Type-2 Fuzzy Sets

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AMS meeting, Akron, October 2012

Overview

The truth value object for type-2 fuzzy sets is the algebra M of all functions from the unit interval to itself. Here we consider a range of topics related to this algebra.

Overview

1. Definition of M.
2. Basic properties of M.
3. The variety V(M) generated by M.
4. Decidability of the equational theory of V(M).
5. Towards an equational basis of V(M).
6. A brief excursion to projectives.
7. Convex normal functions — a subalgebra of M.
8. Other orders and the finite analog.

1. Definition of M

Definition $\mathbb{I}$ is the unit interval.

Definition M is the set of all functions $f : \mathbb{I} \to \mathbb{I}$ equipped with operations $\sqcup, \sqcap, *, \bar{0}, \bar{1}$ given by

$$\begin{aligned} (f \sqcup g)(x) &= \sup \{f(y) \wedge g(z) : y \vee z = x\} \\ (f \sqcap g)(x) &= \sup \{f(y) \wedge g(z) : y \wedge z = x\} \\ f^*(x) &= \sup \{f(y) : \neg y = x\} \end{aligned}$$

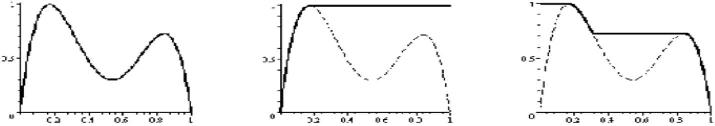
The constants $\bar{0}(x)$, $\bar{1}(x)$ are characteristic functions of $\{0\}$, $\{1\}$.

Note These are convolutions of $\wedge, \vee, \neg, 0, 1$ on $\mathbb{I}$ in the sense that polynomial multiplication is a convolution.

2. Basic properties of M

Definition For $f : \mathbb{I} \to \mathbb{I}$ let

1. f^L = the least increasing function above f.
2. f^R = the least decreasing function above f.



f $\qquad$ f^L $\qquad$ f^R

Note L and M are not part of the type of M, neither are pointwise meet and join $\wedge, \vee$. Enriching M this way would be of interest.

2. Basic properties of M

Using these auxiliary operations L, R and pointwise join and meet, we have much tidier expressions for our operations.

Proposition

1. $f \sqcup g = (f \vee g) \wedge f^L \wedge g^L$.
2. $f \sqcap g = (f \vee g) \wedge f^R \wedge g^R$.

The operation $*$ on M is computed directly to be $f^*(x) = f(1-x)$.

While this makes working with M more tractable, we can do better.

2. Basic properties of M

Definition A bisemilattice is an algebra $(L, +, \cdot)$ where

1. $+$ and $\cdot$ are commutative and associative.
2. $x + x = x$ and $x \cdot x = x$.

It is a Birkhoff system if it also satisfies $x + (x \cdot y) = x \cdot (x + y)$.

Notes

A bisemilattice is two unconnected semilattice operations on the same set. It can be described by any two Hasse diagrams on the set. In a Birkhoff system, the semilattice operations are connected.

A lattice is a Birkhoff system where $x + (x \cdot y) = x = x \cdot (x + y)$.

2. Basic properties of M

Definition A De Morgan bisemilattice is an algebra $(L, +, \cdot, *, 0, 1)$ consisting of a Birkhoff sysytem with additional operations where

1. $*$ is period two.
2. $(x + y)^* = x^* \cdot y^*$.
3. 0 and 1 are units for $+$ and $\cdot$ respectively.

Birkhoff systems have a large literature, and have been studied since the late 60's. De Morgan bisemilattices are more recent, since about 2000 (Brzozowski).

3. The variety V(M) generated by M

Brzozowski showed ...

Theorem $(\mathsf{M}, \sqcap, \sqcup, *, \bar{0}, \bar{1})$ is a De Morgan bisemilattice.

Notes

M is not a lattice, and the partial orders given by its semilattice operations $\sqcup$ and $\sqcap$ do not agree. We will call these orders the join and meet order of M.

3. The variety V(M) generated by M

Definition A variety of algebras is a class of algebras defined to be those satisfying some set of equations.

Examples Abelian groups, rings, lattices, Birkhoff systems and De Morgan bisemilattices all form varieties.

For any algebra A, there is a smallest variety containing it, the class of all algebras satisfying the same equations as A.

Definition V(A) is the variety generated by A.

3. The variety V(M) generated by M

Proposition Let $\mathcal{F}$ be a set of homomorphisms from A to B and

1. For each $x \neq y$ in A there is $f \in \mathcal{F}$ with $f(x) \neq f(y)$.
2. Some $f \in \mathcal{F}$ is onto.

Then V(A) = V(B).

