Quantum structures

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Introduction

Quantum structures is a broad term for a range of areas involving mathematical structures related to quantum mechanics (QM).

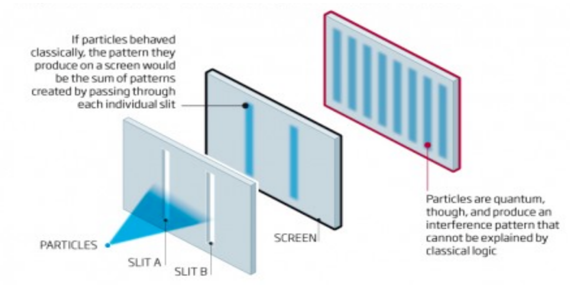
These structures are studied for their connections to QM, and for their own interest. Many are related to other areas. Often, they are non-commutative, or non-distributive versions of classical objects.

I'll talk about some of my results in this area. A bit of a mixture, hopefully something of interest for various tastes.

Introduction

To start, two ideas from physics that influence matters.

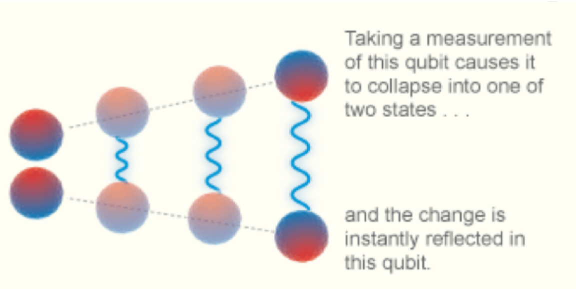
Two key aspects of QM — superposition



If observed, a single particle will pass through one of the two slits.
If not observed, it passes through both.

People say it is a superposition of the two alternatives.

Two key aspects of QM — entanglement



If two particles have interacted (created from the same source), then a measurement on one particle instantaneously affects the other, no matter the distance. “Spooky action at a distance”.

Definitions

Next, definitions of some basic quantum structures, orthomodular posets and lattices.

In some non-trivial ways, these structures are tied to superposition. Explaining this will be one of our aims.

Definitions

Definition An ortholattice (OL) is a bounded lattice with unary operation $\perp$ that satisfies the following.

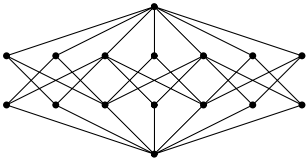
1. $x \wedge x^{\perp} = 0$ and $x \vee x^{\perp} = 1$.
2. $x \leq y \Rightarrow y^{\perp} \leq x^{\perp}$.
3. $x^{\perp\perp} = x$.

An orthomodular lattice (OML) is an OL that satisifes

4. $x \leq y^{\perp} \Rightarrow x \vee (x \vee y)^{\perp} = y^{\perp}$.

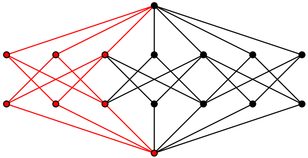
An orthomodular poset (OMP) is a poset satisfying similar conditions, except we require joins only for x, y with $x \leq y^{\perp}$.

Examples



This OML is built from three 8-element Boolean algebras.

Examples



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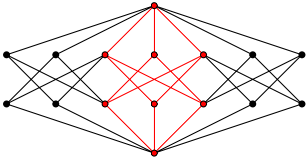

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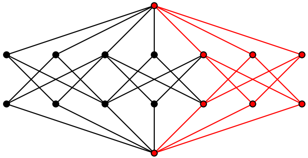

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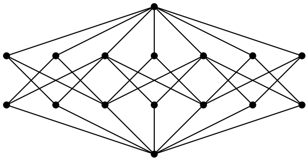

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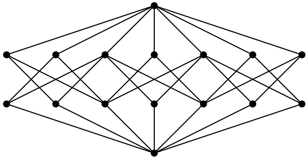

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Theorem Every OL is the union of its Boolean subalgebras. The OMLs are exactly the OLs where $x \leq y$ iff $x \leq y$ in some Boolean subalgebra.

