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DUALITY THEORY FOR THE CATEGORY OF STABLE COMPACTIFICATIONS

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Abstract. We introduce the category of stable compactifications of T_0-spaces and obtain a dual description of it in terms of what we call Raney extensions of proximity frames. These are proximity frame embeddings of a regular proximity frame into a Raney lattice, i.e. the lattice of upsets of a poset. This duality generalizes the duality between compactifications of completely regular spaces involving de Vries extensions given in [7]. It also specializes to give a duality between T_0-spaces and Raney extensions that are maximal in a certain sense. This duality is related to the duality for T_0-spaces given in [6] using the notion of a Raney algebra, i.e. a Raney lattice with a certain type of interior operator.

To the memory of Ralph Kopperman

1. Introduction

This paper contributes to the study of compactifications of non-Hausdorff spaces. It is done through the lens of proximity, a central notion in the historical development of topology. These represent some of the many interests in topology that we shared with our colleague and friend Ralph Kopperman. We were involved in a number of conferences with Ralph. In particular, he was an invited speaker at the BLAST conference organized at NMSU in 2009. It is with sadness and many fond memories that we dedicate this work to him.

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We start by recalling a classic result of Smirnov [16] that compactifications of a completely regular space X can be characterized by Efremovič proximities on the powerset $\wp(X)$ of X. The definition of an Efremovič proximity on $\wp(X)$ is not point-free. De Vries [9] provided a point-free definition of a proximity on the Boolean algebra $\mathcal{RO}(X)$ of regular open sets of X and gave an alternate proof of Smirnov's result. In doing so he developed what later became known as a de Vries algebra. This is a complete Boolean algebra equipped with a binary relation that satisfies his point-free definition of proximity. This yielded a duality between the category KHaus of compact Hausdorff spaces and the category DeV of de Vries algebras.

It is well known that the (equivalence classes of) compactifications of a given completely regular space X form a poset whose largest element is the Stone-Čech compactification. In [7] a more general approach was taken by considering the category Comp of compactifications of completely regular spaces. Objects of this category are compactifications $e : X \to Y$ where X is completely regular, and morphisms are pairs (f, g) which make the obvious squares commute. De Vries duality was generalized to a duality between Comp and the category DeVe of de Vries extensions. The objects of DeVe are certain embeddings $\alpha : A \to B$ in DeV with B complete and atomic, and the morphisms in DeVe are certain pairs of de Vries morphisms that make the obvious squares commute. The full subcategory of Comp consisting of Stone-Čech compactifications is equivalent to the category of completely regular spaces, thus yielding a duality between the category of completely regular spaces and a full subcategory of de Vries extensions. This further restricts to yield dualities for normal spaces, locally compact spaces, Lindelöf spaces, and so forth [7, 8].

Smyth [17] generalized the theory of compactifications of completely regular spaces to that of stable compactifications of T_0-spaces. He generalized Smirnov's theorem by proving that stable compactifications of a T_0-space can be characterized by a more general notion of proximity. The (equivalence classes of) stable compactifications of a given T_0-space form a poset, whose largest element was described by Smyth and is a generalization of the Stone-Čech compactification. We call it the *Smyth compactification*. We note that even for a completely regular space, the Smyth compactification is usually different from the Stone-Čech compactification.

In this paper we consider the category StComp of stable compactifications $e : X \to Y$ of T_0-spaces, allowing different base spaces X and using as morphisms pairs which make the obvious squares commute (see Section 3). In particular, we obtain a duality between StComp and a category of extensions that generalize de Vries extensions. We call these extensions Raney extensions for reasons that we now describe.

The duality between the categories of compact Hausdorff spaces and de Vries algebras can be extended to a duality between the categories of stably compact spaces and what we call regular proximity frames [4]. A *proximity frame* is a point-free description of the proximities developed by Smyth [17] much as de Vries algebras are point-free description of Efremovič proximities. Each proximity frame L has a nucleus $j : L \to L$ and a proximity frame is *regular* if j is the identity. The j-fixed points of a proximity frame form a regular proximity frame and L is isomorphic in the category of proximity frames to its regularization [4]. De Vries algebras are the regularization of compact regular frames.

A *Raney lattice* is a lattice isomorphic to the lattice of all upsets of a poset. The name was introduced in [6] in honor of Raney who investigated these lattices extensively in [15]. Equipped with the partial order as its proximity, Raney lattices are regular proximity frames. A *Raney extension* $\alpha : L \to K$ is a certain embedding of a regular proximity frame L into a Raney lattice K. Our main result establishes a duality between StComp and the category RE of Raney extensions. This duality generalizes the duality of [7] between Comp and DeVe, which utilizes de Vries duality between KHaus and DeV and Tarski duality between the categories of complete and atomic Boolean algebras and sets. In turn, our duality utilizes the duality of [4] between the categories of stably compact spaces and regular proximity frames and Raney duality between the categories of Raney lattices and posets.

The duality between StComp and RE can be restricted to obtain a duality for T_0-spaces. This is done by considering certain Raney extensions which correspond to Smyth compactifications. We call these extensions maximal Raney extensions for reasons that will be made clear in Section 7. This approach is similar to the way the duality between Comp and DeVe is restricted using Stone-Čech compactifications to obtain a duality for completely regular spaces.

An alternate duality between the category of T_0-spaces and a category consisting of what we call Raney algebras was established in [6]. A *Raney algebra* is a pair $(K, \Box)$ where K is a Raney lattice and $\Box$ is a certain interior operator on K. We show that the category of Raney algebras is equivalent to the subcategory of RE consisting of maximal Raney extensions. This shows that the two approaches to a duality for T_0-spaces essentially coincide.

The duality between StComp and RE can also be restricted to a duality involving spectral compactifications of T_0-spaces. These are stable compactifications $e : X \to Y$ where Y is a spectral space, i.e. the prime spectrum of a bounded distributive lattice. This duality allows us to place Raney algebras in the wider setting of dense sublattices of Raney lattices (see Section 8).

2. Preliminaries

We assume the reader is familiar with basics of point-free topology, including the notions of frames, frame homomorphisms, etc (see [13, 14]). To make the paper relatively self-contained, we recall facts about stable compactifications [17, 5], proximity frames [4], and Raney duality [15, 6]. We begin with a discussion of stably compact spaces.

Definition 2.1. A subset S of a T_0-space X is *saturated* if it is an intersection of open sets.

It is well known that S is saturated iff it is an upset in the specialization order on X, where the *specialization order* is defined by $x \le y$ if x belongs to the closure of $\{y\}$. For the following two definitions see [12, Def. VI.6.7, Def. VI.6.17, and Thm. VI.6.18].

Definition 2.2. A topological space X is *stably compact* if it is compact, locally compact, sober, and the intersection of any two compact saturated sets is again compact.

Definition 2.3. Let (X, τ) be stably compact. The *co-compact topology* τ^k is the topology whose closed sets are exactly the compact saturated subsets of X. The *patch topology* π is given by $\pi = \tau \vee \tau^k$; that is, the smallest topology containing τ and τ^k.

It is well known that (X, π) is compact Hausdorff. We recall that a map $f : X \to Y$ between two stably compact spaces is *proper* if it is continuous with respect to the patch topologies.

Definition 2.4. Let Top_0 be the category of T_0-spaces and continuous maps, and StKSp the category of stably compact spaces and proper maps.

We next turn our attention to a characterization of those frames that arise as the opens of a stably compact space. We recall a few well-known notions.

Definition 2.5. Let L be a frame. For $a, b \in L$, we say that b is *way-below* a, written $b \ll a$, if $a \le \bigvee S$ implies $b \le \bigvee T$ for some finite $T \subseteq S$. The way-below relation $\ll$ is *stable* if $a \ll b, c$ implies $a \ll b \wedge c$ and the frame L is *stable* if $\ll$ is stable. Also, L is *compact* if $1 \ll 1$ and *locally compact* if $\ll$ is approximating, meaning that $a = \bigvee\{b : b \ll a\}$ for each $a \in L$. A frame homomorphism is *proper* if it preserves $\ll$.