Strategy To find V(M) find a simpler algebra B and family $\mathcal{F}$ of homomorphisms that separates point, to show V(M) = V(B). We use this repeatedly to get ever simpler such B.

3. The variety V(M) generated by M

Definition Let $\mathbb{I}^+$ be the power set of $\mathbb{I}$ with operations

1. $S \sqcup T = \{s \vee t : s \in S \text{ and } t \in T\}$.
2. $S \sqcap T = \{s \wedge t : s \in S \text{ and } t \in T\}$.
3. $S^* = \{\neg s : s \in S\}$.
4. $\bar{0} = \{0\}$.
5. $\bar{1} = \{1\}$.

We call $\mathbb{I}^+$ the complex algebra of $\mathbb{I}$. This idea is used extensively in logic, and dates back ≈ 100 years to complex algebras of groups.

3. The variety V(M) generated by M

Proposition $V(M, \sqcup, \sqcap) = V(\mathbb{I}^+)$.

Proof The maps $\varphi_a : M \to \mathbb{I}^+$ with $\varphi_a(f) = \{x \in \mathbb{I} : a < f(x)\}$ separate points.

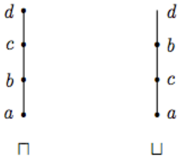
Proposition $V(\mathbb{I}^+) = V(3^+)$ where 3 is a three-element chain.

Proof Homomorphisms from $\mathbb{I}$ to 3 lift to ones from $\mathbb{I}^+$ to 3^+ providing a separating family of maps.

3. The variety V(M) generated by M

So $V(M, \sqcup, \sqcap)$ is generated by a finite (8-element) algebra 3^+ With some basic universal algebra, we can show $V(3^+)$ is generated by a 4-element algebra.

Theorem $V(M, \sqcup, \sqcap)$ is generated by the 4-element algebra below.

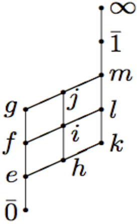


This kind of bisemilattice is called a bichain.

3. The variety V(M) generated by M

Similar results hold when all the operations are considered.

Theorem $\mathsf{V}(\mathsf{M}, \sqcup, \sqcap, *, \bar{0}, \bar{1})$ is generated by 5^+ and by the algebra



$$
\begin{aligned}
\infty^* &= \infty \\
\bar{1}^* &= \bar{0} \\
m^* &= h \\
l^* &= e \\
k^* &= k \\
j^* &= j \\
i^* &= g \\
h^* &= m \\
g^* &= i \\
f^* &= f \\
e^* &= l \\
\bar{0}^* &= \bar{1}
\end{aligned}
$$

Decidability of the equational theory of V(M)

Theorem There is an effective algorithm to determine whether a given equation holds in V(M).

Proof Check if it holds in the finite algebra that generates V(M).

Notes

For an equation with just $\sqcup, \sqcap$ we use a 4-element algebra. If there are n variables, its order is 4^n. Otherwise, it is 12^n, still not bad.

There is another version of this decidability result based on finding a normal form for terms. It seems to be of the same order.

Decidability of the equational theory of V(M)

Corollary M satisfies the generalized distributive law

$$p \sqcap q = p \sqcup q$$

where $p = x \sqcap (y \sqcup z)$ and $q = (x \sqcap y) \sqcup (x \sqcap z)$.

Notes

The usual distributive law is $p = q$.

This implies that any subalgebra of M that is a lattice is a distributive lattice, and therefore a De Morgan algebra.

Towards an equational basis of V(M)

As with any variety, V(M) can be defined by a set of equations. We wish to find such a set. This is still open, but ...

Conjecture V(M) equals the variety S defined by the equations true in all bichains and the generalized distributive law.

Notes

- V(M) $\subseteq$ S $\subseteq$ BiCh (the variety generated by all bichains).
- V(M) and S contain exactly the same bichains.
- S is the splitting variety of a 3-element bichain.

Fancy tools of universal algebra seem of little help in solving this. A primary trouble is that V(M) is not congruence distributive.

A brief excursion to projectives

There is a lot more to this story. Very briefly ...

Definition P is projective in a variety V if for any $A \in V$ and onto $f : A \to P$ there is $B \leq A$ with $f : B \to P$ an isomorphism.

Example We show the 2-element bichain C below is projective in the variety of Birkhoff systems.

Say x_1, x_2 are generators of a free Birkhoff system F and they are mapped to $1, 2$ respectively. We must build a copy of C in F that is mapped isomorphically onto C.

A brief excursion to projectives

Fix the $\cdot$ operation (the left).

Now fix the $+$ operation (the right).

