Projective Bichains

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Basics

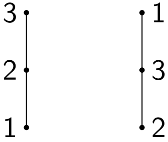
Definition A bisemilattice is an algebra $(B, \cdot, +)$ where $\cdot$ and $+$ are commutative, associative, and idempotent. A Birkhoff system is a bisemilattice that satisfies the weak absorption law

$$x \cdot (x + y) \;=\; x + (x \cdot y).$$

Let $\mathrm{B{\scriptsize IRK}}$ be the variety of Birkhoff systems.

Basics

Definition A bichain is a bisemilattice where the the two partial orderings given by $\cdot$ and $+$ are chains.



N

Proposition Every bichain is a Birkhoff system.

Main Theorem

Theorem For a finite bichain C these are equivalent.

1. C is weakly projective in $\textsc{Birk}$.
2. C does not contain a subalgebra isomorphic to N.

Note Weakly projective means wrt onto homomorphisms.

Corollary A recursive description of all finite weakly projective bichains, and all s.i. weakly projective (hence splitting) bichains.

Example

Show The two-element bichain C is weakly projective in BIRK.

C

Say x_1, x_2 are generators of a free Birkhoff system F and they are mapped to $1, 2$ respectively. We must build a copy of C in F that is mapped isomorphically onto L.

Example

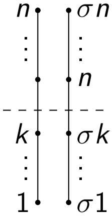
Fix the $\cdot$ operation (the left).

Now fix the $+$ operation (the right).

We don't have to fix the left again because we can prove it is okay. Indeed, $x_2(x_2 + x_1x_2) = x_2 + x_2x_1x_2 = x_2 + x_1x_2$.

Sketch of Main Proof

Say C is an n-element bichain on the elements $1, \ldots, n$.



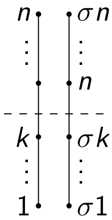
Assume the left side is ordered $1, \ldots, n$, the right $\sigma 1, \ldots, \sigma n$.

No N as a subalgebra, means no medium, big, small on right.

So below dashed line permutes $1, \ldots, k$, above $k + 1, \ldots, n$.

Sketch of main proof

When we fix left, then the right, then left, etc. we can show ...



When fixing up the left, then right, then left, etc.

After a certain number of steps the terms on the top of the dashed line do not occur when fixing the bottom, and the terms on the bottom of the dashed line do not occur when fixing the top.

Induction!

Notes on the main proof

All this involves a horrible amount of axiomatics.

We used PROVER 9 on medium sized examples (8 elements) to guide us to conjectures of what general identities might be true, then proved our conjectures by hand.

PROVER 9 could not have done this without us, and we could not have done this without it. An interesting use of machine.

Notes on the main proof

Of the many identities needed are ones such as the following:

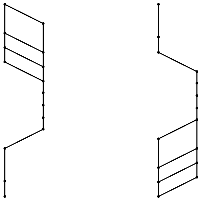
Lemma In any Birkhoff system, if $x_1 \leq \cdots \leq x_n$ under meet, then

$$(x_1 + \cdots + x_k)(x_1 + \cdots + x_n) = x_1 + \cdots + x_k$$

The main proof — other direction

Suppose C contains a copy of N. We construct from C a Birkhoff system that has C as a quotient, but not as a subalgebra.

It is a bit unpleasant.



The origins of this work

The basic object of type-II fuzzy sets is the following algebra M

Definition On the set M of all functions from $[0,1]$ to itself define

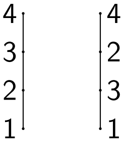
$$\begin{aligned} (f \cdot g)(x) &= \sup\{\min\{f(y), g(z)\} : \min\{y,z\} = x\} \\ (f + g)(x) &= \sup\{\min\{f(y), g(z)\} : \max\{y,z\} = x\} \end{aligned}$$

Note Often, one considers other operations too such as $0, 1, \neg$.

Note These operations are certain *convolutions* of those on $[0,1]$.

The origins of this work

Theorem The variety $V(M)$ generated by M is generated by the 4-element bichain below



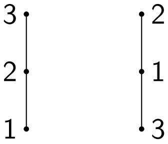
Corollary The equational theory of M is decidable.

Question Find an equational basis for $V(M)$.

Towards an equational basis

Theorem For a bichain C these are equivalent.

1. C belongs to $V(M)$.
2. C satisfies $(x(y+z))(xy+xy) = (x(y+z)) + (xy+xz)$.
3. C does not contain the splitting bihcain S below.



Towards an equational basis

We still don't know a basis, but the above leads us to guess ...

Conjecture $V(M)$ is the splitting variety of S in the variety $\textsc{Bichains}$ generated by the bichains. So a basis of $V(M)$ is the equations defining $\textsc{Bichains}$ and the generalized distributive law

$$(x(y+z))(xy+xy) = (x(y+z)) + (xy+xz).$$

Question What are equations defining $\textsc{Bichains}$?

Thank you for listening.

Papers at www.math.nmsu.edu/$\sim$jharding