Definition 2.6. [13, Sec. VII.4.6] A frame L is *stably compact* if it is compact, locally compact, and stable.

A space Y is stably compact iff its frame of opens is stably compact. Assuming the axiom of choice, stably compact frames are exactly the frames that are isomorphic to frames of opens of stably compact spaces. In fact, this correspondence yields a dual equivalence of the appropriate categories.

Theorem 2.7. [12, 13] *The category* StKSp *of stably compact spaces and proper maps is dually equivalent to the category* StKFrm *of stably compact frames and proper frame homomorphisms.*

We next turn our attention to stable compactifications of T_0-spaces.

Definition 2.8. [17, Sec. 3] Let X be a T_0-space, Y a stably compact space, and $e : X \to Y$ a homeomorphism from X to a subspace of Y. For U open in Y, let $\overline{U}$ be the largest open set of Y whose intersection with the image of X is contained in U. We call $e : X \to Y$ a *stable compactification* of X if $U \ll V \Rightarrow \overline{U} \ll V$ for all U, V open in Y, where $U \ll V$ means U is way below V in the frame of open sets of Y.

Theorem 2.9. [5, Thm. 3.5] *For a T_0-space X, an embedding $e : X \to Y$ into a stably compact space Y is a stable compactification of X iff the image of X is dense in the patch topology of Y.*

Definition 2.10. Two stable compactifications $e_1 : X \to Y_1$ and $e_2 : X \to Y_2$ of a T_0-space X are *equivalent* if there is a homeomorphism $f : Y_1 \to Y_2$ such that $f \circ e_1 = e_2$.

For two stable compactifications $e_1 : X \to Y_1$ and $e_2 : X \to Y_2$ of a T_0-space X, set $e_1 \leq e_2$ provided there is a proper map $f : Y_2 \to Y_1$ with $f \circ e_2 = e_1$. It is not difficult to show that $e_1 \leq e_2$ and $e_2 \leq e_1$ iff e_1 and e_2 are equivalent. Thus, $\leq$ is a quasi-order on the class of stable compactifications of X, giving a partial ordering of the equivalences classes of stable compactifications. We refer to this as *the poset of stable compactifications of X*.

Remark 2.11. While we do not wish to become too formal in treating issues of sets versus proper classes, we remark that for any stable compactification $e : X \to Y$, the image of X is dense in the patch topology of Y. Therefore, there is a bound on the cardinality of Y. Thus, we can canonically construct representatives of each equivalence class and use these to form a poset instead.

Smyth [17, Prop. 16] showed that the poset of stable compactifications of X has a largest element that is constructed as the spectrum of prime filters of the frame of opens of X. It plays the role of the Stone-Čech compactification for T_0-spaces.

Definition 2.12. For a T_0-space X, we call its largest stable compactification the *Smyth compactification* and denote it by $\sigma_X : X \to \sigma X$.

In Smyth's treatment of stable compactifications of T_0-spaces, he generalized the notion of (Efremovič) proximity on a set to that of quasi-proximity and proved that stable compactifications of a T_0-space X can be characterized by the quasi-proximities on X that are compatible with the topology on X [17]. This motivated the notion of proximity on a frame given in [4].

Definition 2.13. A *proximity* on a frame L is a binary relation $\prec$ on L satisfying

(1) $0 \prec 0$ and $1 \prec 1$.
(2) $a \prec b$ implies $a \leq b$.
(3) $a \leq b \prec c \leq d$ implies $a \prec d$.
(4) $a, b \prec c$ implies $a \vee b \prec c$.
(5) $a \prec b, c$ implies $a \prec b \wedge c$.
(6) $a \prec b$ implies there exists $c \in L$ with $a \prec c \prec b$.
(7) $a = \bigvee\{b \in L : b \prec a\}$.

If $\prec$ is a proximity on L, we call the pair $(L, \prec)$ a *proximity frame*, but refer to it as L.

Remark 2.14. This notion generalizes Banaschewski's strong inclusions on a frame [2] and Frith's proximal frames [11]. For further discussion see [4, Rem. 7.2].

Definition 2.15. A *proximity morphism* is a map $\varphi : L \to K$ between two proximity frames that preserves bounds $0, 1$, finite meets, and satisfies

(1) $a_1 \prec b_1$ and $a_2 \prec b_2$ imply $\varphi(a_1 \vee a_2) \prec \varphi(b_1) \vee \varphi(b_2)$.
(2) $\varphi(a) = \bigvee\{\varphi(b) : b \prec a\}$.

Definition 2.16. Let PrFrm be the category of proximity frames and proximity morphisms, where $1_L : L \to L$ is the identity map and the composite $\psi \star \varphi$ of two proximity morphisms $\varphi : L \to K$ and $\psi : K \to M$ is given by

$$(\psi \star \varphi)(a) = \bigvee\{\psi\varphi(b) : b \prec a\}.$$

A stably compact frame L is naturally a proximity frame with the way-below relation as its proximity. Moreover, proper frame homomorphisms between stably compact frames are exactly the proximity morphisms between them. Remarkably, each proximity frame is isomorphic in PrFrm to a stably compact frame (see [4, Sec. 4] for details). This is possible because composition in PrFrm is not the usual function composition, and this allows isomorphisms that are not structure-preserving bijections. In conjunction with Theorem 2.7, this provides the following.

Theorem 2.17. StKFrm *is a full subcategory of* PrFrm *that is equivalent to* PrFrm*. Therefore, both* StKFrm *and* PrFrm *are dually equivalent to* StKSp.

As we have just noted, each proximity frame L is canonically isomorphic in PrFrm to a stably compact frame. There is also a different canonical representative in the isomorphism class of L, which we call a regular proximity frame, and will now describe.

Definition 2.18. For a proximity frame L and $S \subseteq L$ define

(1) $\twoheaddownarrow S = \{a \in L : a \prec s \text{ for some } s \in S\}$
(2) $\twoheaduparrow S = \{a \in L : s \prec a \text{ for some } s \in S\}$.

For $a \in L$ we write $\twoheaddownarrow a$ and $\twoheaduparrow a$ for $\twoheaddownarrow\{a\}$ and $\twoheaduparrow\{a\}$, respectively.

For a proximity frame L, define operations k and j on L as follows. Set $k(a) = \bigwedge \twoheaduparrow a$ and then, using Heyting implication $\to$ of a frame, set

$$j(a) = \bigwedge\{(a \to k(b)) \to k(b) : b \in L\}.$$

Call $a \in L$ *regular* if it is a fixpoint of j, let L_j be the set of regular elements of L, and call the proximity frame L *regular* if $L = L_j$.

Proposition 2.19. [4, Sec. 5] *For a proximity frame L, the restriction $\prec_j$ of $\prec$ to L_j is a proximity on L_j, the proximity frame L_j is regular, and L is isomorphic to L_j in* PrFrm.

We call L_j the *regularization* of L. It is again the nature of the composition in PrFrm that allows non-structural isomorphisms, and allows a proximity frame that is not regular to be isomorphic to one that is regular.

Corollary 2.20. [4, Sec. 5] *The category* PrFrm *is equivalent to its full subcategory* RPrFrm *consisting of regular proximity frames.*

We point out that unlike PrFrm, in RPrFrm isomorphisms are structure-preserving bijections [4, Prop. 6.5]. The construction of the regularization via algebraic means through j is mysterious. However, it has a very natural topological meaning.

Definition 2.21. Let (X, τ) be a stably compact space, int_τ the interior in τ and cl_π the closure in the patch topology π. We set

$$\mathcal{RO}(X) = \{U : U = \mathsf{int}_\tau \mathsf{cl}_\pi(U)\}.$$

For $U, V \in \mathcal{RO}(X)$, define $U \prec V$ iff $\mathsf{cl}_\pi(U) \subseteq V$.

Theorem 2.22. [4, Sec. 6] *Let X be a stably compact space and L the stably compact frame of its opens. Then $(\mathcal{RO}(X), \prec)$ is a regular proximity frame and is the regularization L_j of L.*

Remark 2.23. As was pointed out in [4, Rem. 4.19], for each open set U of a stably compact space X, we have $U = \bigcup\{V \in \mathcal{RO}(X) : V \prec U\}$. Therefore, $\mathcal{RO}(X)$ is a basis for the topology on X. We will utilize this several times in the paper.