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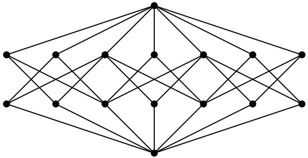

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Theorem Every OL is the union of its Boolean subalgebras. The OMLs are exactly the OLs where $x \leq y$ iff $x \leq y$ in some Boolean subalgebra.

So OMLs are locally classical (Boolean algebras). This is common of many quantum structures.

Examples

The motivating example of an OML is the projection lattice $\mathcal{P}(\mathcal{H})$ of a Hilbert space.

More generally, the projections of any von Neumann algebra A form an OML that nearly determines A ...

Theorem (Dye) The OML of projections of A determines A up to Jordan isomorphism.

von Neumann algebras and C* algebras are again locally classical, being built from function rings $C(X)$.

Result I — varieties of OMLs

The algebraic study of OMLs for their own sake is quite interesting, a bit like the study of modular lattices. Still open ...

- Is the free word problem for OMLs decidable?
- Can every OML be embedded in a complete OML?

One of my results in this area ...

Result I — varieties of OMLs

Theorem Let A be in a variety $\mathcal{V}$ generated by a class of OMLs of finite height at most n. Then there is a sheaf $\mathcal{S}$ of OMLs over a Boolean space X such that

1. A is isomorphic to the global sections of $\mathcal{S}$.
2. The stalks of $\mathcal{S}$ have height at most n on a dense open set.

Further, the MacNeille completion of A is the algebra of sections over dense open sets, modulo equivalence on a dense open set, hence belongs to $\mathcal{V}$.

Note This applies, for example, to the variety generated by $\mathcal{P}(\mathbb{R}^3)$.

The QM motivation for OMLs

The work of von Neumann is still the textbook treatment of QM ...

- To each system one associates a Hilbert space $\mathcal{H}$
- Pure states are vectors in $\mathcal{H}$, general ones density operators.
- Observables (ex. position) are self adjoint operators A on $\mathcal{H}$
- Time evolution is given by a family of unitary operators on $\mathcal{H}$.

Going from $\mathcal{H}$ to $\mathcal{P}(\mathcal{H})$ is a lot of fancy mathematics ...

The QM motivation for OMLs

Spectral Theorem

Self adjoint operators on $\mathcal{H}$ (observables) correspond to σ-additive homomorphisms from Borel$(\mathbb{R}) \to \mathcal{P}(\mathcal{H})$.

Gleason's Theorem

States on $\mathcal{H}$ correspond to σ-additive maps $\mathcal{P}(\mathcal{H}) \to [0, 1]$.

Wigner's Theorem

Automorphisms of $\mathcal{P}(\mathcal{H})$ correspond to unitary and anti-unitary operators on $\mathcal{H}$.

The QM motivation for OMLs

$$\text{Borel}\,(\mathbb{R}) \underbrace{\xrightarrow{\quad A \quad}}_{\text{spectral theorem}} \mathcal{P}\,(\mathcal{H}) \underbrace{\xrightarrow{\quad \sigma \quad}}_{\text{Gleason's theorem}} [0,1]$$

$\sigma \circ A(B) =$ probability that a measurement of the observable A will yield a result in the Borel subset B when the system is in state σ

The old quantum logic program aimed to explain why (if?) the OML in the middle should be $\mathcal{P}(\mathcal{H})$.

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The old quantum logic program aimed to explain why (if?) the OML in the middle should be $\mathcal{P}(\mathcal{H})$. Some results ...

Why the OML $\mathcal{P}(\mathcal{H})$?

- Birkhoff & vN argued that $\mathcal{P}(\mathcal{H})$ gave the Yes/No questions of a system, and these give a non-distributive logic.
- Mackey gave plausible physical arguments why the questions $\mathcal{Q}$ of a quantum system form an OMP.
- Amemiya & Araki showed the closed subspaces of an inner product space form an OML iff the space is a Hilbert space.

