup_{s \in S} A_s$ of lattices, and $s \leq_* t$. Then for any sublattice T of A_t, $T \cup A_s$ is a subalgebra of A.*

In dealing with 3-element subalgebras, the following result is most useful. It can be verified by hand, or by using Prover9/Mace4.

Proposition 5.12 *Up to isomorphism, there are exactly nine 3-element Birkhoff systems, the six bichains and the three shown in Fig. 4.*

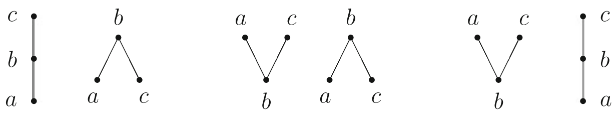


Fig. 4 The 3-element Birkhoff systems that are not bichains

From the first part of this paper [1], the bichains $\mathbf{2_l}$, $\mathbf{2_s}$, $\mathbf{3_d}$, and $\mathbf{3_n}$ belong to $V(\mathbf{3_n})$, and Proposition 3.2 shows that the 4-element bichain $\mathbf{4_n}$ of Fig. 2 belongs to $V(\mathbf{3_n})$. Each of these bichains is subdirectly irreducible. Since these algebras belong to $V(\mathbf{3_n})$, each is a semilattice sum of distributive lattices $\bigsqcup_{s\in S} A_s$. These semilattice sums are described completely by the following.

$\mathbf{2_l}$: S is 1 with $|A_1| = 2$.
$\mathbf{2_s}$: S is $1 <_* 2$ with $|A_1| = 1$ and $|A_2| = 1$.
$\mathbf{3_d}$: S is $1 <_* 2$ with $|A_1| = 1$ and $|A_2| = 2$.
$\mathbf{3_n}$: S is $1 <_* 2$ with $|A_1| = 2$ and $|A_2| = 1$.
$\mathbf{4_n}$: S is $1 <_* 2 <_* 3$ with $|A_1| = 1$, $|A_2| = 2$, and $|A_3| = 1$.

We come to our key result.

Theorem 5.13 *The finite subdirectly irreducible Birkhoff systems that are semilattice sums of distributive lattices and do not have subalgebras isomorphic to either G or G^d are the bichains $\mathbf{2_l}, \mathbf{2_s}, \mathbf{3_d}, \mathbf{3_n}$ and $\mathbf{4_n}$.*

Proof Suppose A is a finite subdirectly irreducible algebra that is the semilattice sum $\bigsqcup_{s\in S} A_s$ of distributive lattices and does not contain a subalgebra isomorphic to G or G^d. We prove by induction on the cardinality $|S|$ of the semilattice S that A is isomorphic to one of the bichains $\mathbf{2_l}, \mathbf{2_s}, \mathbf{3_d}, \mathbf{3_n}, \mathbf{4_n}$.

The base case when S has a single element, hence A is the semilattice sum of a single distributive lattice, leads to A being isomorphic to $\mathbf{2_l}$. We suppose $|S| > 1$ and our result holds for all smaller cases.

Let m be a maximal element of S. The set $S \setminus \{m\}$ is a downset of S, so by Corollary 5.10, $\bigsqcup_{s\in S\setminus\{m\}} A_s$ is subdirectly irreducible or has one element.

In the case that $\bigsqcup_{s\in S\setminus\{m\}} A_s$ has one element, we have that S is $1 <_* m$ and A_1 is a 1-element lattice. Then for any congruence θ on A_m, there is a congruence on A that agrees with θ on A_m and is the identity otherwise. Since A is subdirectly irreducible, it follows that A_m has either one or two elements, and these lead to A being isomorphic to $\mathbf{2_s}$ and $\mathbf{3_d}$, respectively.

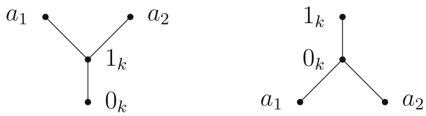
Now assume that $\bigsqcup_{s\in S\setminus\{m\}} A_s$ is non-trivial. Then for each maximal element m of S the sum $\bigsqcup_{s\in S\setminus\{m\}} A_s$ is subdirectly irreducible. Hence, by the inductive hypothesis, it is one of the bichains $\mathbf{2_l}, \mathbf{2_s}, \mathbf{3_d}, \mathbf{3_n}, \mathbf{4_n}$. It follows that the semilattice S is either of the form $1 <_* \cdots <_* k <_* m$, if it has only one maximal element, or $1 <_* \cdots <_* k <_* m_1, m_2$, if it has two maximal elements.

