The Fell compactification of a poset

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Abstract Each poset P naturally forms a locally compact T_0-space in its Alexandroff topology. We may therefore consider the hit-or-miss topology on the closed sets of P and the associated Fell compactification of P. Here we give a purely order-theoretic description of the Fell compactification of P. We note that the Fell compactification naturally gives rise to a stable compactification of P, and place this in the general theory of stable compactifications. When P is a chain, we show that this stable compactification is simply the sobrification of P, and is the least stable compactification of P.

1 Introduction

Say nice things about Hung, then transition into his interests in hit-or-miss topology, etc. **Discuss the history and some connections, then transition to the Fell compactification. Connect with stable compactifications and Nachbin compactifications.**

2 Preliminaries

Here we consider topological spaces that are not necessarily Hausdorff. A *compact space* is the one in which every open cover has a finite subcover, and a *locally compact space* is one in which compact sets form a neighborhood base.

In a topological space, a closed set is *irreducible* if it cannot be written as the union of two proper closed subsets. A space X is *sober* if each irreducible closed set

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is the closure of a unique singleton. Sober spaces are T_0. A set that is an intersection of open sets is *saturated*.

Definition 2.1. (see, e.g., [3, Def. VI-6.7]) A space is *stably compact* if it is compact, locally compact, sober, and the intersection of any family of compact saturated sets is compact.

For a stably compact space (X,τ), the *co-compact topology* τ^k on X has as opens the complements of compact saturated sets, and the *patch topology* $\pi = \tau \vee \tau^k$ is the smallest topology on X containing the original and co-compact topologies.

Definition 2.2.
1. An *ordered topological space* $(X,\pi,\leq)$ is a set X with a partial ordering $\leq$ and topology π.
2. An ordered topological space $(X,\pi,\leq)$ is a *Nachbin space* if π is compact and $\leq$ is closed in the product topology.

Remark 2.3. The study of ordered topological spaces in general, and of Nachbin spaces in particular was pioneered by Nachbin in the 1940s (see [6]); the name Nachbin space appears to originate from [1, Def. 2.5].

Every Nacbin space is Hausdorff. A Nachbin space has an *upper topology* π_u, and a *lower topology* π_ℓ. To define these, we recall that a subset S of a poset is an *u*pset if $x \in S$ and $x \leq y$ implies $y \in S$, and is a *d*ownset if $x \in S$ and $y \leq x$ implies $y \in S$. Then π_U is defined as open upsets of $(X,\pi,\leq)$ and π_ℓ is defined as open downsets. Both (X,π_u) and (X,π_ℓ) are stably compact spaces. We use $\uparrow S$ for the smallest upset containing S, $\downarrow S$ for the smallest downset containing S, and for $x \in S$ we use $\uparrow x$ for $\uparrow\{x\}$ and $\downarrow x$ for $\downarrow\{x\}$.

The *specialization order* of a topological space is defined by setting $x \leq y$ if x is in the closure of y. This is a partial ordering on X iff X is T_0. For a stably compact space (X,τ) with specialization order $\leq$ and patch topology $\pi = \tau \vee \tau^k$, we have that $(X,\pi,\leq)$ is a Nachbin space with upper topology τ and lower topology τ^k. This provides a 1-1 correspondence between stably compact spaces and Nachbin spaces (see, e.g., [3, Sec. VI-6]).

We next turn to the definition of the well-known hit-or-miss topology. For a topological space X, let $\mathscr{O}(X)$ be the set of open sets, $\mathscr{F}(X)$ the set of closed sets, and $\mathscr{K}(X)$ the set of compact sets in X.

Definition 2.4. Let X be a topological space.

1. For $S \subseteq X$, define

$$\square_S = \{F \in \mathscr{F}(X) \mid F \cap S = \varnothing\} \text{ and } \lozenge_S = \{F \in \mathscr{F}(X) \mid F \cap S \neq \varnothing\}.$$

2. Let $\tau_\square$ be the topology on $\mathscr{F}(X)$ given by the subbasis $\{\lozenge_K \mid K \in \mathscr{K}(X)\}$.
3. Let $\tau_\lozenge$ be the topology on $\mathscr{F}(X)$ given by the subbasis $\{\lozenge_U \mid U \in \mathscr{O}(X)\}$.
4. Let $\pi = \tau_\square \vee \tau_\lozenge$.

We call $\tau_\Diamond$ the *hit topology*, $\tau_\Box$ the *miss topology*, and π the *hit-or-miss topology*.

It is easily seen that for any collection $\{S_i \mid i \in I\}$ of subsets of X, we have

$$\bigcap_{i\in I} \Box_{S_i} = \Box_{\bigcup_{i\in I} S_i} \text{ and } \bigcup_{i\in I} \Diamond_{S_i} = \Diamond_{\bigcup_{i\in I} S_i}.$$

Therefore, the subbasis for $\tau_\Box$ is actually a basis, and the hit-or-miss topology has a basis of sets of the form

$$\{\Box_K \cap \Diamond_{U_1} \cap \cdots \cap \Diamond_{U_n} \mid K \in \mathscr{K}(X) \text{ and } U_1, \ldots, U_n \in \mathscr{O}(X)\}.$$

If X is locally compact, then the hit-or-miss topology π on $\mathscr{F}(X)$ is compact Hausdorff [2, Thm. 1], $(\mathscr{F}(X), \pi, \subseteq)$ is a Nachbin space [4, p. 57], the lower topology π_ℓ of this Nachbin space is the hit topology $\tau_\Diamond$, and the upper topology π_u is the miss topology $\tau_\Box$. Moreover, if X is compact Hausdorff, then it is easy to see that the hit-or-miss topology coincides with the Vietoris topology [5, Sec. III-4]. The next result is well-known.