There are a number of subcategories of RPrFrm that will be of interest to us. To begin, note that if (X, τ) is compact Hausdorff, then $\pi = \tau$ and Definition 2.21 becomes the standard definition of regular open sets. Thus, $\mathcal{RO}(X)$ is a de Vries algebra. This yields the following.

Theorem 2.24. [4, Sec. 7] DeV *is a full subcategory of* RPrFrm.

For any frame L, its partial ordering $\leq$ is a proximity on L. It is not difficult to show that the proximity frame $(L, \leq)$ is regular. These are exactly the regular proximity frames whose proximities are reflexive [4, Sec. 9]. It follows from Definition 2.15 that proximity morphisms between the proximity frames $(L, \leq)$ are exactly the frame homomorphisms between them. This gives the following.

Theorem 2.25. Frm *naturally forms a full subcategory of* RPrFrm.

We will have particular use for a subcategory of frames in this context. We use the term *Raney lattice* for a complete distributive lattice in which each element is a join of completely join-prime elements. These were first called Raney lattices in [6] in honor of Raney, who in [15] showed that a lattice L is a Raney lattice iff L is the lattice of upsets of a poset. It follows that a Raney lattice is completely distributive, and is in particular a frame. Thus, Theorem 2.25 yields the following.

Theorem 2.26. *The category* Ran *of Raney lattices and complete lattice homomorphisms is a (non-full) subcategory of* Frm *and hence of* RPrFrm.

Figure 1 summarizes various categories of proximity frames and their relationships. Each of the indicated subcategories is full, except the inclusion of Ran into Frm. The inclusions of StKFrm and RPrFrm into PrFrm are equivalences. The inclusions of StKFrm and Frm into PrFrm are of a different nature. We view a stably compact frame L as the proximity frame $(L, \ll)$, and a general frame L as the regular proximity frame $(L, \leq)$. This is why the diagram does not show StKFrm included in Frm.

We need several dual equivalences between categories in Figure 1 and categories of a topological nature. We begin with Raney duality, an extension of Tarski duality between sets (discrete spaces) and complete atomic Boolean algebras, to a duality for the category Pos of posets and order preserving maps (Alexandroff T_0-spaces).

Theorem 2.27 (Raney duality)**.** *There is a dual equivalence between* Pos *and* Ran.

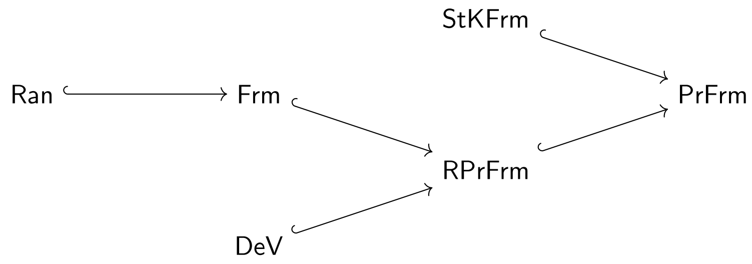


FIGURE 1. Subcategories of proximity frames

The contravariant functors $\mathcal{U}$: $\mathsf{Pos} \to \mathsf{Ran}$ and $\mathcal{J}$: $\mathsf{Ran} \to \mathsf{Pos}$ are defined as follows. For X a poset, $\mathcal{U}(X)$ is its lattice of upsets, and for $f : X \to X'$ order preserving, $\mathcal{U}(f)$ is the complete lattice homomorphism f^{-1}. For K a Raney lattice, $\mathcal{J}(K)$ is the poset X_K of completely join-prime elements of K with the ordering of X_K dual to that of K. For $\varphi : K \to K'$ a complete lattice homomorphism,

$$\mathcal{J}(\varphi)(x') = \bigwedge\{x : x' \leq \varphi(x)\}. \tag{2.1}$$

The natural isomorphisms $\gamma : 1_{\mathsf{Ran}} \to \mathcal{U} \circ \mathcal{J}$ and $\varepsilon : 1_{\mathsf{Pos}} \to \mathcal{J} \circ \mathcal{U}$ are given by

$$\gamma_K(a) = \{x \in X_K : x \leq a\} \text{ and } \varepsilon_X(x) = {\uparrow}x. \tag{2.2}$$

For the following result, we first recall that for a proximity frame L, a filter F of the underlying frame L is *round* provided $F = \Uparrow F$. An *end* is a meet-prime element of the lattice of round filters of L. We topologize the set Y_L of ends of L by taking as a basis all sets $\{F : a \in F\}$ where a ranges over elements of L.

Theorem 2.28. [4, Sec. 6] *There is a dual equivalence between the categories* StKSp *and* RPrFrm*.*

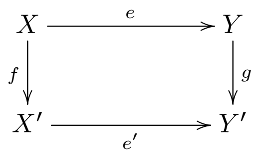
The contravariant functors $\mathcal{RO} : \mathsf{StKSp} \to \mathsf{RPrFrm}$ and $\mathcal{E} : \mathsf{RPrFrm} \to \mathsf{StKSp}$ are defined as follows. For Y a stably compact space, $\mathcal{RO}(Y)$ is as in Definition 2.21, and for $f : Y \to Y'$ a proper map, $\mathcal{RO}(f) = \mathsf{int}_\tau \mathsf{cl}_\pi f^{-1}$. For L a regular proximity frame, $\mathcal{E}(L)$ is the stably compact space Y_L of its ends, and for $\varphi : L \to L'$ a proximity morphism, $\mathcal{E}(\varphi) = \Uparrow h^{-1}$. The natural isomorphisms $\zeta : 1_{\mathsf{RPrFrm}} \to \mathcal{RO} \circ \mathcal{E}$ and $\eta : 1_{\mathsf{StKSp}} \to \mathcal{E} \circ \mathcal{RO}$ are given by

$$\zeta_L(a) = \{F \in Y_L : a \in F\} \text{ and } \eta_Y(y) = \{U \in \mathcal{RO}(Y) : y \in U\}. \tag{2.3}$$

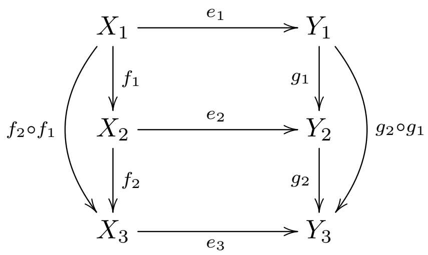
3. The category of stable compactifications

In this section we introduce the category of stable compactifications, which is our first main category of interest.

Definition 3.1. Let StComp be the category whose objects are stable compactifications $e : X \to Y$ and whose morphisms are pairs (f, g) where $f : X \to X'$ is continuous, $g : Y \to Y'$ is proper, and the following diagram commutes:



Identity morphisms are pairs $(1_X, 1_Y)$ of identity morphisms and the composition of two morphisms (f_1, g_1) and (f_2, g_2) is defined to be $(f_2 \circ f_1, g_2 \circ g_1)$.



It is straightforward to check that StComp indeed forms a category, and that the category Comp of compactifications of completely regular spaces is a full subcategory of StComp.

Lemma 3.2. *For a morphism (f, g) in* StComp*, the following conditions are equivalent.*

1. *(f, g) is an isomorphism in* StComp.
2. *Both $f : X \to X'$ and $g : Y \to Y'$ are homeomorphisms.*
3. *$f : X \to X'$ is a bijection and $g : Y \to Y'$ is a homeomorphism.*
4. *$f : X \to X'$ is an isomorphism with respect to the specialization orders on X and X' and $g : Y \to Y'$ is a homeomorphism.*

Proof. That (1)⇔(2) and (4)⇒(3) are obvious. That (2)⇒(4) follows from the fact that $f : X \to X'$ continuous implies f is order preserving with respect to the specialization orders on X and X'. Finally, to see that (3)⇒(2), since (f, g) is a morphism in StComp, f is continuous, so it is sufficient to see that $f : X \to X'$ is open. Let U be open in X. Then there is V open in Y such that $U = e^{-1}(V)$. Because g is a homeomorphism, $g(V)$ is open in Y'. We show that $f(U) = (e')^{-1}g(V)$. Let $x \in X$. If $x \in U$, then $e(x) \in V$, so $ge(x) \in g(V)$, and hence $e'f(x) \in g(V)$.

