Decompositions in quantum mechanics — an overview

John Harding

Abstract There is current interest in using ideas from quantum mechanics in the study of economics. We give an overview of an approach to quantum mechanics rooted not necessarily in Hilbert space, but in the primitive mathematical idea of direct products. This approach includes the standard von Neumann Hilbert space approach. It provides a conceptually simpler understanding of issues from standard quantum mechanics, and offers possibilities beyond the standard Hilbert space formulation. These further possibilities may be of particular interest in consideration of economics where the aim is to exploit quantum principles rather than specific physical situations.

1 Introduction

The standard mathematical treatment of quantum mechanics took its modern form through the work of Dirac [4] and von Neumann [24]. It was von Neumann who fully expressed the theory in terms of Hilbert spaces. A considerable amount of sophisticated mathematics is put into the first steps of the mathematical formulation of the quantum theory of an electron.

In quantum computing, one is content with finite-dimensional Hilbert spaces $\mathbb{C}^n$. But even to consider position and momentum of a single electron, one requires an infinite-dimensional Hilbert space $\mathscr{H}$. This is a complex vector space with inner product $u \cdot v$, that gives a norm $\|u\| = \sqrt{u \cdot u}$, that in turn gives a metric $d(u,v) = \|u-v\|$ under which the vector space is a complete metric space. Observables, such as position and momentum, correspond to self-adjoint operators on $\mathscr{H}$; and states correspond to density operators on $\mathscr{H}$.

John Harding
New Mexico State University, Las Cruces, New Mexico 88003, USA
e-mail: jharding@nmsu.edu

A number of substantial mathematical results are the backbone of the theory. The spectral theorem [17, 21] relates a self-adjoint operator to a family of projection operators of $\mathscr{H}$. Gleason's theorem [5, 6, 16, 20] relates density operators to probability measures on the collection $\mathscr{P}(\mathscr{H})$ of all projection operators. Wigner's theorem [23, 25] relates symmetries of the system to automorphisms of $\mathscr{P}(\mathscr{H})$, and Stone's theorem [17, 22, 23] relates the time evolutions of the system to these symmetries.

In this sea of mathematical terminology and results, the question we should not lose is "why?" Why should we attach a Hilbert space to a system, why are self-adjoint operators used for observables, and so forth. The most popular answer to these questions is is because it works; but this is not really an answer at all, it is an instruction to not ask the question.

Mackey [18, p. 61-71] gave a simple set of six physically motivated axioms that showed that the collection $\mathscr{Q}$ of Yes/No questions that can be asked of a physical system form a type of structure known as an orthomodular poset (OMP). This is a structure $(\mathscr{Q},\leq,',0,1)$ comprised of a family of Boolean algebras "glued together", and allows one to reason about events as is done in a Boolean algebra, but allowing for the possibility that certain events might not have a conjunction or disjunction. Mackey's seventh axiom was completely unmotivated, and simply stated that the OMP of questions are the projections $\text{Proj}(\mathscr{H})$ of a Hilbert space $\mathscr{H}$.

In this note, we show how simple properties of direct products can be used to construct an OMP of questions from virtually any type of structure such as a set, or group, or topological space, in place of a Hilbert space. Using the premise of direct products as basic, we give operationally motivated constructions of observables and dynamics. When applied to a Hilbert space, this recreates standard quantum theory. In terms of motivation of the use of direct products — a Yes/No question splits the system into a product of two, one where the event occurs, one where the alternate occurs.

This note is a compendium of results obtained and presented in detail in a series of papers [7, 8, 9, 10, 11, 12, 13, 14, 15].

2 Direct product decompositions of Hilbert spaces

We first consider the relationship between projections and products for a Hilbert space. A projection operator P has a closed subspace A as its range, and each closed subspace is the range of a unique projection. A fundamental property of closed subspaces is that each vector $w\in\mathscr{H}$ can be uniquely expressed as $w=u+v$ for some vectors $u\in A$ and $v\in A^\perp$, where $A^\perp$ is the set of vectors orthogonal to all vectors in A. Thus $\mathscr{H}$ is in bijective correspondence with the set of ordered pairs (u,v) where $u\in A$ and $v\in A^\perp$, hence with $A\times A^\perp$. This is the idea behind the following result.

Theorem 1. *A closed subspace A of a Hilbert space $\mathscr{H}$ gives a direct product decomposition $\mathscr{H} \simeq A \times A^{\perp}$, and a direct product decomposition $\mathscr{H} \simeq \mathscr{H}_1 \times \mathscr{H}_2$ gives a closed subspace $A = \{(u,0) : u \in \mathscr{H}_1\}$.*

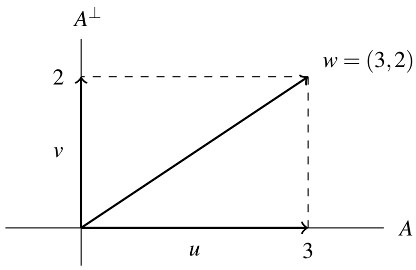

Fig. 1 A closed subspace A inducing a direct product decomposition

We are interested not just in the set of projections of $\mathscr{H}$, but also in its structure as an OMP.

