The Type-2 Truth algebra

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Foreword

Sorry, but I will use the F – word.

I know that some don't think it belongs in talks.

Fuzzy sets

Basic idea

Ask an expert what they think of the statement

70 F is COLD

They say they believe this to the "degree" 0.4.

Fuzzy sets

Repeat this for each temperature in $X = \{50, 60, 70, 80, 90\}$ and we get a fuzzy subset of X, a map $\textsc{Cold} : X \to [0, 1]$

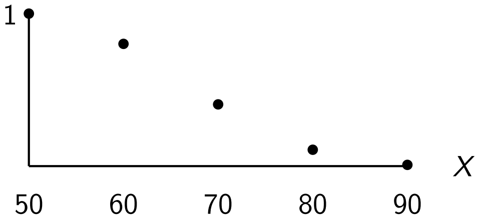


Figure: A fuzzy subset of X for $\textsc{Cold}$

Interval valued fuzzy sets

Rather than a number for degree of belief, an interval. In interval valued fuzzy subset is a map $\textsc{Cold} : X \to \{(a, b) : a \leq b\}$.

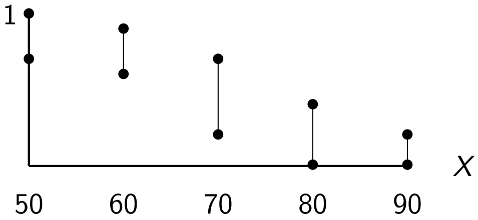
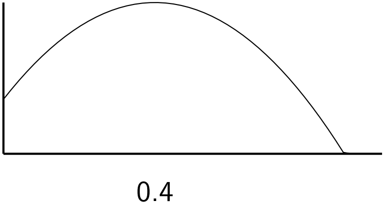


Figure: An interval valued fuzzy subset of X for $\textsc{Cold}$

Type-2 fuzzy sets

To describe the statement 70 is COLD, experts give a function $f : [0,1] \to [0,1]$.



Expert(s) are confident that 70 is COLD with truth degree 0.4, not confident that 70 is COLD with degree 0.9.

Type-2 fuzzy sets

A type-2 fuzzy subset is $\textsc{Cold} : X \to \{f \mid f : [0,1] \to [0,1]\}$

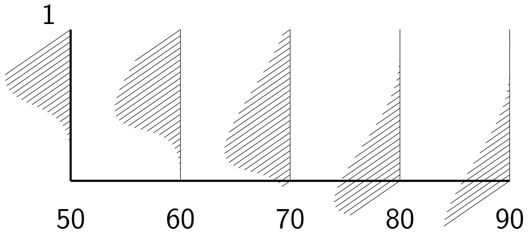


Figure: A type-2 fuzzy subset of X for $\textsc{Cold}$

Truth value algebras

The truth value algebras for fuzzy sets, interval valued fuzzy sets, and type-2 fuzzy sets are

$$\mathbf{I} = [0,1]$$

$$\mathbf{I}^{[2]} = \{(a,b) : a \leq b \in \mathbf{I}\}$$

$$\mathrm{M} = \{f \mid f : \mathbf{I} \to \mathbf{I}\}$$

$\mathbf{I}$ and $\mathbf{I}^{[2]}$ sit in M as characteristic functions of points and intervals

Operations

To make computations with fuzzy sets we use operations on these algebras.

Both $\mathbb{I}$ and $\mathbb{I}^{[2]}$ are bounded distributive lattices with negation, De Morgan algebras.

Sometimes in place of $\wedge$ one uses a t-norm $\triangle$, a commutative, associative, order preserving operation with 1 as an identity. For example, regular multiplication on $\mathbb{I}$.

Operations on M

Definition (Zadeh) Define the following operations on M

1. $(f \sqcap g)(x) = \bigvee\{(f(y) \wedge g(z)) : y \wedge z = x\}$
2. $(f \sqcap g)(x) = \bigvee\{(f(y) \wedge g(z)) : y \vee z = x\}$
3. $f^*(x) = f(1 - x)$
4. $0(x) = 1$ if $x = 0$ and 0 otherwise
5. $1(x) = 1$ if $x = 1$ and 0 otherwise

Operations on M

Are they natural?

Are they useful?

Are they tractable?

Operations on $\mathbb{M}$

Are they natural?

They are convolutions like $(f \star g)(x) = \int_0^1 f(t)g(x-t)dt$

They are all convolutions of the operations of $\mathbb{I}$ using $\wedge$ and $\bigvee$

They are also related to the complex algebra of $\mathbb{I}$.

Operations on $\mathbf{M}$

Are they useful?

Both $\mathbf{I}$ and $\mathbf{I}^{[2]}$ embed as subalgebras into $\mathbf{M}$.

They allow the categorical mathematics to build fuzzy controllers for a natural subset of type-2 fuzzy sets.

The answer depends on whether such controllers work well.

Operations on M

Are they tractable

Yes.

We see how to calculate them easily, determine algebraic properties of M, solve its free word problem, determine its automorphisms.