Proposition 2.5. *The map $e : X \to \mathscr{F}(X)$ that sends x to its closure $\overline{\{x\}}$ has the following properties.*

(1) *e is 1-1 iff X is T_0.*
(2) *If $U \in \mathscr{O}(X)$, then $e^{-1}(\Diamond_U) = U$; hence e is continuous with respect to $\tau_\Diamond$.*
(3) *If X is T_1 and $K \in \mathscr{K}(X)$, then $e^{-1}(\Box_K) = X \setminus K$; hence if X is Hausdorff, then e is continuous with respect to $\tau_\Box$.*
(4) *If X is Hausdorff, then e is continuous with respect to π.*

An *embedding* of a space X into a space Y is a 1-1 map $e : X \to Y$ that is a homeomorphism from X to the image $e(X)$ given the subspace topology from Y. Classically, a *compactification* of a space X is an embedding of X into a compact Hausdorff space Y where the image of X is dense in Y. Smyth [7] introduced stable compactifications to generalize the classical theory of compactifications to the setting of T_0-spaces. Using [1, Thm. 3.5] Smyth's definition can be formulated as follows.

Definition 2.6. A *stable compactification* of a T_0-space X is an embedding of X into a stably compact space Y where the image of X is dense in the patch topology of Y.

A related notion is that of an *order-compactification* of an ordered topological space $(X, \pi, \leq)$. This consists of a Nachbin space $(Y, \pi, \leq)$ and a mapping $e : X \to Y$ that is both a topological embedding and an order embedding.

Definition 2.7. (see, e.g., [4, p. 57]) For X a locally compact T_0-space, its *Fell compactification* $H(X)$ is the closure of the image of X in the hit-or-miss topology of $\mathscr{F}(X)$.

Since X is locally compact, the hit-or-miss topology is compact Hausdorff, so the closed subset $H(X)$ of $\mathscr{F}(X)$ is a compact Hausdorff space. When X is a non-compact locally compact Hausdorff space, $e : X \to H(X)$ is an embedding, and the Fell compactification is the one-point compactification of X (see [2, p. 475]). When X is non-Hausdorff, $e : X \to H(X)$ is no longer an embedding. So in this setting the term Fell compactification is somewhat of a misnomer. However, there are two ways to rectify this, by altering the topology of either $\mathscr{F}(X)$ or X.

Proposition 2.8. *If X is locally compact T_0, then the Fell compactification $H(X)$ with the restriction of the hit topology is a stable compactification of X.*

Proof. Since X is locally compact, $(\mathscr{F}(X), \pi, \subseteq)$ is a Nachbin space, and since $H(X)$ is a closed subset, it naturally forms a Nachbin space as well. The upper topology of $\mathscr{F}(X)$ is the hit topology $\tau_\Diamond$, and it follows that the restriction of $\tau_\Diamond$ to $H(X)$ is its upper topology. So under the restriction of $\tau_\Diamond$ we have that $H(X)$ is a stably compact space. By definition, $H(X)$ is the closure of the image of X under the topology π, hence this image is dense in the patch topology of the stably compact space $H(X)$. □

Proposition 2.9. *Let (X, τ) be a locally compact T_0-space, $\leq$ its specialization order, and σ the smallest topology on X making $e : X \to \mathscr{F}(X)$ continuous with respect to the hit-or-miss topology. Then $\tau \subseteq \sigma$ and $e : (X, \sigma, \leq) \to (H(X), \pi, \subseteq)$ is an order-compactification of $(X, \sigma, \leq)$.*

Proof. By Proposition 2.5(2), $\tau \subseteq \sigma$. Also, since e is 1-1 by Proposition 2.5(1), e is a topological embedding of (X, σ) into $(H(X), \pi)$. Therefore, $e : (X, \sigma) \to (H(X), \pi)$ is a compactification of (X, σ). To see that it is an order-compactification, observe that $e(x) \subseteq e(y)$ iff $\mathsf{cl}_\tau\{x\} \subseteq \mathsf{cl}_\tau\{y\}$ iff ${\downarrow}x \subseteq {\downarrow}y$ iff $x \leq y$. □

3 The Fell compactification of a poset

Throughout this section P is a poset with partial ordering $\leq$. The collection of upsets of P is closed under arbitrary intersections and arbitrary unions, and in particular forms a topology on P called the Alexandroff topology. We denote it τ_A. Clearly the closed sets of τ_A are the downsets of P. It is known [**?**], and easily seen, that τ_A is T_0 and that the specialization order on P given by τ_A is the given partial ordering $\leq$ of P. The following is easily seen.

Proposition 3.1.

4 The lattice of order-compactifications of a chain

Describe all order-compactifications of a chain by means of proximities; show there is always a least one, so it is a complete lattice. Give examples showing how Fell can sometimes be the least one, sometimes the largest one, sometimes neither.

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