Definition 1. An orthomodular poset (OMP) [18, 20, 23] is a structure $(\mathscr{Q}, \leq, ', 0, 1)$ that consists of a partially ordered set with least element 0, largest element 1, and an order reversing unary operation $' : \mathscr{Q} \to \mathscr{Q}$ of period two called orthocomplementation such that the following hold.

1. if $x \leq y'$, then x, y have a least upper bound that is denoted $x \oplus y$
2. $x \oplus x' = 1$
3. if $x \leq y$ then $x \oplus (x \oplus y')' = y$

We say that x, y are orthogonal when $x \leq y'$.

Example 1. The closed subspaces of a Hilbert space $\mathscr{H}$ form an orthomodular poset Closed$(\mathscr{H})$. The partial ordering is set containment $\subseteq$, the orthocomplementation A' is given by the orthogonal subspace $A^{\perp}$, the least element 0 is the trivial subspace $\{0\}$, and the largest element 1 is the whole space. When $S \subseteq T^{\perp}$, the least upper bound of S and T is their span $S \oplus T$. It turns out that the closed subspaces are a lattice, that is, that any two closed subspaces have a least upper bound and greatest lower bound, but in physical considerations it is difficult to give meaning to these for arbitrary closed subspaces.

Closed subspaces of $\mathscr{H}$ correspond to direct product decompositions of $\mathscr{H}$. So we can consider the OMP structure in terms of direct product decompositions. The least closed subspace $\{0\}$ and largest closed subspace $\mathscr{H}$ correspond to the direct product decompositions

$$\mathscr{H} \simeq \{0\} \times \mathscr{H} \quad \text{and} \quad \mathscr{H} \simeq \mathscr{H} \times \{0\}$$

The orthocomplement of the decomposition $\mathscr{H} \simeq A \times A^{\perp}$ for A is given by the decomposition for $A^{\perp}$, which is $\mathscr{H} \simeq A^{\perp} \times A^{\perp\perp}$. Since $A^{\perp\perp} = A$, this is the decomposition $\mathscr{H} \simeq A^{\perp} \times A$. Orthocomplements of decompositions are very simple, we just switch the order of the factors.

The situation for the ordering $A \subseteq B$ of closed subspaces is the most interesting of all. Consider the example shown in Figure 2 of $\mathscr{H}$ being $\mathbb{R}^3$ with X, Y, Z being the three coordinate axes. Let A be the closed subspace X consisting of the x-axis, and let B be the x, y-plane. Then indeed $A \subseteq B$. Note that $\mathscr{H} \simeq X \times Y \times Z$ is a ternary direct product decomposition. Since $A^{\perp} = Y \times Z$ and $B = X \times Y$, we have

$$\mathscr{H} \simeq A \times A^{\perp} \quad \text{is} \quad \mathscr{H} \simeq X \times (Y \times Z)$$

$$\mathscr{H} \simeq B \times B^{\perp} \quad \text{is} \quad \mathscr{H} \simeq (X \times Y) \times Z$$

The partial ordering $\leq$ of binary decompositions is given by all instances of the following that arise from a ternary decomposition $\mathscr{H} \simeq X \times Y \times Z$

$$\mathscr{H} \simeq X \times (Y \times Z) \quad \leq \quad \mathscr{H} \simeq (X \times Y) \times Z$$

That every instance of $A \subseteq B$ is captured by this follows from the third condition in Definition 1, known as the orthomodular law. If $A \subseteq B$, then $B = A \oplus C$ where $C = (A \oplus B^{\perp})^{\perp}$. So $\mathscr{H} = A \times C \times B^{\perp}$ is the required ternary product decomposition.

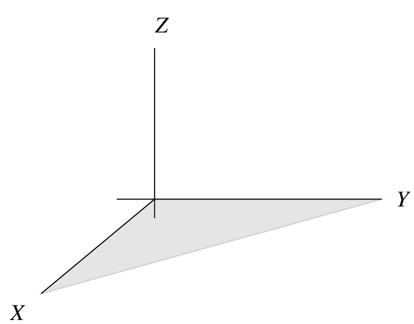

Fig. 2 A ternary decomposition given rise to a comparability between binary decompositions

3 Direct product decompositions in a general setting

In the previous section, we showed that one can naturally define the structure of an OMP on the binary direct product decompositions $\mathscr{H} \simeq \mathscr{H}_1 \times \mathscr{H}_2$ of a Hilbert space so that the resulting structure is isomorphic to the OMP of closed subspaces Closed($\mathscr{H}$) that plays such a vital role in standard quantum theory. This can be extended to many types of structure. We begin by considering matters for a set X.

Definition 2. A n-ary direct product decomposition of a set X is an indexed family of sets $X_1,\dots,X_n$ and a bijection $\alpha : X \to X_1 \times \cdots \times X_n$. This n-ary direct product decomposition is equivalent to another $\beta : Y_1 \times \cdots \times Y_n$ if for $i = 1,\dots,n$ there are bijections $\gamma_i : X_i \to Y_i$ so that $\beta = (\gamma_1 \times \cdots \times \gamma_n) \circ \alpha$. See Figure 3.