Suppose A_k has only a single element. If S has two maximal elements m_1 and m_2, then there is a congruence θ_1 collapsing $A_k \cup A_{m_1}$ and nothing else, and a congruence θ_2 collapsing $A_k \cup A_{m_2}$ and nothing else. This would contradict that A is subdirectly irreducible. If S has only a single element m, then again there is a congruence θ collapsing $A_k \cup A_m$ and nothing else. But by Corollary 5.9 there is a congruence ψ collapsing $\bigsqcup_{i\leq_* k} A_i$

and nothing else, and our assumption that $\bigsqcup_{s\in S\setminus\{m\}} A_s$ is not a singleton, provides that ψ is non-trivial. This again contradicts that A is subdirectly irreducible. So A_k has more than one element, and from our descriptions of $\mathbf{2_l}$, $\mathbf{2_s}$, $\mathbf{3_d}$, $\mathbf{3_n}$, $\mathbf{4_n}$, we have that A_k has exactly two elements 0_k, 1_k, and we assume $0_k < 1_k$ in the lattice A_k.

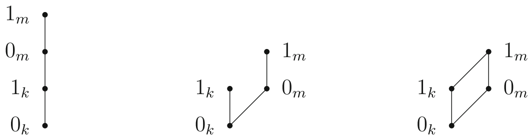
We next show that S has only one maximal element by assuming that S has two maximal elements m_1, m_2 and deriving a contradiction. Since A_k has 2 elements, Corollary 5.10 and the inductive hypothesis imply that for each $i = 1, 2$, that $\bigsqcup_{s\leq_* m_i} A_s$ is isomorphic to either $\mathbf{3_n}$ or $\mathbf{4_n}$ since these are the only alternatives whose semilattice sum has a lattice with 2 elements that does not occur at the top of the semilattice order. Therefore A_{m_1} and A_{m_2} must both have a single element. We let a_1 be the element of A_{m_1} and a_2 be that of A_{m_2}.

Note that $k = m_1 * m_2$, and that the intersection of the subalgebras $\bigsqcup_{s\leq_* m_1} A_s$ and $\bigsqcup_{s\leq_* m_2} A_s$ is $\bigsqcup_{s\leq_* k} A_s$. It follows that these first two subalgebras are both isomorphic to $\mathbf{3_n}$, or are both isomorphic to $\mathbf{4_n}$. This leads to $A_{m_1} \cup A_{m_2} \cup A_k$ being the subalgebra shown below. This is a contradiction since this algebra fails (BS) because $a_1(a_1+a_2) = a_1 \cdot 0_k = 0_k$ and $a_1 + a_1 a_2 = a_1 + 1_k = 1_k$.



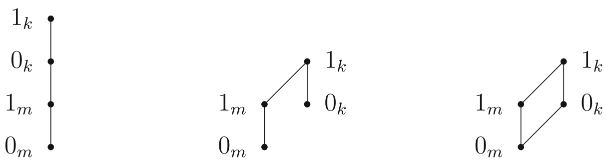
We have seen that A_k has 2 elements 0_k and 1_k, and now know that S has a single maximal element m, hence is a chain. We will show that A_m has a single element. Since the inductive hypothesis implies that $\bigsqcup_{i\leq_* k} A_i$ is isomorphic to either $\mathbf{2_l}$ or $\mathbf{3_d}$, this will show that A is isomorphic to either $\mathbf{3_n}$ or $\mathbf{4_n}$.

Assume that A_m has at least 2 elements, and that 0_m and 1_m are its smallest and largest elements. By Proposition 5.11 we have that $\{0_k, 1_k, 0_m, 1_m\}$ is a subalgebra of A. We consider the possibilities (m1), (m2), (m3) for its meet order:



(m1) (m2) (m3)

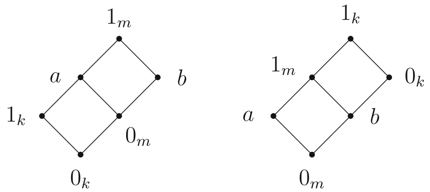
And the possibilities (j1), (j2), (j3) for the join order:



(j1) (j2) (j3)

Recall that there is a congruence ψ collapsing $\bigsqcup_{i\leq_* k} A_i$ and nothing else. If (m1) and (j1) occur, then there is a congruence collapsing just A_m and nothing else. This contradicts that A is subdirectly irreducible. The same is true if (m1) and (j2) occur, and if (m2) and (j2) occur. Having (m1) and (j3) occur gives a subalgebra isomorphic to G, an impossibility. Having (m2) and (j3) occur gives a subalgebra $\{0_k, 1_k, 1_m\}$ that is not a Birkhoff system. The only case not following from one of these by symmetry is (m3) and (j3). We note that the assumption that G^d is not a a subalgebra is used in consideration of these symmetric cases.

