On topological Boolean algebras

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Ames, September 2014

Organization

Part I — a combinatorial conjecture

Part II — what the conjecture has to do with top Boolean algebras.

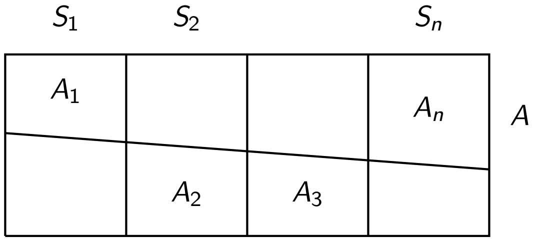
Part I

A Combinatorial Conjecture — a Better Boolean Bogoliuboff!

The conjecture

X is a finite set, partitioned into n pieces $S_1, \ldots, S_n$.

U is a collection of subsets of X such that for any $A \subseteq X$, at least one of the 2^n sets built from A belongs to U.



Then there are two sets in U whose union contains all but at most one element from each S_i.

The conjecture — weakened

Our conjecture asks to prove there are 2 sets in U whose union contains all but at most one element from each S_i.

Lemma There are 4 sets in U whose union contains all but at most 2^{2n} elements of X.

Proof This follows from the proof of Bogoliuboff's Lemma for finite abelian groups and uses group characters. It gives no understanding of why it is true (at least to me).

The conjecture — why we care

The conjecture admits a much nicer statement than the lemma. It seems more likely to admit a simple combinatorial proof.

The Lemma is sufficient for all we say later, but its lack of an elementary proof is what concerns us. In algebraic form ...

Lemma Let F be a finite Boolean algebra, $S \leq F$ and $U \subseteq F$ be such that $U + S = F$. Then there are $|S|^2$ coatoms of F whose meet belongs to $U + U + U + U$.

Part II — Topological Boolean algebras

Definition A topological Boolean algebra is a Boolean algebra B equipped with a topology making the basic operations continuous.

Theorem (Pappert Strauss) The compact Hausdorff topological Boolean algebras are exactly the 2^X where 2 is discrete.

Proof (key step) B is a topological abelian group. By Pontryagin duality it has continuous characters $\chi : B \to \mathbb{C}$ separating points. For each $x \in B$, $x + x = 0$, so χ maps into $\{-1, 1\}$. Then $\chi^{-1}[-1]$ and $\chi^{-1}[1]$ are disjoint clopen sets. ...

Aim

We would like an elementary proof of this result using only basic concepts of Boolean algebras and topological lattices. Others have been interested in this problem ...

- Marcel Erné
- Mamuka Jibladze
- Dito Pataraia

We achieve this using basic order theoretic arguments and ideas from Dikranjan's *An Elementary proof of the Peter Weyl Theorem*.

Basics

Proposition Let $x \in B$ and $V \subseteq B$ be open with $0 \in V$.

1. $x\downarrow$ and $x\uparrow$ are closed.
2. $V\downarrow$ is open.
3. $x + V$ open.
4. The closure $\Gamma V \subseteq V + V$.
5. V a downset $\Rightarrow$ $V + V = V \vee V$.
6. V a downset $\Rightarrow$ $\exists$ an open downset U with $U + U \subseteq V$.

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Proof 1 $x\downarrow = f^{-1}[\{x\}]$ where $f(\cdot) = \cdot \vee x$.

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Proof 2 $x \in V\downarrow \Rightarrow x \leq v$ for some $v \in V$. Set $f(\cdot) = \cdot \vee v$. Then $f(x) = v \in V$. V is open, so continuity of f gives an open $x \in W$ with $f(W) \subseteq V$. Then $W \subseteq V\downarrow$.

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Proof 3 $f(\cdot) = x + \cdot$ is continuous and its own inverse, therefore is a homeomorphism.

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Proof 4 Let $x \in$ the closure of V. Since $x + V$ is an open nhbd of x, then $x + V$ intersects V. Say $x + v = y \in V$. Then $x = y + v$, so $x \in V + V$.

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Proof 5 This is just basic Boolean algebra.

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Proof 6 $0 \vee 0 \in V$ so continuity gives an open W with $0 \in W$ and $W \vee W \subseteq V$. Let $U = W\downarrow$. Then $U + U = U \vee U = (W \vee W)\downarrow \subseteq V$.

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One last item that is literally the first theorem one proves about topological lattices (Johnstone).

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Theorem Each ideal in a compact Hausdorff topological lattice L has a join, and this join belongs to its closure. Thus L is complete.

Proof of Pappert Strauss' Theorem

Theorem If B is a compact Hausdorff topological Boolean algebra, then B is isomorphic and homeomorphic to 2^A for some set A.

Proof For isomorphism, it is enough to show B is atomic. Since B is complete, this implies B is isomorphic to 2^A for A its atoms.

For homeomorphism, let $a \in A$. Then $a'\!\downarrow$ and $a\!\uparrow$ are closed. It follows that they are clopen. So the topology of B is finer than the product topology of 2^A. Both are compact Hausdorff, hence equal.

We must prove atomicity ...

The proof — atomicity

Suppose $x \neq 0$.

Step 1 $x{\uparrow}$ and $\{0\}$ are disjoint and closed.

Step 2 There are disjoint open C, D with $x{\uparrow} \subseteq C$ and $0 \in D$.

Step 3 Set $V = D{\downarrow}$. Then V is open and $x \notin V$.

Step 4 Exists open downset U with $\Gamma(\underbrace{U + U + U + U}_{U_4}) \subseteq V$.

The proof — atomicity

Step 5 For each $b \in B$ we have $b + U$ is open. Use compactness!

Step 6 Exists finite $S \leq B$ so that $\{s + U : s \in S\}$ covers B.

Step 7 Set $\mathfrak{F} = \{F : F \text{ is finite and } S \leq F \leq B\}$.

Now we use our combinatorial lemma. Let $k = |S|^2$.

Step 8 For each $F \in \mathfrak{F}$ there are k prime ideals of F whose meet is contained in U_4.

Step 9 By a standard compactness argument from logic, there are prime ideals $P_1, \ldots, P_k$ of B whose meet is contained in U_4.

The proof — atomicity

Step 10 Let I be the ideal $P_1 \wedge \cdots \wedge P_k$.

Step 11 $I \subseteq U_4$ and $\Gamma U_4 \subseteq V$ and $V \cap x{\uparrow} = \varnothing$.

Step 12 $\bigvee I \in \Gamma I \subseteq V$ (from basics) $\Rightarrow x \nleq \bigvee I$.

Step 13 $\bigvee I = (\bigvee P_1) \wedge \cdots \wedge (\bigvee P_k)$ ($\wedge$-continuity for BAs)

Step 14 $x \nleq \bigvee P_i$ for some $1 \leq i \leq k$.

In a Boolean algebra, the join of a prime ideal is either 1 or a coatom. So there is a coatom c of B that does not lie above x. Therefore its complement c' is an atom beneath x.

Thank you for listening.

Papers at www.math.nmsu.edu/~jharding