Example 2. Let $X = \{a,b,c,d\}$ be a 4-element set. Up to equivalence, there is one binary decomposition of $X \simeq \{*\} \times X$ as the product of a 1-element set and a 4-element set X; one binary decomposition $X \simeq X \times \{*\}$ as the produce of a 4-element set and a 1-element set; and 6 binary decompositions $X \simeq \{p,q\} \times \{r,s\}$ as the product of two 2-element sets. The specific elements in the two 2-element sets are of no importance. In each of these six decompositions, each of a,b,c,d is represented as an ordered pair in $\{p,q\} \times \{r,s\}$. The decomposition is specified by stating which element has the same first component as a, and then which has the same second component as a.

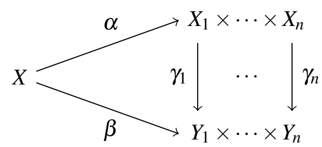

Fig. 3 Equivalence of n-ary decompositions

Definition 3. Let X be a set. For an n-ary decomposition $\alpha : X \to X_1 \times \cdots \times X_n$, let $[\alpha : X \to X_1 \times \cdots \times X_n]$ be the class of all equivalent decompositions. Let $\mathscr{Q}(X)$ be the set of all equivalence classes of binary decompositions of X, which we call questions.

Definition 4. For a set X, put structure $(\mathscr{Q}(X), \leq, ', 0, 1)$ by setting

1. $0 = [X \simeq \{*\} \times X]$ and $1 = [X \simeq X \times \{*\}]$
2. $[X \simeq X_1 \times X_2]' = [X \simeq X_2 \times X_1]$
3. $[X \simeq X_1 \times (X_2 \times X_3)] \leq [X \simeq (X_1 \times X_2) \times X_3]$ for each $[X \simeq X_1 \times X_2 \times X_3]$

In this definition, we have suppressed the specific bijections, but they are the obvious ones. For $[\alpha : X \to X_1 \times X_2]$, the orthocomplement is $[\alpha' : X \to X_2 \times X_1]$ where $\alpha'(x) = (x_2, x_1)$ if $\alpha(x) = (x_1, x_2)$.

Theorem 2. *For a set X, the structure* $(\mathscr{Q}(X), \leq, ', 0, 1)$ *is an* OMP.

This theorem is established in [8] where it is generalized to other types of structure such as groups, rings, modules, G-sets, topological spaces, uniform spaces, normed groups, topological groups, and so forth. For a group G, an n-ary direct product decomposition consists of an indexed family $G_1,\dots,G_n$ of groups and a group isomorphism $\alpha : G \to G_1 \times \cdots \times G_n$. The idea for other types of structures are similar, using appropriate notions of products and isomorphisms.

4 Questions

It was Mackey's original argument [18] that showed that the questions of a quantum system should form an OMP. Here we consider matters for the OMP of questions $\mathscr{Q}(X)$ for a set X, but our comments apply equally to the questions of of group, topological space, and importantly, to the OMP $\mathscr{Q}(\mathscr{H})$ of questions of a Hilbert space $\mathscr{H}$ that is used so extensively in standard treatments of quantum mechanics.

Definition 5. An n-ary experiment of a physical system is an experiment that has n mutually exclusive and exhaustive outcomes labelled Outcome 1, ..., Outcome n. A question is a binary experiment. We usually use Yes for Outcome 1, and No for Outcome 2 of a question.

Remark 1. An n-ary experiment might consist of n detector bulbs placed in different areas with the guarantee that exactly one bulb will go off. If one wishes to push against this definition and say there will always be the possibility that more than one, or none, of the bulbs will flash, then view an experiment with n bulbs as a 2^n-ary experiment, with one outcome for each possible set of bulbs that can flash. One might also wish to consider experiments with infinitely many outcomes. We don't believe these are physical, and will treat them later as limits of families of finitary experiments.

The fundamental idea of the decompositions approach is that one associates to a physical system a mathematical object such as a set X, and that each n-ary experiment of the system corresponds to an equivalence class of n-ary direct product decompositions $[X \simeq X_1 \times \cdots \times X_n]$.

Definition 6. Let e be an n-ary experiment $[X \simeq X_1 \times \cdots \times X_n]$. Then for a sequence $\sigma = S_1, \ldots, S_k$ of pairwise disjoint sets with union $\{1, \ldots, n\}$ define $\sigma(e)$ to be the k-ary experiment $[X \simeq Y_1 \times \cdots \times Y_k]$ where $Y_j = \prod_{i \in S_j} X_i$.

We say that an experiment f is built from e if $f = \sigma(e)$ for some σ, and let $B(e)$ be the set of all binary experiments, that is, questions, that are built from e. For example, if e is the 4-ary experiment $e = [X \simeq X_1 \times X_2 \times X_3 \times X_4]$, then some of the questions built from e are the following.

$$\begin{aligned}(\{1,2\},\{3,4\})e &= [X \simeq (X_1 \times X_2) \times (X_3 \times X_4)] \\ (\{3\},\{1,2,4\})e &= [X \simeq X_3 \times (X_1 \times X_2 \times X_4)]\end{aligned}$$

Note, that the use of the empty set is allowed. Since the product of the empty family is a singleton $\{*\}$, we have $(\emptyset, \{1,2,3,4\})e = [X \simeq \{*\} \times X]$.