So we assume that (m3) and (j3) describe the meet and join semilattices for $\{0_k, 1_k, 0_m, 1_m\}$. We note that there may be more elements in A_m beside 0_m and 1_m. The set of all elements of the finite lattice A_m that lie above 1_k in the meet order is a principal filter, say $\uparrow a$, and the set of elements of A_m that lie beneath 0_k in the join order is a principal ideal, say $\downarrow b$. Note that $1_m \in \uparrow a$ and $0_m \in \downarrow b$ from our assumption that (m3) and (j3) describe a portion of the meet and join orders.



We note that $\uparrow a \cap \downarrow b = \emptyset$. Indeed, if c belongs to this intersection, then $c \neq 1_m$ since $1_m \not\leq_+ 0_k$ since we have assumed that the join order is described by (j3). But then $\{0_k, 1_k, c, 1_m\}$ is a subalgebra of A that is isomorphic to G. Also, $\uparrow a \cup \downarrow b = A_m$. Suppose that d is an element of A_m that is not in this union. Then $d(0_k + d) = d \cdot 1_k = 0_k$, with the first equality since $d \notin \downarrow b$ and the second since $d \notin \uparrow a$; while $d + 0_k \cdot d = d + 0_k = 1_k$, with the first equality obvious, and the second since $d \notin \downarrow b$.

Thus $\uparrow a$ and $\downarrow b$ are a disjoint pair consisting of a prime ideal and its complementary prime filter. So there is a congruence γ that collapses $\{0_k\} \cup \downarrow b$ and $\{1_k\} \cup \uparrow a$ and nothing else. With the congruence ψ described above, this contradicts that A is subdirectly irreducible. So A_m has only a single element, concluding the proof. □

Corollary 5.14 *Up to isomorphism, the finite subdirectly irreducible members of the variety $V(\mathbf{3_n})$ are $\mathbf{2_l}$, $\mathbf{2_s}$, $\mathbf{3_d}$, $\mathbf{3_n}$, and $\mathbf{4_n}$.*

Proof The results of the first part of this paper show that $\mathbf{2_l}$, $\mathbf{2_s}$, $\mathbf{3_d}$ and obviously $\mathbf{3_n}$ belong to $V(\mathbf{3_n})$, and Proposition 3.2 shows that $\mathbf{4_n}$ also belongs to $V(\mathbf{3_n})$. Corollary 4.9 provides that each member of $V(\mathbf{3_n})$ is a semilattice sum of distributive lattices. Proposition 5.2 gives that (N) and (N^d) are valid in $V(\mathbf{3_n})$, so it is a consequence of Lemma 5.5 that neither G nor G^d belongs to $V(\mathbf{3_n})$. So each finite subdirectly irreducible algebra in $V(\mathbf{3_n})$ is a semilattice sum of distributive lattices that does not contain G or G^d as a subalgebra. Hence by Theorem 5.13 it is one of $\mathbf{2_l}$, $\mathbf{2_s}$, $\mathbf{3_d}$, $\mathbf{3_n}$ or $\mathbf{4_n}$. □

We next apply these results to resolve a number of gaps from the fifth section of first part of the paper [1] concerning the varieties $V(\mathbf{3_n})$, $V(\mathbf{3_m}, \mathbf{3_n})$ and $V(\mathbf{3_j}, \mathbf{3_n})$.