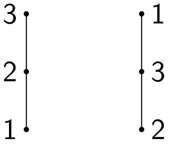
We don't have to fix the left again because we can prove it is okay. Indeed, $x_2(x_2 + x_1x_2) = x_2 + x_2x_1x_2 = x_2 + x_1x_2$.

A brief excursion to projectives

Note Projective + subdirectly irreducible ⇒ splitting.

We became interested in projectives to show the one 3-element bichain not contained in the 4-element bichain that generates V(M) is splitting. We hope its splitting variety is V(M).

We thought that every finite bichain would be projective. That is the case with finite chains. However, the 3-element bichain below is not projective (not so easy to show)!



N

A brief excursion to projectives

Theorem For a finite bichain C, these are equivalent.

1. C is projective in the variety of Birkhoff systems.
2. C does not contain the 3-element bichain N as a subalgebra.

This then leads to a characterization of finite splitting bichains.

This used nasty computations proved by humans with intuition guided by computer (Prover9).

Convex normal functions — a subalgebra of M

Definition Call $f \in \mathrm{M}$ convex if $f = f^L \wedge f^R$ and normal if $\sup f = 1$.

In plain terms, convex functions are ones that go up then go down. Concave might be a better name.

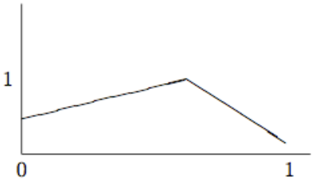
Theorem Let L be the convex normal functions. Then

1. L is a subalgebra of M.
2. The orders from $\sqcup$ and $\sqcap$ agree on L.
3. L is a distributive lattice and a De Morgan algebra.
4. L is complete, but not meet or join continuous.

Convex normal functions — a subalgebra of M

There is a nice way to untangle the crazy operations on L.

For a convex normal f, let $S(f)$ be the result of "flipping f up", i.e. taking the reflection in the line $y = 1$ of the increasing part.



Convex normal functions — a subalgebra of M

Theorem The flipping map S is an isomorphism from $(\mathsf{L}, \sqcap, \sqcup, \bar{0}, \bar{1})$ to the lattice of decreasing functions from $\mathbb{I}$ to $[0, 2]$ whose range has 1 as an accumulation point.

This opens the way to a new idea, defining an equivalence relation of "almost everywhere" on functions. But care is needed.

Warning The bottom and top $\bar{0}, \bar{1}$ of L are the characteristic functions of $\{0\}, \{1\}$, so agree almost everwhere.

Convex normal functions — a subalgebra of M

Definition Define a relation on L by setting $f \cong g$ if the flipped up versions $S(f)$ and $S(g)$ agree almost everywhere.

Theorem $\cong$ is a congruence and for D being the quotient $L/\cong$

1. D is a complete, completely distributive lattice.
2. D has a natural metric (from $\int |f-g|\,dx$).
3. With this metric D is a compact Hausdorff topological lattice.
4. This topology agrees with the Lawson topology on D.
5. D further carries a continuous De Morgan structure.

Other orders and the finite analog

The key point in defining M is that $\mathbb{I}$ is a complete bounded chain. (We really just need complete).

Definition For finite integers $m, n \geq 0$ let

1. M(m,n) be all maps from $\{1, \ldots, m\}$ to $\{1, \ldots, n\}$.
2. L(m,n) be all convex maps from $\{1, \ldots, m\}$ to $\{1, \ldots, n\}$.

Again, we define operations $\sqcup, \sqcap, *, \bar{0}, \bar{1}$ to be convolutions of the join, meet, involution, and bounds of the n-element chain.

Proposition The situation is as before. The L(m,n) are subalgebras of the M(m,n) and they form de Morgan algebras.

Other orders and the finite analog

The structures L(m,n) and M(m,n) have interesting combinatorial and order theoretic properties. Briefly (still not finished) . . .

Theorem The L(m,n) are related to projectives.

Each M(m,n) has two orders, one from each semilattice operation. As they are finite bounded semilattices, each is a lattice order. But neither order makes the map * on M(m,n) an involution.

Theorem The intersection of the meet and join orders on M(m,n) is a (non-distributive) lattice order that makes * an involution.

Note I have no idea why either of these two theorems are true.

Unfinished business

Here are some open problems and other directions ...

1. Find an equational basis for V(M).
2. For any complete lattice L and set X, operations on L can be convoluted to operations on L^X. Investigate.
3. Which t-norms on the convex normal functions are compatible with the congruence $\cong$ of equivalence almost everywhere?
4. Make sense of the combinatorics of the structures M(m,n).

Many thanks to the organizers!

Thank you for listening.

Papers at www.math.nmsu.edu/~jharding