Therefore, $f(x) \in (e')^{-1}g(V)$, which yields that $f(U) \subseteq (e')^{-1}g(V)$. Conversely, if $f(x) \in (e')^{-1}g(V)$, then $e'f(x) \in g(V)$, so $ge(x) \in g(V)$. As g is a homeomorphism, $e(x) \in V$. Thus, $x \in U$, yielding the other inclusion. Consequently, $f(U) = (e')^{-1}g(V)$, so $f(U)$ is open, and hence f is open, completing the proof. □

Clearly if $e : X \to Y$ and $e' : X \to Y'$ are equivalent stable compactifications of X (see Definition 2.10), then they are isomorphic in StComp. The converse is not true in general. This for example follows from the fact that Comp is a full subcategory of StComp, and this result is not true in Comp (see [7, Exmp. 3.2]). However, if $e : X \to Y$ is the Smyth compactification $\sigma_X : X \to \sigma X$, then the converse is also true. For this, we first prove a result of some independent interest.

Theorem 3.3. *The ideal functor* $\mathfrak{I}$: Frm $\to$ StKFrm *is a coreflector. Hence* StKFrm *is a (non-full) coreflective subcategory of* Frm.

Proof. Let L be a frame. It is well known that $\mathfrak{I}L$ is a stably compact frame, where $I \ll J$ iff there is $a \in J$ with $I \subseteq {\downarrow}a$, and $\bigvee \cdot : \mathfrak{I}L \to L$ is a frame homomorphism. By the dual statement to [1, Thm. I.18.2] it is enough to show for any stably compact frame M and a frame homomorphism $\varphi : M \to L$, that there is a unique proper frame homomorphism $\psi : M \to \mathfrak{I}L$ with $\bigvee \psi(a) = \varphi(a)$ for each $a \in M$. Set

$$\psi(a) = {\downarrow}\{\varphi(b) : b \ll a\}.$$

It is straightforward to see that ψ is well defined. Let $a, b \in M$. We have $x \in \psi(a) \cap \psi(b)$ iff $x \le \varphi(c)$ for some $c \ll a$ and $x \le \varphi(d)$ for some $d \ll b$. Since $y \ll c$ and $y \ll d$ iff $y \ll c \wedge d$, the last condition is equivalent to $x \le \varphi(e)$ for some $e \ll a \wedge b$, which means that $x \in \psi(a \wedge b)$. Thus, $\psi(a) \cap \psi(b) = \psi(a \wedge b)$. Let $S \subseteq M$. Since ψ is order preserving, we have $\bigvee \psi[S] \subseteq \psi(\bigvee S)$. For the reverse inclusion, let $x \in \psi(\bigvee S)$. Then $x \le \varphi(b)$ for some $b \ll \bigvee S$. Since M is stably compact, $s = \bigvee\{t : t \ll s\}$ for each $s \in S$. So $b \ll \bigvee\{t : t \ll s \text{ for some } s \in S\}$. By the definition of $\ll$, we have $b \le t_1 \vee \cdots \vee t_n$ for some $t_i \ll s_i \in S$. Therefore, $x \le \varphi(b) \le \varphi(t_1) \vee \cdots \vee \varphi(t_n)$. Since $\varphi(t_i) \in \psi(s_i)$, we have $x \in \bigvee \psi[S]$. Thus, ψ is a frame homomorphism. To see that ψ is proper, suppose $a \ll c$. Then $\psi(a) \subseteq {\downarrow}\varphi(a)$ and $\varphi(a) \in \psi(c)$. Therefore, $\psi(a) \ll \psi(c)$ by the nature of $\ll$ on $\mathfrak{I}L$. To see that the diagram commutes, let $a \in M$. Then

$$\bigvee \psi(a) = \bigvee {\downarrow}\{\varphi(b) : b \ll a\} = \bigvee\{\varphi(b) : b \ll a\} = \varphi\left(\bigvee\{b : b \ll a\}\right) = \varphi(a).$$

Finally, to see that ψ is unique, let $\psi' : M \to \mathfrak{I}L$ be a proper frame homomorphism such that $\bigvee \psi'(a) = \varphi(a)$ for each $a \in M$. We show that $\psi(a) = \psi'(a)$. Let $b \ll a$. Since ψ' is proper, $\psi'(b) \ll \psi'(a)$, so $\psi'(b) \subseteq {\downarrow}c$ for some $c \in \psi'(a)$. Therefore, $\varphi(b) = \bigvee \psi'(b) \le c \in \psi'(a)$, so $\varphi(b) \in \psi'(a)$.

Thus, $\psi(a) \subseteq \psi'(a)$. For the reverse inclusion, since $a = \bigvee\{b : b \ll a\}$, we have $\psi'(a) = \bigvee\{\psi'(b) : b \ll a\}$. So it is enough to show that $\psi'(b) \subseteq \psi(a)$ for each $b \ll a$. We have $b \ll a$ implies $\psi'(b) \subseteq {\downarrow}\varphi(b) \subseteq \psi(a)$, completing the proof. □

Remark 3.4. When talking about a (co)reflective subcategory B of A, it is common to assume that B is a full subcategory of A. In this case, if $r : \mathsf{A} \to \mathsf{B}$ is the (co)reflector, then $r(B)$ is isomorphic to B for each object B of B. In Theorem 3.3, as well as in Corollary 3.6, we do not have the fullness assumption. Indeed, in either of these cases, $r(B)$ is not necessarily isomorphic to B.

Remark 3.5. Banaschewski and Mulvey [3, Prop. 2] proved that the category of compact completely regular frames is a full coreflective subcategory of Frm. Theorem 3.3 extends this to the category of stably compact frames, but without fullness.

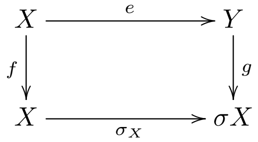
Corollary 3.6. *The Smyth compactification yields a reflector* $\sigma : \mathsf{Top_0} \to \mathsf{StKSp}$. *Hence* StKSp *is a (non-full) reflective subcategory of* $\mathsf{Top_0}$.

Proof. Let X be a T_0-space with L its frame of open sets. Since the Smyth compactification σX is constructed as the prime spectrum of L and $\mathfrak{I}L$ is isomorphic to the frame of opens of its prime spectrum, we obtain that $\mathfrak{I}L$ is isomorphic to the frame of opens of σX. To establish the result, by [1, Thm. I.18.2] it is enough to show for a stably compact space Y and a continuous map $f : X \to Y$, that there is a unique proper map $g : \sigma X \to Y$ with $g \circ \sigma_X = f$. Using the open set functor from $\mathsf{Top_0}$ to Frm, this follows from Theorem 3.3 using the dual equivalence of Theorem 2.7. □

Remark 3.7. It is well known that KHaus is a full reflective subcategory of the category of completely regular spaces and continuous maps. The reflector is the Stone-Čech compactification. Corollary 3.6 generalizes this result to $\mathsf{Top_0}$ and StKSp, but without fullness. Its origins can be traced back to Smyth [17, Prop. 16].

Theorem 3.8. *If* $e : X \to Y$ *is isomorphic to* $\sigma_X : X \to \sigma X$ *in* StComp, *then* e *is equivalent to* σ_X.

Proof. Since $e : X \to Y$ is isomorphic to $\sigma_X : X \to \sigma X$ there is (f, g) making the diagram commute, and by Lemma 3.2 both f, g are homeomorphisms.