Definition 7. A subset B of an OMP P is a Boolean subalgebra if it is closed under orthocomplementation, closed under the join $x \oplus y$ of orthogonal elements, contains the bounds $0, 1$, and with the inherited order forms a Boolean algebra under these operations.

If B is a Boolean subalgebra of P, then two elements x,y in B have a join $x \vee y$ and meet $x \wedge y$, and these are their join and meet as taken in B.

Theorem 3. *The finite Boolean subalgebras of the* OMP $\mathscr{Q}(X)$ *are exactly the sets* $B(e)$ *of questions built from some common experiment e. Further, if e has n factors that are not singletons* $\{*\}$*, then* $B(e)$ *has n atoms.*

In quantum mechanics, two questions are compatible if they lie in a Boolean subalgebra of the OMP of questions. Physically, this is interpreted to mean that they can be conducted at the same time. But conducting two binary questions e,f at the same time is the same as conducting one 4-ary experiment g whose outcomes are given by the pairs of outcomes Yes-Yes, Yes-No, No-Yes, and No-No to the questions e,f. This common sense physical reasoning is born in the mathematics.

Proposition 1. *Two questions* e,f *are compatible, i.e. lie in a Boolean subalgebra, iff there is a unique 4-ary experiment* $g = [X \simeq X_1 \times X_2 \times X_3 \times X_4]$ *with*

$$e = (\{1,2\},\{3,4\})g = [X \simeq (X_1 \times X_2) \times (X_3 \times X_4)]$$
$$f = (\{1,3\},\{2,4\})g = [X \simeq (X_1 \times X_3) \times (X_2 \times X_4)]$$

In this case their meet and join are given by

$$e \wedge f = (\{1\},\{2,3,4\})g = [X \simeq X_1 \times (X_2 \times X_3 \times X_4)]$$
$$e \vee f = (\{1,2,3\},\{4\})g = [X \simeq (X_1 \times X_2 \times X_3) \times X_4]$$

Remark 2. There is similarity between Boolean operations on classical questions, which are modeled as subsets of an event space, and Boolean operations on compatible quantum questions viewed as binary direct product decompositions. Instead of taking the union and intersection of sets as in the classical case, we take the union and intersection of indexing sets of the factors of a direct product decomposition.

There is a further property of questions arising from decompositions that points to a physical property of quantum mechanics that is not obvious. Of course, this property applies also to the standard Hilbert space setting, but does not generally hold for questions modeled by an arbitrary OMP. In alternate terminology, the following result [9] shows that the OMP $\mathscr{Q}(X)$ is regular.

Theorem 4. *Let X be a set and S be a set of n questions in* $\mathscr{Q}(X)$*. If any two questions in S can be asked simultaneously, then all of the questions in S can be asked simultaneously. In fact, they can all be built from an experiment with* 2^n *outcomes. This holds also for a group, module, topological space, and so forth.*

5 States

A finite set of pairwise orthogonal elements in an OMP has a join. An OMP is called σ-complete if every countable set of pairwise orthogonal elements has a join [20].

Definition 8. A state on an OMP P is a map $s : P \to [0,1]$ with

1. $s(0) = 0$ and $s(1) = 1$.
2. If x is orthogonal to y, then $s(x \oplus y) = s(x) + s(y)$.

A state on a σ-OMP is countably additive if the second condition applies to joins of countable pairwise orthogonal families.

Remark 3. Depending on one's perspective, what we call a state is called a finitely additive state, and a state refers to a countably additive state. We shall retain the idea of finite additivity as the primitive notion defining a state. The use of countable additivity is a means to deal with limiting processes of physical situations. As we will see, there are other means.

The idea of a state $s : P \to [0,1]$ is that when restricted to a Boolean subalgebra B of P it yields a finitely additive probably measure on B. A countably additive state yields a σ-additive probability measure on any Boolean σ-subalgebra of P. The physical interpretation of a state is that for a question e we have

$$s(e) \;=\; \text{the probability that } e \text{ yields a Yes answer when in state } s$$

There are some issues [11] with the existence of a good supply of states in the decompositions approach. It seems that one must begin with a structure that has some contact with the real numbers, and to take decompositions that interact well with this structure. While this is an area that requires more investigation, there are several directions that show a wide scope beyond the setting of Hilbert spaces [10].

Definition 9. An η-set is a set X with a map $\eta : X \to [0,\infty)$ and element $0 \in X$ that satisfies $\eta(0) = 0$. A product of a family $(X_1,\eta_1,0),\ldots,(X_n,\eta_n,0)$ of η-sets is the set $X = X_1 \times \cdots \times X_n$ with $\eta(x_1,\ldots,x_n) = \sum_{i=1}^{n} \eta_i(x_i)$ and $0 = (0,\ldots,0)$.

Theorem 5. *For an η-set X, the collection $\mathscr{Q}(X)$ of equivalence classes of it binary direct product decompositions is an* OMP*.*

The following shows that questions of an η-set have a large supply of states.