Everyone knew of "toy" OMLs, but there was a feeling that orthomodularity captured an essential aspect of Hilbert spaces.

Result II — options for $\mathcal{Q}$

Change our viewpoint! Orthomodularity comes not from Hilbert spaces, but what we do with them. Projections of $\mathcal{H}$ correspond to direct product decompositions $\mathcal{H} \simeq \mathcal{H}_1 \times \mathcal{H}_2$.

Theorem For A any set, group, ring, vector space, topological space, partially ordered topological group, ... its direct product decompositions $\mathcal{Q}(A)$ form an OMP where

1. $$(A \simeq A_1 \times A_2)^{\perp} \quad = \quad A \simeq A_2 \times A_1.$$
2. $$A \simeq A_1 \times (A_2 \times A_3) \quad \leq \quad A \simeq (A_1 \times A_2) \times A_3.$$

Result II — options for $\mathcal{Q}$

This gives a path to many alternatives to $\mathcal{P}(\mathcal{H})$. Among those close to Hilbert space setting ...

- $\mathcal{Q}(A)$ for A a normed group with operators.
- $\mathcal{Q}(A)$ for A a f.d. vector bundle.

In both cases there is a rich theory for $\text{Borel}(\mathbb{R}) \to \mathcal{Q}(A) \to [0,1]$.

A version of Wigner's theorem for $\mathcal{Q}(X)$ for a finite set X feels like the fundamental theorem of projective goemetry.

Result II — options for $\mathcal{Q}$

Why decompositions should arise? Dirac used the superposition principal as motivation for the use of vector spaces in QM.

> *The superposition process is a kind of additive process and implies that states can in some way be added to give new states. The states must therefore be connected with mathematical quantities of a kind which can be added together to give other quantities of the same kind. The most obvious of such quantities are vectors.*

We view the superposition of v_1 and v_2 not as $v_1 + v_2$, but as the ordered pair (v_1, v_2). Does one add non-orthogonal states?

Recent trends in QM — PVMs

Interest in quantum information changed emphasis from (sharp) observables to unsharp ones given by projection valued measures. Models for unsharp questions are effect algebras.

Definition An effect algebra $(E, \oplus, 0, 1)$ is a set with partial binary operation $\oplus$ such that

1. $\oplus$ is commutative and associative.
2. For each x there is a unique x' with $x \oplus x' = 1$.
3. $x \oplus 1$ defined $\Rightarrow x = 0$.

For a Hilbert space, $\mathcal{E}(\mathcal{H}) = \{A : A \text{ is self-adjoint and } 0 \leq A \leq I\}$.

Result III — Effect algebras in hindsight

Effect algebras were defined for their role in QM. There is a nice algebraic motivation for the definition of effect algebras.

Theorem

1. The forgetful functor from OMPs to POSETS has an adjoint given by the Kalbach construction (gluing Boolean algebras generated by chains of the poset).
2. Effect algebras are algebras for the resulting monad.

I did the first bit, Jenca the second.

Recent trends in QM — categorical QM

Recent interest in quantum computation has put primary focus on entanglement of quantum systems. For Hilbert spaces, the space for the compound system is given by the tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$.

For the OMP approach to questions, this is a huge problem since for OMPs $\mathcal{Q}_1$ and $\mathcal{Q}_2$, there need not be an OMP with the properties required to be questions $\mathcal{Q}_1 \otimes \mathcal{Q}_2$ of the compound system.

A recent path is the so-called categorical quantum mechanics. It owes much to the computer science view of information flow.

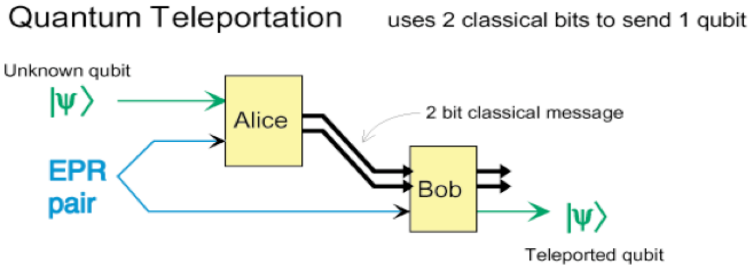
Recent trends in QM — categorical QM

Definition Abramsky and Coecke's categorical QM has

1. A strongly compact closed monoidal category $\mathcal{C}$ with tensor $\otimes$.
2. Objects in $\mathcal{C}$ are quantum systems.
3. Morphisms in $\mathcal{C}$ are processes.