By Corollary 3.6, σ is a reflector. So the following diagram commutes.

$$\begin{array}{ccc} X & \xrightarrow{\sigma_X} & \sigma X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \sigma f} \\ X & \xrightarrow[\sigma_X]{} & \sigma X \end{array}$$

Moreover, σf is a homeomorphism because σ is a functor and f is a homeomorphism. Let $h = g^{-1} \circ \sigma f$. Then h is a homeomorphism since both g^{-1} and σf are homeomorphisms. Furthermore,

$$h \circ \sigma_X = (g^{-1} \circ \sigma f) \circ \sigma_X = g^{-1} \circ (\sigma f \circ \sigma_X) = g^{-1} \circ (\sigma_X \circ f) = g^{-1} \circ (g \circ e) = e.$$

This shows that $e : X \to Y$ is equivalent to $\sigma_X : X \to \sigma X$. □

4. The category of Raney extensions

In this section we introduce the category of Raney extensions, which is our other main category of interest.

Definition 4.1. Let K be a complete lattice and $S \subseteq K$. We say that S is *join-meet dense* in K if each element of K is a join of meets of elements from S; that S is *meet-join dense* in K if each element of K is a meet of joins of elements from S; and that S is *dense* in K if S is both join-meet and meet-join dense in K.

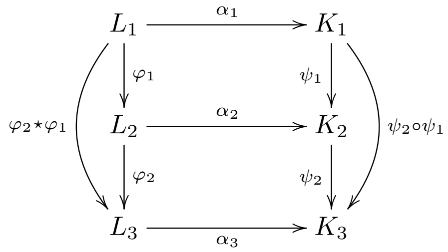
Remark 4.2. If K is completely distributive, then it is easy to see that $S \subseteq K$ being dense, join-meet dense, and meet-join dense are all equivalent.

Definition 4.3. Let L be a regular proximity frame, K a Raney lattice, and $\alpha : L \to K$ a one-to-one proximity morphism. We call $\alpha : L \to K$ a *Raney extension* if $\alpha[L]$ is dense in K.

Definition 4.4. Let RE be the category whose objects are Raney extensions $\alpha : L \to K$ and whose morphisms are pairs (φ, ψ) such that $\varphi : L \to L'$ is a proximity morphism, $\psi : K \to K'$ is a complete lattice homomorphism, and $\psi \circ \alpha = \alpha' \star \varphi$.

$$\begin{array}{ccc} L & \xrightarrow{\alpha} & K \\ {\scriptstyle \varphi}\downarrow & & \downarrow{\scriptstyle \psi} \\ L' & \xrightarrow[\alpha']{} & K' \end{array}$$

Identity morphisms are pairs $(1_L, 1_K)$ of identity morphisms and the composition $(\varphi_2, \psi_2) \star (\varphi_1, \psi_1)$ of two morphisms (φ_1, ψ_1) and (φ_2, ψ_2) is defined to be $(\varphi_2 \star \varphi_1, \psi_2 \circ \psi_1)$.



Remark 4.5. In the above definition, since ψ preserves joins, $\psi \circ \alpha = \psi \star \alpha$ [4, Lem. 3.7]. Because RPrFrm is a category, it is then straightforward to check that RE is a category.

In [7] the category Comp of compactifications of completely regular spaces and DeVe of de Vries extensions were introduced. The category Comp is a full subcategory of StComp, and DeVe is a full subcategory of RE consisting of those Raney extensions $\alpha : L \to K$ where L is a de Vries algebra and K is a complete and atomic Boolean algebra. It was shown that Comp is dually equivalent to DeVe. Our focus is to extend this result to a duality between StComp and RE. We begin with the following result that motivates the definition of a Raney extension.

Theorem 4.6. *For $e : X \to Y$ a stable compactification, $e^{-1} : \mathcal{RO}(Y) \to \mathcal{U}(X)$ is a Raney extension.*

Proof. We have that $\mathcal{U}(X)$ is a Raney lattice by Raney duality, and that $\mathcal{RO}(Y)$ is a regular proximity frame by Theorem 2.28. To simplify proof, we view X as a subspace of Y, so $e^{-1}(U) = U \cap X$. That e^{-1} is well defined is clear since $U \in \mathcal{RO}(Y)$ is open in Y, so $U \cap X$ is open in X, hence an upset in the specialization order on X.

To see that e^{-1} is one-to-one, let $U \in \mathcal{RO}(Y)$. Since X is patch-dense in Y, we have $\mathsf{cl}_\pi(U) = \mathsf{cl}_\pi(U \cap X)$, so $U = \mathsf{int}_\tau \mathsf{cl}_\pi(U) = \mathsf{int}_\tau \mathsf{cl}_\pi(U \cap X)$. Thus, for $U, V \in \mathcal{RO}(Y)$, we have $U \subseteq V$ iff $U \cap X \subseteq V \cap X$, yielding that e^{-1} is one-to-one.

To show that e^{-1} is a proximity morphism, it clearly preserves the bounds and finite meets. If $U_1 \prec V_1$ and $U_2 \prec V_2$, then $\mathsf{cl}_\pi(U_1) \subseteq V_1$ and $\mathsf{cl}_\pi(U_2) \subseteq V_2$. Since the join in $\mathcal{RO}(Y)$ is $U_1 \vee U_2 = \mathsf{int}_\tau \mathsf{cl}_\pi(U_1 \cup U_2)$, we have $U_1 \vee U_2 \subseteq \mathsf{cl}_\pi(U_1) \cup \mathsf{cl}_\pi(U_2)$. Thus,

$$(U_1 \vee U_2) \cap X \subseteq (\mathsf{cl}_\pi(U_1) \cap X) \cup (\mathsf{cl}_\pi(U_2) \cap X) \subseteq (V_1 \cap X) \cup (V_2 \cap X).$$

Since proximity in $\mathcal{U}(X)$ is set inclusion, $(U_1 \vee U_2) \cap X$ is proximal to $(V_1 \cap X) \cup (V_2 \cap X)$. For the final condition, suppose $U \in \mathcal{RO}(Y)$.

Since U is open in Y, it follows from Remark 2.23 that $U = \bigcup\{V \in \mathcal{RO}(Y) : V \prec U\}$, so $U \cap X = \bigcup\{V \cap X : V \prec U\}$.

Finally, to see that $e^{-1}[\mathcal{RO}(Y)]$ is dense in $\mathcal{U}(X)$, by Remark 4.2 it is sufficient to see that $e^{-1}[\mathcal{RO}(Y)]$ is join-meet dense in $\mathcal{U}(X)$. Since every upset is a join of principal upsets $\uparrow x$, it is sufficient to see that each $\uparrow x$ is a meet of elements from $e^{-1}[\mathcal{RO}(Y)]$. We claim that $\uparrow x = \bigcap\{V \cap X : x \in V \in \mathcal{RO}(Y)\}$. The left-to-right inclusion is clear. For the right-to-left inclusion, if $y \notin \uparrow x$, there is U open in Y with $x \in U$ and $y \notin U$. Since $U = \bigcup\{V \in \mathcal{RO}(Y) : V \prec U\}$, there is $V \in \mathcal{RO}(Y)$ with $x \in V$ and $y \notin V$, completing the proof. □

The correspondence of Theorem 4.6 extends to a contravariant functor.

Proposition 4.7. *Define* $(-)^* : \mathsf{StComp} \to \mathsf{RE}$ *by sending* $e : X \to Y$ *to* $e^{-1} : \mathcal{RO}(Y) \to \mathcal{U}(X)$ *and* (f,g) *to* $(\mathcal{RO}(g), \mathcal{U}(f))$. *Then* $(-)^*$ *is a well-defined contravariant functor.*

Proof. That $(-)^*$ is well defined on objects follows from Theorem 4.6. That $(-)^*$ is well defined on morphisms follows from Raney duality and Theorem 2.28. Finally, it is easy to see that $(-)^*$ sends identity morphisms to identity morphisms and $[(f',g') \circ (f,g)]^* = (f,g)^* \star (f',g')^*$ because $\mathcal{RO}$ and $\mathcal{U}$ are functors. Thus, $(-)^*$ is a well-defined contravariant functor. □

5. Duality between StComp and RE

In this section we define a contravariant functor $(-)_* : \mathsf{RE} \to \mathsf{StComp}$ and prove our main result that this functor and the contravariant functor $(-)^* : \mathsf{StComp} \to \mathsf{RE}$ of Proposition 4.7 yield a dual equivalence.