Theorem 6. *Let X be an η-set and $x \in X$ with $\eta(x) \neq 0$. There is a finitely additive additive state s_x on $\mathscr{Q}(X)$ such that for a question $e = [X \simeq X_1 \times X_2]$ with $x = (x_1,x_2)$*

$$s_x(e) = \frac{\eta_1(x_1)}{\eta(x)}$$

The standard Hilbert space approach is related to that of η-sets. Given a Hilbert space $\mathscr{H}$, define $\eta(x) = \|x\|^2$. Then questions given by closed subspaces satisfy $\|x\|^2 = \|x_1\|^2 + \|x_2\|^2$, a version of the standard Pythagorean theorem. So $\mathscr{Q}(\mathscr{H})$ is a sub-OMP of that given by considering $\mathscr{H}$ as an η-set. We next consider a situation intermediate to that of η-sets and that of Hilbert spaces.

Definition 10. A norm on a group G is a mapping $\|\cdot\| : G \to [0,\infty)$ with

1. $\|x\| = 0$ iff $x = 0$.
2. $\|x\| = \|-x\|$.
3. $\|x+y\| \leq \|x\| + \|y\|$.

We have written the operations of G additively, but G need not be abelian.

A norm on a group induces a metric in the usual way. In fact, there are close connections between normed groups and metric and topological groups [1]. By a complete normed group, we mean one whose metric is complete. A question of a normed group is an equivalence class $[G \simeq G_1 \times G_2]$ that is simultaneously a group decomposition and an η-set decomposition with respect to the map $\eta(g) = \|g\|^2$. For details see [8].

Theorem 7. *For a normed group G, its questions form an* OMP $\mathscr{Q}(G)$ *and for each* $g \in G$ *with* $\|g\| \neq 0$ *there is a finitely additive state* s_g *with*

$$s_g(e) = \frac{\|g_1\|^2}{\|g\|^2}$$

If G is complete, then $\mathscr{Q}(G)$ *is* σ*-complete and the states* s_g *are* σ*-additive.*

6 Observables

To begin, suppose we have an n-ary experiment $e = [X \simeq X_1 \times \cdots \times X_n]$ for a set X, or some other structure such as a group, or η-set, and so forth. For $i \leq n$ we have a question $e_i = [X \simeq X_i \times \prod_{j \neq i} X_j]$. Then for a state $s : \mathscr{Q}(X) \to [0,1]$ we interpret

$$s(e_i) \;=\; \text{the probability of the } i^{th} \text{ outcome of } e \text{ when in state } s$$

An n-ary experiment is intended to be a purely physical thing, such as detectors placed in various regions with exactly one guaranteed to go off. It is common to introduce a human element into matters as well by associating numerical values to various outcomes of an experiment. This choice of numerical values is a human activity, not a physical one. For instance, deflection through a Stern Gerlach divice may have outcomes labelled as -3/2, -1/2, 1/2, 3/2. This may be for what we consider to be a good reason, but it is still a matter of human choice in giving these numerical values of outcomes. In the following, we separate the physical aspects from the human chosen numerical ones.

Definition 11. A finite Boolean subalgebra of $\mathscr{Q}(X)$ is a finite observable quantity. If A is the set of atoms of a Boolean subalgebra of $\mathscr{Q}(X)$, then a map $f : A \to \mathbb{R}$ is a scaling of this finite observable quantity.

By Theorem 3, there is a correspondence between finite Boolean subalgebras of $\mathscr{Q}(X)$ with n atoms and n-ary experiments $e = [X \simeq X_1 \times \cdots X_n]$ with non-trivial factors. For such B, its atoms correspond to the questions $e_i = [X \simeq X_i \times \prod_{j\neq i} X_j]$. So a scaling of a finite observable quantity amounts to the same thing as an assignment of real numbers to the outcomes of an n-ary experiment.

Definition 12. A finite observable $O = (B, f)$ consists of a finite observable quantity B and a scaling f of B. The expected value of O in state s is $\sum_{i=1}^{n} s(e_i)f(e_i)$.

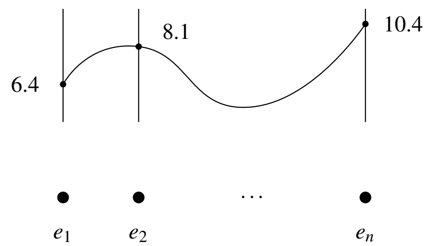

Fig. 4 A finite observable (B, f) where B has atoms $e_1, \ldots, e_n$ and the values of the scaling f on these atoms.

Remark 4. For Hilbert space quantum mechanics, a finite observable $O = (B, f)$ consists of a family of pairwise orthogonal projection operators $P_1, \ldots, P_n$ that sums to unity paired with a family $\lambda_1, \ldots, \lambda_n$ of real numbers. If different numerical values λ_i are given to different events P_i, then this information can be conveyed by the self-adjoint operator $A = \lambda_1 P_1 + \cdots \lambda_n P_n$. The eigenvalues and eigenspaces of A give back the projections and their scaling.

We take the view that finite observable quantities, which correspond to n-ary experiments, are the truly physical entities. We enrich these finite observable quantities with a scaling, but this process of attaching numerical values is a human process. There is a further abstraction of the purely physical to deal with limiting processes.

Definition 13. An observable quantity is a Boolean subalgebra of the OMP $\mathscr{Q}(X)$.