Such categories have a graphical calculus (from Penrose) that is very useful to study *protocols*. An example ...

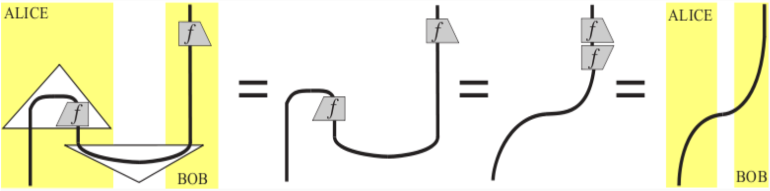
Recent trends in QM — categorical QM



Alice gets one half of an entangled pair, Bob the other. Alice's half is measured with a particle in state $|\Psi>$. One of 4 outcomes is obtained. Alice tells Bob which one. Bob performs one of 4 adjustments on his half. Bob's half ends up a duplicate of $|\Psi>$.

Recent trends in QM — categorical QM

The graphical calculus of strongly compact closed categories gives the correctness of the teleportation calculus in the following simple way. It is obviously useful in creating other protocols.



This categorical setup also gives basic reasons why "no-cloning" theorems, and other basic results.

Result IV — questions $\mathcal{Q}$ in categorical QM

Here $\mathcal{C}$ is a strongly compact closed category with biproducts (the original formulation of categorical QM).

Definition For an object $A \in \mathcal{C}$, let $\mathcal{Q}(A)$ be the binary biproduct decompositions of A.

Theorem For objects $A, A_1, A_2 \in \mathcal{C}$

1. $\mathcal{Q}(A)$ forms an OMP.
2. $\mathcal{Q}(A_1 \otimes A_2)$ is (nearly) an OMP tensor product $\mathcal{Q}(A_1) \otimes \mathcal{Q}(A_2)$.

This ties categorical QM with the Birkhoff von Neumann appraoch

Recent trends in QM — the topos approach

Butterfield and Isham have recently created a view of QM based in topos theory. We just give a flavor.

To a QM system, associate a von Neumann algebra $\mathcal{A}$. Its abelian subalgebras $\mathcal{S}$ give sets of observables that can be simultaneously observed, and behave classically together.

To each such $\mathcal{S}$ associate its dual space. This gives a presheaf Σ over the (dual of the) poset of abelian subalgebras of $\mathcal{A}$.

Theorem The Kochen Specker Theorem (one can't simultaneously assign values to all observables of a QM system) is equivalent to Σ failing to have a global section.

Recent trends in QM — the topos approach

Definition For a von Neumann algebra $\mathcal{A}$, let $\mathrm{Ab}(\mathcal{A})$ be its poset of abelian subalgebras, ordered by inclusion.

Definition For an OMP P, let $\mathrm{Bool}(P)$ be its poset of Boolean subalgebras.

A central ingredient of the topos approach is the Bohr topos, whose internal logic is the Heyting algebra of downsets of $\mathrm{Ab}(\mathcal{A})$. Look at nLab on the web for more details.

Result V — The Bohr topos

Theorem Let P be an OMP, and $\mathcal{A}$ be a von Neumann algebra.

1. P is determined up to isomorphism by $\mathrm{Bool}(P)$.
2. $\mathcal{A}$ is determined up to Jordan isomorphism by $\mathrm{Ab}(\mathcal{A})$.

Current work Extending to categorical dualities, and finding a nice way to use orientations to change Jordan isomorphism to isomorphism.

Thank you for listening.

Papers at www.math.nmsu.edu/~jharding