We begin by showing that each Raney extension naturally gives rise to a stable compactification. We recall from Section 2 that Y_L is the space of ends of a regular proximity frame L, and that X_K is the poset of completely join-prime elements of a Raney lattice K. Let $\alpha : L \to K$ be an arbitrary proximity morphism. Since the proximity on K is $\le$, we note that if $x \in X_K$, then $\uparrow x$ is an end of K, so $\mathcal{E}(\alpha) = \Uparrow\alpha^{-1}[\uparrow x]$ is an end of L. Thus, $e_\alpha : X_K \to Y_L$ in the next definition is well defined.

Definition 5.1. Let L be a regular proximity frame, K a Raney lattice, and $\alpha : L \to K$ an arbitrary proximity morphism. Define $e_\alpha : X_K \to Y_L$ by

$$e_\alpha(x) = \Uparrow\alpha^{-1}[\uparrow x].$$

Lemma 5.2. *For* $\alpha : L \to K$ *as described above we have:*

1. $\alpha[L]$ *is dense in* K *iff* $(\forall x, y \in X_K)(x \le y \Leftrightarrow e_\alpha(y) \subseteq e_\alpha(x))$.
2. α *is one-to-one iff* $e_\alpha[X_K]$ *is patch-dense in* Y_L.

Proof. (1) First suppose that $\alpha[L]$ is dense in K. Let $x, y \in X_K$. Clearly $x \le y$ implies $e_\alpha(y) \subseteq e_\alpha(x)$. Suppose that $x \nleq y$. Since $\alpha[L]$ is dense and y is completely join-prime in K, we have that y is a meet of elements from $\alpha[L]$. Therefore, there is $a \in L$ with $x \nleq \alpha(a)$ and $y \le \alpha(a)$. Because α is a proximity morphism, $\alpha(a) = \bigvee\{\alpha(b) : b \prec a\}$. As y is completely join-prime, there is $b \prec a$ with $y \le \alpha(b)$ and $x \nleq \alpha(a)$. Thus, $a \in \twoheaduparrow\alpha^{-1}[\uparrow y]$ but $a \notin \alpha^{-1}[\uparrow x]$, so $a \notin \twoheaduparrow\alpha^{-1}[\uparrow x]$. Consequently, $e_\alpha(y) \nsubseteq e_\alpha(x)$.

Conversely, suppose that $(\forall x, y \in X_K)(x \le y \Leftrightarrow e_\alpha(y) \subseteq e_\alpha(x))$. To see that $\alpha[L]$ is dense in K, it is enough to show that each $x \in X_K$ is a meet of elements from $\alpha[L]$. Clearly $x \le \bigwedge\{\alpha(a) : a \in L \text{ and } x \le \alpha(a)\}$. To see the equality, since each element of K is a join of completely join-primes, it is sufficient to show that if $y \nleq x$ in X_K, then there is $a \in L$ with $x \le \alpha(a)$ and $y \nleq \alpha(a)$. From $y \nleq x$ it follows from the assumption that $e_\alpha(x) \nsubseteq e_\alpha(y)$. Therefore, there is $b \in e_\alpha(x) \smallsetminus e_\alpha(y)$. So $b \notin \twoheaduparrow\alpha^{-1}[\uparrow y]$ and there is $a \prec b$ with $x \le \alpha(a)$. Thus, $x \le \alpha(a)$ and $y \nleq \alpha(a)$.

(2) First suppose that α is one-to-one. To see that $e_\alpha[X_K]$ is patch-dense in Y_L, we must show that each nonempty set that is open in the patch topology of Y_L has nonempty intersection with $e_\alpha[X_K]$. The patch topology on Y_L is the join of the stably compact topology and its co-compact topology. By Remark 2.23, $\mathcal{RO}(Y_L)$ is a basis for the stably compact topology on Y_L. Since $\mathcal{RO}(Y_L) = \zeta_L[L]$ (see (2.3)), each open set in the stably compact topology is a union of sets of the form $\zeta_L(a)$. By [4, Rem. 4.22] and Hofmann–Mislove Theorem [12, Theorem II-1.20], each compact saturated set is an intersection of sets of the form $\zeta_L(b)$. Because open sets in the co-compact topology are complements of compact saturated sets, each open set in the co-compact topology is a union of complements of sets $\zeta_L(b)$. So to show that $e_\alpha[X_K]$ is patch-dense in Y_L it is sufficient to show that $a \nleq b$ in L implies $(\zeta_L(a) \smallsetminus \zeta_L(b)) \cap e_\alpha[X_K] \neq \varnothing$. Since α is one-to-one, $a \nleq b$ implies $\alpha(a) \nleq \alpha(b)$. Therefore, there is $x \in X_K$ with $x \le \alpha(a)$ and $x \nleq \alpha(b)$. Because α is a proximity morphism and x is completely join-prime in K, there is $c \prec a$ with $x \le \alpha(c)$. Thus, $e_\alpha(x) \in (\zeta_L(a) \smallsetminus \zeta_L(b)) \cap e_\alpha[X_K]$.

Conversely, suppose that $e_\alpha[X_K]$ is patch-dense in Y_L. To see that α is one-to-one, let $a \nleq b$ in L. Therefore, $\zeta_L(a) \smallsetminus \zeta_L(b) \neq \varnothing$. Since $e_\alpha[X_K]$ is patch-dense, there is $x \in X_K$ such that $e_\alpha(x) \in \zeta_L(a) \smallsetminus \zeta_L(b)$. Thus, $a \in e_\alpha(x)$ and $b \notin e_\alpha(x)$. From $a \in e_\alpha(x) = \twoheaduparrow\alpha^{-1}[\uparrow x]$ it follows that $x \le \alpha(a)$. From $b \notin \twoheaduparrow\alpha^{-1}[\uparrow x]$, for each $d \prec b$ we have $x \nleq \alpha(d)$. Since $\alpha(b) = \bigvee\{\alpha(d) : d \prec b\}$ and x is completely join-prime, we conclude that $x \nleq \alpha(b)$. Consequently, $\alpha(a) \nleq \alpha(b)$, and so α is one-to-one. □

Definition 5.3. For $\alpha : L \to K$ a Raney extension, let τ_α be the least topology on X_K that makes $e_\alpha : X_K \to Y_L$ continuous with respect to the stably compact topology on Y_L.

For the following result, we remind the reader of the natural isomorphism $\gamma : 1_{\mathsf{Ran}} \to \mathcal{U} \circ \mathcal{J}$ (see (2.2)).

Lemma 5.4. *Let $\alpha : L \to K$ be a Raney extension.*

(1) *$\gamma_K[\alpha[L]]$ is a basis for (X_K, τ_α).*
(2) *The specialization order of τ_α on X_K is the dual of the restriction of the order on K to X_K.*

Proof. (1) This follows from the facts that $\mathcal{RO}(Y_L)$ is a basis for the stably compact space Y_L, that τ_α is the least topology on X_K that makes e_α continuous, and that $e_\alpha^{-1}(\zeta_L(a)) = \gamma_K(\alpha(a))$ for each $a \in L$. To see the last equality, observe that $x \in \gamma_K(\alpha(a))$ iff $x \le \alpha(a)$. Since $\alpha(a) = \bigvee\{\alpha(b) : b \prec a\}$ and x is completely join-prime, the last inequality is equivalent to the existence of $b \prec a$ satisfying $x \le \alpha(b)$, which is equivalent to $x \in e_\alpha^{-1}(\zeta_L(a))$.