Definition 14. For an observable quantity B, a set $\mathscr{F}$ of questions in B is consistent any finite set of questions in $\mathscr{F}$ can possibly all have a Yes answer at the same time. In other words, $\mathscr{F}$ is consistent if the meet of any finite set of questions in $\mathscr{F}$ is non-zero. An ideal question is a maximally consistent set of questions in B.

The idea of an ideal question is simple, it is a type of limiting process of families of every finer collections of questions. Giving a physical example is challenging, not because of the idea of ideal questions, but because of a general lack of precision when we talk about physical measurements.

Example 3. Suppose we have a classical situation where a particle is located at some point along the real line. Any experiment we conduct might only tell us if the particle is within some given region. Following in this sprit, there should not be difference between testing for the region $(2,3)$ and the region $[2,3]$. A plausible mathematical setup would be a test for each equivalence class, up to sets of measure zero, of regions that are finite unions of possibly infinite intervals with rational endpoints. So our Boolean algebra of questions is the Boolean subalgebra of the algebra of Lebesgue measurable sets modulo measure zero that is generated by the intervals $(-\infty,q)$ for a rational q. For a real number x, the family of questions consisting of equivalences classes of intervals (p,q) containing x is then consistent.

Those familiar with Stone duality for Boolean algebras [3] will recognize that ideal questions of B are the ultrafilters of B. Much is known about this topic. If we let Z be the set of all ideal questions (i.e. ultrafilters) of B, then there is a topology on Z that has as a basis all sets of the form

$$\phi(e) = \{\mathscr{F} : \mathscr{F} \text{ is an ideal question and } e \in \mathscr{F}\}$$

With this topology, Z is a compact Hausdorff space, and the clopen subsets of this space are exactly the sets $\phi(e)$ for $e \in B$.

Definition 15. Let B be a Boolean algebra with Stone space Z. The Borel algebra of Z is the σ-algebra generated by its open sets, and the small Borel algebra of Z is the σ-algebra generated by the clopen sets of Z.

Definition 16. Let B be an observable quantity with Stone space Z. A scaling of B is an extended real valued function $f : Z \to \mathbb{R} \cup \{\pm\infty\}$ that is measurable with respect to the Borel algebra of the extended reals and the small Borel algebra of Z. An observable is a pair $O = (B, f)$ consisting of an observable quantity and a scaling.

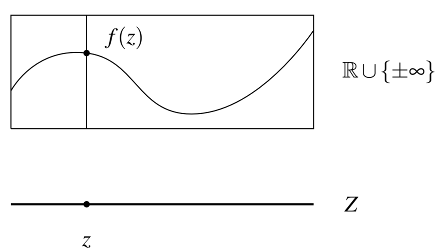

Fig. 5 An observable $O = (B, f)$ where B has Stone space Z and scaling $f : X \to \mathbb{R} \cup \{\pm\infty\}$.

Proposition 2. *For a finitely additive state $s : B \to [0,1]$ on B, there is a unique probability measure μ_s on the small Borel algebra of Z such that for each $e \in B$*

$$\mu_s(\Phi(e)) = s(e)$$

Thus $\mu_s(e)$ gives the probability of a Yes outcome to the question e when in state s.

Proof. The collection F of subsets of Z of the form $\phi(e)$ for a question e is a field a sets. Members of F are clopen subsets of a compact space, so have the finite intersection property. So the map $\mu_s : F \to [0,1]$ given by $\mu_s(\phi(e)) = s(e)$ is a probability in the sense of [19, p. 10]. By [19, p. 23], this extends uniquely to a probability measure on the σ-algebra generated by F, which is the small Borel algebra of Z.

Definition 17. Let $O = (B, f)$ be an observable, let s be a finitely additive state on B, and let T be a Borel subset of the extended reals. Then we interpret

$$\begin{aligned} \mu_s(f^{-1}(T)) &= \text{probability a measurement of } O \text{ lies in } T \text{ when in state } s \\ \int_Z f(z)\, d\mu_s &= \text{expected value of } O \text{ when in state } s \end{aligned}$$

Further, there is an obvious functional calculus of observables that are based on the same observable quantity B.

We relate this to the notion of an observable in standard Hilbert space quantum mechanics as a self-adjoint operator. First, we make an observation.

Proposition 3. *Let B be a complete Boolean algebra with Stone space Z and let $f : Z \to \mathbb{R} \cup \{\pm\infty\}$. If f is continuous, then it is measurable with respect to Borel algebra of the extended reals and the small Borel algebra of Z.*

Proof. It is enough to show that for any real number λ, that $A = f^{-1}[-\infty, \lambda)$ is in the small Borel algebra of Z. For each n let $A_n = f^{-1}[-\infty, \lambda - 1/n)$. Then $A = \bigcup_n A_n$. Since f is continuous, each A_n is open. Also the closure $\overline{A_n}$ is contained in the closed set $f^{-1}[-\infty, \lambda - 1/n]$ which is contained in A. So $A = \bigcup_n \overline{A_n}$. Since Z is the Stone space of a complete Boolean algebra, the closure of an open set in Z is clopen. So each $\overline{A_n}$ belongs to the small Borel algebra, and hence so does A.