For a natural subalgebra of M we show it is a frame, that t-norms lift to it, develop nice topological properties, and use this in a categorical setting to work with fuzzy controllers.

Working with $\sqcap$ and $\sqcup$

These are not pointwise operations, but they can be computed pointwise with the aid of two auxiliary operations.

Definition For $f : \mathbf{I} \to \mathbf{I}$ let f^L and f^R be the least increasing and least decreasing functions above f.

Theorem

1. $f \sqcap g = (f \vee g) \wedge f^R \wedge g^R$
2. $f \sqcup g = (f \vee g) \wedge f^L \wedge g^L$

Equations

Theorem M satisfies the equations for De Morgan algebras except that absorption and distributivity are weakened to the following.

1. $x \sqcap (x \sqcup y) = x \sqcup (x \sqcap y)$
2. $(x \sqcap y) \sqcup (x \sqcap z) \sqcup (y \sqcap z) = (x \sqcup y) \sqcap (x \sqcup z) \sqcap (y \sqcup z)$

M is not a lattice.

The unbalanced distributive laws do not hold.

M is a type of thing known as a De Morgan Birkhoff system.

Complex algebras

Definition For a bounded chain C with negation $*$ define C^+ to be the power set of C with operations

1. $A \sqcap B = \{a \wedge b : a \in A, b \in B\}$
2. $A \sqcup B = \{a \vee b : a \in A, b \in B\}$
3. $A^* = \{a^* : a \in A\}$
4. $0 = \{0\}$
5. $1 = \{1\}$
6. $A \wedge B = A \cap B$
7. $A \vee B = A \cup B$
8. $A^L = A \uparrow$ (the upset of A)
9. $A^R = A \downarrow$ (the downset of A)

Varieties

Theorem If C has at least 5 elements, then M and C^+ generate the same variety.

The variety generated by M is generated by a 32 element algebra. So its free word problem is solvable. Using the UA calculator, we can find smaller algebras that generate the same variety.

Corollary The variety generated by $(M, \sqcap, \sqcup)$ is generated by

$\sqcap$ $\sqcup$

Automorphisms

Theorem Automorphisms α and β of I given an automorphism Γ of M, and all automorphisms of M arise this way.

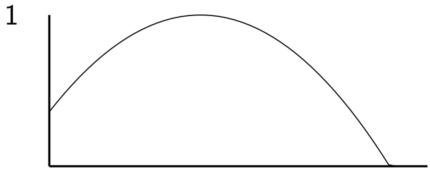
$$\Gamma(f) = \beta \circ f \circ \alpha$$

One half of this is quite difficult to prove.

Convex normal functions

Definition For $f \in \mathrm{M}$ we say it is

1. convex if $f = f^L \wedge f^R$ (it goes up then down)
2. normal if $\sup f = 1$



There is an argument that the functions used for assigning type-2 degrees of belief to $\textsc{Cold}$ could be assumed convex normal.

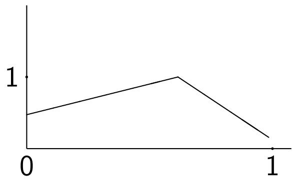
Convex normal functions

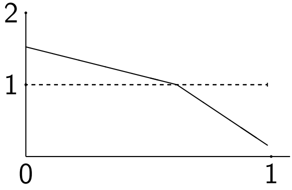
Note It was fuzzy engineers that came up with the name convex. It is not the notion used in calculus for a couple hundred years.

Theorem The convex normal functions L are a subalgebra of M that is a lattice, hence an actual De Morgan algebra.

Straightening

Working with L is still a challenge because its operations are not pointwise. For $f \in \mathrm{L}$ define its straightening by flipping up its increasing part.





Straightening the order

Definition Let D be the De Morgan algebra of all $g : [0, 1] \to [0, 2]$ that are decreasing and have 1 in the closure of their image.

Theorem L is isomorphic to D.

Now things are pointwise, so easy to work with.

Agreement almost everwhere

The top and bottom of L agree almost everywhere. But their straightened versions do not!

Definition Two members of L agree convexly almost everywhere (c.a.e.) if their straightened versions agree almost everywhere.

Agreement almost everywhere

Theorem Agreement c.a.e. is a congruence on L. The quotient algebra has the following properties.

1. It is complete, and completely distributive.
2. $\int_0^1 |f(x) - g(x)|\, dx$ is a metric on it.
3. It is a compact Hausdorff topological De Morgan algebra.

It is easy to argue why members of the quotient algebra L (c.a.e.) should serve as truth values for things such as $\textsc{Cold}$.

Lifting t-norms

The operations $\sqcap$ and $\sqcup$ were lifted from those of I by convolution. We can so lift any operation on I to M.