(2) Let $\le_S$ denote the specialization order of τ_α on X_K. Then $x \le_S y$ iff $(\forall U \in \tau_\alpha)(x \in U \Rightarrow y \in U)$. By (1), $\gamma_K[\alpha[L]]$ is a basis for (X_K, τ_α), so $x \le_S y$ iff $(\forall a \in L)(x \in \gamma_K(\alpha(a)) \Rightarrow y \in \gamma_K(\alpha(a)))$. This means that $(\forall a \in L)(x \le \alpha(a) \Rightarrow y \le \alpha(a))$. Since $\alpha[L]$ is dense in K and x, y are completely join prime in K, the last statement is equivalent to $y \le x$. □

Theorem 5.5. *If $\alpha : L \to K$ is a Raney extension, then $e_\alpha : X_K \to Y_L$ is a stable compactification.*

Proof. Since L is a proximity frame, Y_L is a stably compact space. As $\alpha[L]$ is dense in K, it follows from Lemma 5.2(1) that e_α is one-to-one. Because α is one-to-one, $e_\alpha[X_K]$ is patch-dense in Y_L by Lemma 5.2(2). Finally, it follows from the definition of τ_α that e_α is a topological embedding of X_K into the stably compact space Y_L. Thus, $e_\alpha : X_K \to Y_L$ is a stable compactification. □

Lemma 5.6. *If (φ, ψ) is a morphism in RE, then $(\mathcal{J}(\psi), \mathcal{E}(\varphi))$ is a morphism in StComp.*

Proof. Since $\varphi : L \to L'$ is a proximity morphism, $\mathcal{E}(\varphi) : Y_{L'} \to Y_L$ is a morphism in StKSp. Because $\psi : K \to K'$ is a complete lattice homomorphism, $\mathcal{J}(\psi) : X_{K'} \to X_K$ is order preserving.

$$
\begin{array}{ccc}
L & \xrightarrow{\ \alpha\ } & K \\
\varphi \downarrow & & \downarrow \psi \\
L' & \xrightarrow[\ \alpha'\ ]{} & K'
\end{array}
\qquad\qquad
\begin{array}{ccc}
X_K & \xrightarrow{\ e_\alpha\ } & Y_L \\
\mathcal{J}(\psi) \uparrow & & \uparrow \mathcal{E}(\varphi) \\
X_{K'} & \xrightarrow[\ e_{\alpha'}\ ]{} & Y_{L'}
\end{array}
$$

It is left to prove that $\mathcal{E}(\varphi) \circ e_{\alpha'} = e_\alpha \circ \mathcal{J}(\psi)$. For this we will use that $\psi \circ \alpha = \alpha' \star \varphi$, which means that for each $a \in L$,

$$\psi(\alpha(a)) = \bigvee\{\alpha'(\varphi(b)) : b \prec a\}. \tag{5.1}$$

Let $x \in X_{K'}$. We have

$$\mathcal{E}(\varphi)(e_{\alpha'}(x)) = \Uparrow\varphi^{-1}\Uparrow(\alpha')^{-1}[\uparrow x]$$

and

$$e_\alpha(\mathcal{J}(\psi)(x)) = \Uparrow\alpha^{-1}[\uparrow \bigwedge\{k \in K : x \le \psi(k)\}].$$

Let $a \in L$. Then

$$a \in \mathcal{E}(\varphi)(e_{\alpha'}(x)) \text{ iff } \exists b \prec a, \exists c \prec \varphi(b) : x \le \alpha'(c) \tag{5.2}$$

and

$$a \in e_\alpha(\mathcal{J}(\psi)(x)) \text{ iff } \exists d \prec a : \bigwedge\{k \in K : x \le \psi(k)\} \le \alpha(d). \tag{5.3}$$

First suppose $a \in e_\alpha(\mathcal{J}(\psi)(x))$. By (5.3), there is $d \prec a$ such that $\bigwedge\{k \in K : x \le \psi(k)\} \le \alpha(d)$. Since ψ is a complete lattice homomorphism, $x \le \psi(\alpha(d))$, so by (5.1), $x \le \bigvee\{\alpha'(\varphi(e)) : e \prec d\}$. Because x is completely join-prime, there is $e \prec d$ with $x \le \alpha'(\varphi(e))$. We set $b = d$ and $c = \varphi(e)$. Then $b \prec a$, $c \prec \varphi(b)$, and $x \le \alpha'(c)$, yielding $a \in \mathcal{E}(\varphi)(e_{\alpha'}(x))$ by (5.2).

Next suppose that $a \in \mathcal{E}(\varphi)(e_{\alpha'}(x))$. By (5.2), there exist $b \prec a$ and $c \prec \varphi(b)$ such that $x \le \alpha'(c)$. Therefore, $x \le \alpha'(c) \prec \alpha'(\varphi(b)) \le \psi(\alpha(a))$ by (5.1). Since $x \le \psi(\alpha(a))$, we have that $\alpha(a) \in \{k \in K : x \le \psi(k)\}$. Thus, $\bigwedge\{k \in K : x \le \psi(k)\} \le \alpha(a)$. As α is a proximity morphism, $\alpha(a) = \bigvee\{\alpha(d) : d \prec a\}$. By Raney duality, $\bigwedge\{k \in K : x \le \psi(k)\}$ is completely join-prime, so there is $d \prec a$ such that $\bigwedge\{k \in K : x \le \psi(k)\} \le \alpha(d)$. Consequently, $a \in e_\alpha(\mathcal{J}(\psi)(x))$ by (5.3). □

Proposition 5.7. *Define* $(-)_* : \mathsf{RE} \to \mathsf{StComp}$ *by sending* $\alpha : L \to K$ *to* $e_\alpha : X_K \to Y_L$ *and* (φ, ψ) *to* $(\mathcal{J}(\psi), \mathcal{E}(\varphi))$*. Then* $(-)_*$ *is a well-defined contravariant functor.*

Proof. That $(-)_*$ is well defined on objects follows from Theorem 5.5. That $(-)_*$ is well defined on morphisms follows from Lemma 5.6. Finally, it is easy to see that $(-)_*$ sends identity morphisms to identity morphisms and $[(\varphi', \psi') \star (\varphi, \psi)]_* = (\varphi, \psi)_* \circ (\varphi', \psi')_*$ because $\mathcal{E}$ and $\mathcal{J}$ are functors. Thus, $(-)_*$ is a well-defined contravariant functor. □

Theorem 5.8 (Main Theorem)**.** *The functors* $(-)^* : \mathsf{StComp} \to \mathsf{RE}$ *and* $(-)_* : \mathsf{RE} \to \mathsf{StComp}$ *yield a dual equivalence of* StComp *and* RE*.*

Proof. Recall that $\varepsilon : 1_{\mathsf{Pos}} \to \mathcal{J} \circ \mathcal{U}$ and $\eta : 1_{\mathsf{StKSp}} \to \mathcal{E} \circ \mathcal{RO}$ are natural isomorphisms. We claim that $(\varepsilon, \eta) : 1_{\mathsf{StComp}} \to (-)_* \circ (-)^*$ is a natural isomorphism. To see this, let $e : X \to Y$ be a stable compactification and set $\alpha = e^{-1}$. Using the isomorphism $\varepsilon_X : X \to X_{\mathcal{U}(X)}$ and the homeomorphism $\eta_Y : Y \to Y_{\mathcal{RO}(Y)}$, it is sufficient to show that the pair (ε_X, η_Y) is an isomorphism in StComp, as naturality follows from the naturality of ε and η. To show that (ε_X, η_Y) is an isomorphism in StComp, by Lemma 3.2, it is enough to show that $e_\alpha \circ \varepsilon_X = \eta_Y \circ e$.

$$\begin{array}{ccc} X & \xrightarrow{\quad e \quad} & Y \\ {\scriptstyle \varepsilon_X}\downarrow & & \downarrow{\scriptstyle \eta_Y} \\ X_{\mathcal{U}(X)} & \xrightarrow[e_\alpha]{\qquad} & Y_{\mathcal{RO}(Y)} \end{array}$$

To see this, let $x \in X$. Note that $e_\alpha \varepsilon_X(x)$ and $\eta_Y e(x)$ are both ends of $Y_{\mathcal{RO}(Y)}$. Therefore, we must show that for each $U \in \mathcal{RO}(Y)$, we have that $U \in e_\alpha \varepsilon_X(x)$ iff $U \in \eta_Y e(x)$. By definition, $U \in \eta_Y e(x)$ iff $e(x) \in U$. On the other hand,

$$\begin{array}{lll} U \in e_\alpha \varepsilon_X(x) & \text{iff} & \exists V \prec U : V \in \alpha^{-1}[\uparrow \varepsilon_X(x)] \\ & \text{iff} & \exists V \prec U : V \in \alpha^{-1}[\{S \in \mathcal{U}X : \uparrow x \subseteq S\}] \\ & \text{iff} & \exists V \prec U : V \in \alpha^{-1}[\{S \in \mathcal{U}X : x \in S\}] \\ & \text{iff} & \exists V \prec U : x \in \alpha(V) \\ & \text{iff} & \exists V \prec U : x \in e^{-1}(V) \\ & \text{iff} & \exists V \prec U : e(x) \in V. \end{array}$$

Since $\mathcal{RO}(Y)$ is a basis, $U = \bigcup\{V \in \mathcal{RO}(Y) : V \prec U\}$. So the last condition is equivalent to $e(x) \in U$. Thus, $\eta_Y \circ e = e_\alpha \circ \varepsilon_X$.