Example 4. To each self-adjoint operator A on a Hilbert space $\mathscr{H}$ there is a complete Boolean subalgebra B of the projection lattice and a σ-homomorphism E_A from the Borel algebra of the reals to B given by the spectral theorem [21]. This E_A is called the spectral measure of A. Let $f_A : Z \to \mathbb{R} \cup \{\pm\infty\}$ given by

$$f_A(z) = \inf\{\lambda : E_A(-\infty, \lambda] \in z\}$$

Then f_A is continuous. In fact [17] shows that the self-adjoint operators affiliated with the von Neumann algebra generated by B are in bijective correspondence with the continuous extended real valued functions on Z, and that this correspondence preserves functional calculus. Since f_A is continuous, by Proposition 3 $O = (B, f)$ is an observable in the sense defined above. A state s in the usual sense in Hilbert space quantum mechanics gives a σ-additive state on B, and as shown in [10], the descriptions of the probability that O yields a result in a Borel set T when in state s, and the description of the expected value of O when in state s, agree with the usual descriptions in terms of the self-adjoint operator A.

Remark 5. The usual Hilbert space treatment of observables via self-adjoint operators fits within our more operationally motivated treatment of observables. But even in the Hilbert space setting, the operationally motivated approach is more general. The standard Hilbert space treatment effectively chooses observables $O = (B, f)$ where B is a complete subalgebra of the projection lattice and f is not only small Borel measurable, but also continuous. The definition of a self-adjoint operator is quite technically complex, involving intricacies with domain, and it is not physically clear why they are the correct notion.

7 Dynamics

Direct product decompositions illuminate dynamics of a quantum system as well. In this section, we will present the main ideas, and leave many technical details for the reader to find in [14]. We will again frame matters for sets, but they apply equally to decompositions of other types of structures as well.

Definition 18. For a set X, let $\mathrm{Aut}(X)$ be the group of permutations of X. A one-parameter group of automorphisms of X is a group homomorphism $U : \mathbb{R} \to \mathrm{Aut}(X)$. We customarily write U_t for the automorphism $U(t)$.

The idea is that if the element $x \in X$ describes the system at some given time, then for any time $t \in \mathbb{R}$, then after passage of time t, the system will be depicted by the element $U_t(x)$. This provides dynamics for the system. The group homomorphism requirement $U_{s+t} = U_s \circ U_t$ says that letting t units of time pass, and then letting s units of time pass, is equivalent to letting $s+t$ units of time pass. The requirement that U_{-t} is the inverse of U_t amounts to time reversal.

Remark 6. Time reversal is a standard feature of Hilbert space quantum mechanics. Our approach can be adapted by considering the monoid $\mathrm{End}(X)$ of endomorphisms of X and a monoid homomorphism $V : \mathbb{R}^+ \to \mathrm{End}(X)$.

Definition 19. A set with natural frequencies is an algebra $A = (X, (E_t)_{\mathbb{R}})$ consisting of a set X and a family of unary operations on X that satisfy $E_{s+t} = E_s \circ E_t$, and E_0 is the identity. These imply that $E_{-t} = E_t^{-1}$.

The physical idea of a set with natural frequencies is that a physical system has some base "vibration" in a certain specified setting. Mathematically, a set with natural frequencies is simply an algebra in the sense of general algebra [3]. So there are obvious notions of homomorphisms and isomorphisms between sets with natural frequencies. There is also a notion of a finite direct product decomposition, which we spell out in the following.

Definition 20. Let $A = (X, (E_t)_{\mathbb{R}})$ be a set with natural frequencies. A finite direct product decomposition $A \simeq A_1 \times \cdots \times A_n$ is a family $A_i = (X_i, (E_t^i)_{\mathbb{R}})$ for $i = 1, \ldots, n$

of sets with natural frequencies and a bijection $\alpha : X \to X_1 \times \cdots \times X_n$ so that for each $x \in X$, if $\alpha(x) = (x_1, \ldots, x_n)$ then

$$\alpha(E_t x) = (E_t^1 x_1, \ldots, E_t^n x_n)$$

We will often be sloppy with the notation and simply write $E_t x = (E_t^1 x_1, \ldots, E_t^n x_n)$ in the above. All previous results about direct product decompositions of sets were valid for general algebras, and in particular apply to sets with natural frequencies. In particular, we have the following.

Theorem 8. *Let $A = (X, (E_t)_{\mathbb{R}})$ be a set with natural frequencies. Then the set $\mathcal{Q}(A)$ of equivalence classes of binary direct product decompositions of A is an* OMP *under the operations described previously.*

We can then use all other results developed for questions, such as notions of states and observables, and apply them to sets with natural frequencies. We next come to the central idea of this section.