Theorem Let $\triangle$ be a continuous t-norm on I. Then

1. Its convolution $\blacktriangle$ is a operation on L
2. $\blacktriangle$ has the properties of a t-norm on L
3. Agreement (c.a.e.) is a congruence with respect to $\blacktriangle$
4. In L (c.a.e.) $f \blacktriangle \bigvee g_i = \bigvee (f \blacktriangle g_i)$

Fuzzy controllers

An example

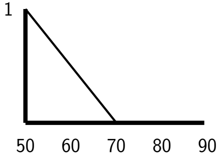
We have a room with a device in it to heat and cool the room and a sensor that measures approximate temperature.

$$X = \{50, 60, 70, 80, 90\} \quad \text{possible temperatures}$$
$$Y = \{-2, -1, 0, +1, +2\} \quad \text{settings of the device}$$

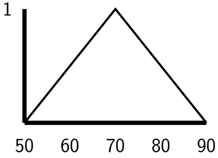
A setting of -2 puts lots of cold air in the room, $+2$ lots of hot air.

Fuzzy controllers

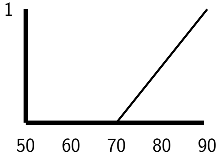
Make linguistic variables COLD, NICE, and HOT for temperature; AIR and FURNACE for settings. Experts give fuzzy sets for these.



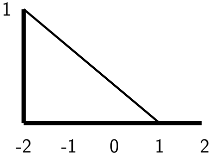
COLD



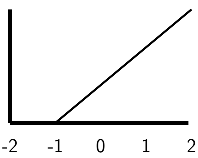
NICE



HOT



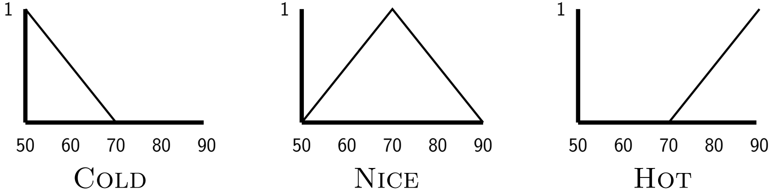
AIR



FURNACE

Fuzzy controllers

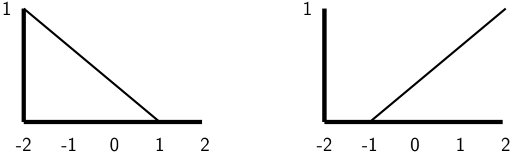
We represent the fuzzy sets for temperature as a matrix



$$P = \begin{pmatrix} 1 & .5 & 0 & 0 & 0 \\ 0 & .5 & 1 & .5 & 0 \\ 0 & 0 & 0 & .5 & 1 \end{pmatrix}$$

	50	60	70	80	90
COLD	1	.5	0	0	0
NICE	0	.5	1	.5	0
HOT	0	0	0	.5	1

And do the same for adjustments



AIR FURNACE

$$Q = \begin{pmatrix} 1 & .7 & .3 & 0 & 0 \\ 0 & 0 & .3 & .7 & 1 \end{pmatrix}$$

	-2	-1	0	1	2
AIR	1	.7	.3	0	0
FURNACE	0	0	.3	.7	1

Fuzzy controllers

We are given a rule base that says what should be done in each case.

$$R = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

	Cold	Nice	Hot
Air	0	1	1
Furnace	1	0	0

Fuzzy controllers

Then if our sensor gives a reading of 80 for temperature we make a column vector $\hat{T}$ with a 1 in the spot for 80 and 0's elsewhere and compute $Q^T RP(\hat{T})$

$$\begin{pmatrix} 1 & 0 \\ .7 & 0 \\ .3 & .3 \\ 0 & .7 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & .5 & 0 & 0 & 0 \\ 0 & .5 & 1 & .5 & 0 \\ 0 & 0 & 0 & .5 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} .5 \\ .3 \\ .2 \\ 0 \\ 0 \end{pmatrix}$$

The result is a fuzzy subset of $Y = \{-2, -1, 0, 1, 2\}$ that we then "defuzzify" to get an adjustment to the device.

Fuzzy controllers

Matrix multiplication computes entries as sums of products.

This multiplication was done using $\cdot$ as product and $\bigvee$ as sum. It can be done using any continuous t-norm $\triangle$ as product and $\bigvee$ as sum. This requires

$$x \triangle \bigvee y_i = \bigvee (x \triangle y_i)$$

to obtain associativity of matrix multiplication.

Symmetric monoidal categories

Ordinary fuzzy controllers live in the symmetric monoidal category of matrices over $(\mathsf{I}, \triangle, \bigvee)$.

Objects: sets
Morphisms: matrices composed by multiplication

Tenor product is ordinary product of sets and Kronecker products of matrices. It allows to have more dependent or independent variables in the controller.

Type-2 fuzzy controllers

Do exactly the same with the category of matrices over $(\mathsf{L}, \blacktriangle, \bigvee)$.

Quantum

Quantum processes are also modeled in such symmetric monoidal categories.

Nobody is saying that fuzzy controllers and quantum processes are the same thing.

But they do have some commonality, and having them in the same setting should allow more direct comparisons.

Thanks for listening.

Papers at www.math.nmsu.edu/~jharding