Theorem 5.15 *The following varieties are equal: $V(\mathbf{3_n})$, the variety $V(\mathrm{N}, \mathrm{N}^d)$, and the variety $V(\mathrm{S}_{3_m}, \mathrm{S}_{3_j}) \cap \mathsf{BCh}$.*

Proof Proposition 5.2 gives that (N) and (N^d) are valid in $V(\mathbf{3_n})$, so $V(\mathbf{3_n})$ is contained in $V(\mathrm{N}, \mathrm{N}^d)$. Proposition 5.4 gives that (N) implies ($\mathrm{S}_{\mathbf{3_m}}$) and ($\mathrm{N}^d$) implies ($\mathrm{S}_{\mathbf{3_j}}$), so $V(\mathrm{N}, \mathrm{N}^d)$ is contained in $V(\mathrm{S}_{\mathbf{3_m}}, \mathrm{S}_{\mathbf{3_j}})$. Thus $V(\mathbf{3_n})$ is contained in $V(\mathrm{S}_{\mathbf{3_m}}, \mathrm{S}_{\mathbf{3_j}})$.

We next see the other inequalities. Corollary 4.8 gives that each member of $V(\mathrm{S}_{\mathbf{3_m}}, \mathrm{S}_{\mathbf{3_j}})$ is a semilattice sum of lattices, so each member of $V(\mathrm{N}, \mathrm{N}^d)$ is also a semilattice sum of lattices. Since each member of BCh that is a lattice is distributive, and Proposition 5.3 gives that each lattice in $V(\mathrm{N}, \mathrm{N}^d)$ is distributive, each member of $V(\mathrm{N}, \mathrm{N}^d)$, and each member of $V(\mathrm{S}_{\mathbf{3_m}}, \mathrm{S}_{\mathbf{3_j}}) \cap \mathsf{BCh}$ is a semilattice sum of distributive lattices. In particular, this shows that each of these varieties is locally finite, hence is generated by its finite subdirectly irreducible members.

Lemma 5.5 shows that neither of G, G^d belong to $V(\mathrm{N}, \mathrm{N}^d)$, and Proposition 3.3 shows that neither of G, G^d belongs to BCh. Thus each member of $V(\mathrm{N}, \mathrm{N}^d)$, and each member of $V(\mathrm{S}_{\mathbf{3_m}}, \mathrm{S}_{\mathbf{3_j}}) \cap \mathsf{BCh}$, is a semilattice sum of distributive lattices that does not have a subalgebra isomorphic to G or G^d. By Theorem 5.13, the finite subdirectly irreducible algebras in $V(\mathrm{N}, \mathrm{N}^d)$, and the finite subdirectly irreducible algebras in $V(\mathrm{S}_{\mathbf{3_m}}, \mathrm{S}_{\mathbf{3_j}}) \cap \mathsf{BCh}$, are among $\mathbf{2_l}, \mathbf{2_s}, \mathbf{3_d}, \mathbf{3_n}$, and $\mathbf{4_n}$. But all of these belong to $V(\mathbf{3_n})$. Thus, as $V(\mathrm{N}, \mathrm{N}^d)$ and $V(\mathrm{S}_{\mathbf{3_m}}, \mathrm{S}_{\mathbf{3_j}}) \cap \mathsf{BCh}$ are generated by their finite subdirectly irreducible algebras, they are contained in $V(\mathbf{3_n})$. $\square$

Remark 5.16 In view of the result above, it is natural to ask whether $V(\mathrm{S}_{\mathbf{3_m}}) \cap \mathsf{BCh}$ is equal to $V(\mathbf{3_j}, \mathbf{3_n})$. But we have seen that this is not the case. In Section 2 we saw an infinite family of subdirectly irreducible, weakly projective bichains that have subalgebras isomorphic to $\mathbf{3_j}$ and $\mathbf{3_n}$, but not to $\mathbf{3_m}$. Each of this will generate a distinct variety that contains $V(\mathbf{3_j}, \mathbf{3_n})$ and is contained in $V(\mathrm{S}_{\mathbf{3_m}}) \cap \mathsf{BCh}$.

We can now complete our study from the first part of the paper [1].

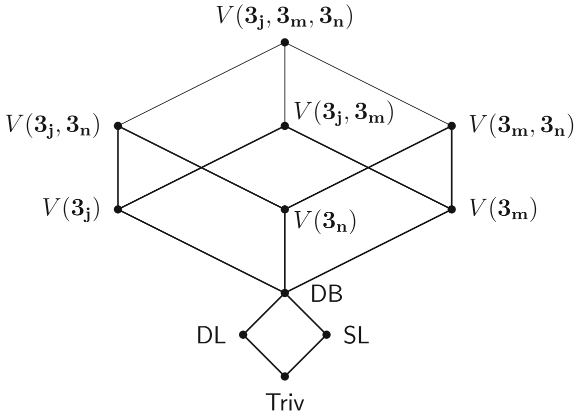
Corollary 5.17 $V(\mathbf{3}_m, \mathbf{3}_n) \cap V(\mathbf{3}_j, \mathbf{3}_n) = V(\mathbf{3}_n)$.