Theorem 9. *Let A be a set with natural frequencies. Suppose $H = (B, f)$ is a finite observable of A where B has n atoms, the scaling f takes values $\lambda_1, \ldots, \lambda_n$ on these atoms, and the n-ary experiment whose questions yield B is given by*

$$h = [A \simeq A_1 \times \cdots \times A_n]$$

Then there is a one-parameter group of automorphisms of A, written E^H, given by

$$E_t^H(x) \; = \; (E_{\lambda_1 t}^1 x_1, \ldots, E_{\lambda_n t}^n x_n)$$

The observable H is called the Hamiltonian and represents the energy of the system. The idea is that the factors of the system "vibrate" at a speed proportional to their energy. We require that the dynamics of the system, as given by the one-parameter group U_t is given in this fashion by the energy, a statement written as an equation called the time-independent Schrödinger equation

$$U_t \; = \; E_t^H$$

Example 5. Consider the standard Hilbert space setting. Each Hilbert space can be given a natural frequency E_t by setting

$$E_t(v) \; = \; e^{-it} v$$

Hilbert space direct product decompositions are given by closed subspaces, and these all respect natural frequencies. So the decompositions of $\mathscr{H}$ as considered being equipped with natural frequencies or not are the same. The standard treatment of time-independent dynamics takes a self-adjoint operator for the energy, called the Hamiltonian, and then gives dynamics by the time-independent Schrödinger equation for Hilbert space quantum mechanics

$$U_t = e^{-iHt}$$

When applied to a self-adjoint operator $H = \lambda_1 P_1 + \cdots + \lambda_n P_n$ that is a finite weighted sum of projectors, this yields exactly the prescription described above because a vector v_i in the range of P_i evolves to $e^{-i\lambda_i t} v_i$. For a Hamiltonian H given by a general unbounded self-adjoint operator, one can show [14] that there is a sequence H_n of finite observables so that for any $v \in \mathscr{H}$ and time t we have

$$E_t^{H_n}(v) \longrightarrow e^{-iHt} v$$

in the topology of $\mathscr{H}$.

References

1. N. H. Bingham and A. J. Ostaszewski, *Normed versus topological groups: dichotomy and duality*, Dissertationes Math. (Rozprawy Mat.) 472 (2010), 138 pp.
2. G. Birkhoff and J. von Neumann, *The logic of quantum mechanics*, Ann. of Math. (2) 37 (1936), no. 4, 823-843.
3. S. Burris and H. P. Sankappanavar, *A Course in Universal Algebra*, Graduate Texts in Mathematics (78), Springer, 1981.
4. P. A. M. Dirac, *The Principles of Quantum Mechanics, fourth ed.*, Oxford University Press, 1958.
5. A. Dvurecenskij, *Gleason's Theorem and its Applications*, Math. and its Applications (East European Series), 60. Kluwer Academic Publishers Group, Dordrecht; Ister Science Press, Bratislava, 1993.
6. A. Gleason, *Measures on the closed subspaces of a Hilbert space*, J. of Math. and Mech., Indiana Univ. Math. J. 6 (1957), No. 4, 885?893.
7. T. Hannan and J. Harding, *Automorphisms of decompositions*, Math. Slovaka 66 (2016), no. 2, 493-526.
8. J. Harding, *Decompositions in quantum logic*, Trans. Amer. Math. Soc. 348 (1996), no. 5, 1839-1862.
9. J. Harding, *Regularity in quantum logic*, Internat. J. of Theoret. Phys. 37 (1998), no. 4, 1173-1212.
10. J. Harding, *Axioms of an experimental system*, Internat. J. of Theoret. Phys. 38 (1999), no. 6, 1643-1675.
11. J. Harding, *States on orthomodular posets of decompositions*, Internat. J. Theoret. Phys. 40 (2001), no. 6, 1061-1069.
12. J. Harding, *Orthomodularity of decompositions in a categorical setting*, Internat. J. of Theoret. Phys. 45 (2006), no. 6, 1117-1127.
13. J. Harding, *A link between quantum logic and categorical quantum mechanics*, Internat. J. Theoret. Phys. 48 (2009), no. 3, 769-802.
14. J. Harding, *Dynamics in the decompositions approach to quantum mechanics*, Internat. J. Theoret. Phys. 56 (2017), no. 12, 3971-3990.
15. J. Harding, *Wigner's theorem for an infinite set*, Math. Slovaca 68 (2018), no. 5, 1173-1222.
16. G. Kalmbach, *Orthomodular Lattices*, London Mathematical Society Monographs, 18. Academic Press, Inc. London, 1983.
17. R. V. Kadison and J. R. Ringrose, *Fundamentals of the Theory of Operator Algebras. Vol. 1: Elementary Theory*, Academic Press, New York, 1983.
18. G. W. Mackey, *The Mathematical Foundations of Quantum Mechanics*, A lecture-note volume by W. A. Benjamin, Inc., New York-Amsterdam, 1963.

19. J. Neveu, *Mathematical Foundations of the Calculus of Probability*, Holden-Day, Inc., San Francisco, Calif.-London-Amsterdam, 1965.
20. P. Pták and S. Pulmannová, *Orthomodular Structures as Quantum Logics*, Fundamental Theories of Physics, 44. Kluwer Academic Publishers Group, Dordrecht, 1991.
21. E. Prugovecki, *Quantum Mechanics in Hilbert Space*, second ed., Academic Press, New York, 1981.
22. M. Stone, *On one-parameter unitary groups in Hilbert Space*, Annals of Math., 33 (1932) no. 3 643-648.
23. V. S. Varadarajan, *Geometry of Quantum Theory*, Second Ed., Springer-Verlag, 1985.
24. J. von Neumann, *Mathematical Foundations of Quantum Mechanics*, Princeton University Press, 1955.
25. E. P. Wigner, *Group Theory and its Applications to the Quantum Mechanics of the Atomic Spectra*, Academic Press, 1959.