Proof Propositions 5.2 and 5.4 show that $V(\mathbf{3_m}, \mathbf{3_n})$ is contained in the variety $V(\mathrm{N}^d)$ and that $V(\mathbf{3_j}, \mathbf{3_n})$ is contained in $V(\mathrm{N})$. Therefore $V(\mathbf{3_m}, \mathbf{3_n}) \cap V(\mathbf{3_j}, \mathbf{3_n}) \subseteq V(\mathrm{N}^d) \cap V(\mathrm{N}) = V(\mathrm{N}, \mathrm{N}^d)$. The result then follows from Theorem 5.15 which shows $V(\mathrm{N}, \mathrm{N}^d) = V(\mathbf{3_n})$. $\square$

Corollary 5.18 *The only subvarieties of $V(\mathbf{3}_m, \mathbf{3}_n)$ are $V(\mathbf{3}_m)$, $V(\mathbf{3}_n)$,* DB, DL, SL, *and the trivial variety.*

Proof By Proposition 5.9 of [1], $V(\mathbf{3_m}, \mathbf{3_n})$ covers $V(\mathbf{3_m})$. Suppose V is a subvariety of $V(\mathbf{3_m}, \mathbf{3_n})$ that does not contain $V(\mathbf{3_m})$. Then $\mathbf{3_m} \notin \mathsf{V}$, so $\mathsf{V} \subseteq V(\mathrm{S}_{\mathbf{3_m}})$. Also, as $V(\mathbf{3_m}, \mathbf{3_n})$ is contained in $V(\mathrm{S}_{\mathbf{3_j}})$, we have $\mathsf{V} \subseteq V(\mathrm{S}_{\mathbf{3_m}}, \mathrm{S}_{\mathbf{3_j}})$. Since V is contained in BCh, it follows from Theorem 5.15 that $\mathsf{V} \subseteq V(\mathbf{3_n})$. The result then follows from the description of the subvarieties of $V(\mathbf{3_n})$ given in [1]. $\square$

Clearly, a symmetric result holds also for the subvarieties of $V(\mathbf{3_j}, \mathbf{3_n})$. This provides the following figure of the subvarieties of $V(\mathbf{3_j}, \mathbf{3_m}, \mathbf{3_n})$ where bold lines indicate covers.



6 Problems

We collect some problems we feel would aid in developing the theory of Birkhoff systems.

1. Solve free word problem for BS.
2. Is BS generated by its finite members?
3. Can every Birkhoff system be embedded into a complete one?
4. Are epimorphisms surjective in BS?
5. Which Birkhoff systems are weakly projective?
6. Is every subvariety of BCh generated by the bichains it contains?
7. Is the equational theory of BCh finitely based?
8. Is the variety generated by the 4-element bichain B finitely based?

References

1. Harding, J., Romanowska, A.: Varieties of Birkhoff systems I, to appear in Order
2. Harding, J., Walker, C., Walker, E.: Projective bichains. Algebra Univ. **67**, 347–374 (2012)
3. Mal'cev, A.I.: Multiplication of classes of algebraic systems. Sibirsk. Mat. Zh. **8**, 346–365 (1967). in Russian. English translation in: The Metamathematics of Algebraic Systems. Collected Papers: 1936–1967
4. Mal'cev, A.I.: The Metamathematics of Algebraic Systems. Collected Papers: 1936–1967. North-Holland, Amsterdam (1971)
5. Mal'cev, A.I.: Algebraic Systems. Springer Varlag, New York (1973)
6. Płonka, J., Romanowska, A.: Semilattice sums, in Universal Algebra and Quasigroup Theory. In: Romanowska, A., Smith, J.D.H. (eds.), pp. 123–158, Heldermann, Berlin (1992)
7. Romanowska, A.: Constructing and reconstructing of algebras. Demonstratio Math. **18**, 209–230 (1985)
8. Romanowska, A.B., Smith, J.D.H.: Modal Theory. Heldermann Verlag, Berlin (1985)
9. Romanowska, A.B., Smith, J.D.H.: Modes. World Scientific, Singapore (2002)