Recall that $\zeta : 1_{\mathsf{RPrFrm}} \to \mathcal{RO} \circ \mathcal{E}$ and $\gamma : 1_{\mathsf{Ran}} \to \mathcal{U} \circ \mathcal{J}$ are natural isomorphisms. We claim that $(\zeta, \gamma) : 1_{\mathsf{RE}} \to (-)^* \circ (-)_*$ is a natural isomorphism. To see this, let $\alpha : L \to K$ be a Raney extension. Using the isomorphisms $\zeta_L : L \to \mathcal{RO}(Y_L)$ and $\gamma_K : K \to \mathcal{U}(X_K)$, it is sufficient to show that the pair (ζ_L, γ_K) is an isomorphism in RE, as naturality follows from the naturality of ζ and γ. For this it is enough to show that $\gamma_K \circ \alpha = e_\alpha^{-1} \star \zeta_L$.

$$\begin{array}{ccc} L & \xrightarrow{\quad \alpha \quad} & K \\ {\scriptstyle \zeta_L}\downarrow & & \downarrow{\scriptstyle \gamma_K} \\ \mathcal{RO}(Y_L) & \xrightarrow[e_\alpha^{-1}]{\qquad} & \mathcal{U}(X_K) \end{array}$$

To see this, let $a \in L$. Note that $(\gamma_K \circ \alpha)(a)$ and $(e_\alpha^{-1} \star \zeta_L)(a)$ are both upsets of X_K. Therefore, we must show that for each $x \in X_K$, we have that $x \in \gamma_K \alpha(a)$ iff $x \in (e_\alpha^{-1} \star \zeta_L)(a)$. By definition, $x \in \gamma_K \alpha(a)$ iff $x \le \alpha(a)$. On the other hand,

$$\begin{aligned} x \in (e_\alpha^{-1} \star \zeta_L)(a) \quad & \text{iff} \quad x \in \bigcup \{ e_\alpha^{-1} \zeta_L(b) : b \prec a \} \\ & \text{iff} \quad x \in e_\alpha^{-1} \zeta_L(\bigvee \{ b : b \prec a \}) \\ & \text{iff} \quad x \in e_\alpha^{-1} \zeta_L(a) \\ & \text{iff} \quad e_\alpha(x) \in \zeta_L(a) \\ & \text{iff} \quad a \in e_\alpha(x) \\ & \text{iff} \quad a \in \Uparrow \alpha^{-1}[\uparrow x] \\ & \text{iff} \quad \exists b \prec a : x \le \alpha(b). \end{aligned}$$

Since $\alpha(a) = \bigvee \{ \alpha(b) : b \prec a \}$, the last condition is equivalent to $x \le \alpha(a)$. Thus, $\gamma_K \circ \alpha = e_\alpha^{-1} \star \zeta_L$. □

6. Orderings of stable compactifications

In Section 2 we considered the poset of stable compactifications of a T_0-space X. In this section we develop a dual view of this poset in the category of Raney extensions. This will lead us to consider the poset of stable compactifications where we fix not the topology of X but rather its specialization order.

Definition 6.1. Let K be a Raney lattice. A subframe of K that is dense in the sense of Definition 4.1 is called a *dense subframe*. Let $\mathsf{Den}(K)$ be the set of dense subframes of K partially ordered by set inclusion.

Recall that X_K is the poset of completely join-prime elements of K and that the partial order on X_K is the dual of the order on K. In order to not confuse the two, we use $\sqsubseteq$ for the partial order of X_K.

Definition 6.2. Let $\mathsf{Top}(X_K)$ be the set of topologies on X_K whose specialization order is $\sqsubseteq$, and partially order these topologies by set inclusion.

Note that each member of $\mathsf{Top}(X_K)$ yields a T_0-space since its specialization order is a partial order. Further, $\mathsf{Top}(X_K)$ has a largest element, the Alexandroff topology τ_A whose open sets are all upsets of $(X_K, \sqsubseteq)$.

Theorem 6.3. *$\mathsf{Den}(K)$ is isomorphic to $\mathsf{Top}(X_K)$.*

Proof. Define $\Gamma : \mathsf{Den}(K) \to \mathsf{Top}(X_K)$ by $\Gamma(S) = \gamma_K[S]$. Since $\gamma_K : K \to \mathcal{U}(X_K)$ is an isomorphism, $\gamma_K[S]$ is a topology τ on X_K, and is clearly a coarsening of τ_A. Let $\le_\tau$ be the specialization order of τ. Since $\tau \subseteq \tau_A$, we have that $\sqsubseteq$ is contained in $\le_\tau$. For the reverse inclusion suppose that $x \not\sqsubseteq y$. Then $y \nleq x$ in K. Since S is dense, there is $s \in S$ with $x \le s$ and $y \nleq s$. Therefore, $x \in \gamma(s)$ and $y \notin \gamma(s)$. Thus, $x \nleq_\tau y$. So Γ is well defined, and it is obviously order preserving.

Next define $\Delta : \mathsf{Top}(X_K) \to \mathsf{Den}(K)$ by $\Delta(\tau) = \gamma_K^{-1}[\tau]$. Since $\gamma_K : K \to \mathcal{U}(X_K)$ is an isomorphism, $\gamma_K^{-1}[\tau]$ is a subframe of K. To see that $\gamma_K^{-1}[\tau]$ is dense in K let $x \in X_K$. We claim that $x = \bigwedge\{\gamma_K^{-1}(U) : x \in U \in \tau\}$. Clearly x is underneath this meet. Thus, it is sufficient to see that each $y \in X_K$ underneath this meet is underneath x. If $y \nleq x$ in K, then $x \notin y$. Since $\sqsubseteq$ is the specialization order of τ, there is $U \in \tau$ with $x \in U$ and $y \notin U$, which contradicts that y is underneath the meet. So Δ is well defined, and it is obviously order preserving.

Finally, that Γ and Δ are inverses of each other is obvious since $\gamma_K : K \to \mathcal{U}(X_K)$ is an isomorphism. □

Proposition 6.4. *For a Raney lattice K, the duals of Raney extensions $\alpha : L \to K$ are stable compactifications of spaces (X_K, τ) where $\tau \in \mathsf{Top}(X_K)$.*

Proof. Let $\alpha : L \to K$ be a Raney extension and $e_\alpha : X_K \to Y_L$ the dual stable compactification. By Lemma 5.4(1), the topology τ_α on X_K is given by $\gamma_K[S]$ where S is the dense subframe of K generated by $\alpha[L]$. Then by Theorem 6.3 we have that $\tau_\alpha \in \mathsf{Top}(X_K)$. □

Smyth [17] considered the poset of stable compactifications of a T_0-space. Viewing this from a dual perspective, it is more natural to first consider what amounts to the stable compactifications of all spaces (X_K, τ) where $\tau \in \mathsf{Top}(X_K)$. This approach is more general in that instead of fixing a T_0-space, we are fixing a poset $(X_K, \sqsubseteq)$ and considering stable compactifications of all T_0-spaces on X_K whose specialization order is $\sqsubseteq$. Dually, this amounts to looking at Raney extensions with the same target. To avoid issues of working with proper classes, we make the following notion of equivalence. Here we recall that isomorphisms of regular proximity frames are structure-preserving bijections [4, Prop. 6.5].

Definition 6.5. We say that two Raney extensions $\alpha : L \to K$ and $\alpha' : L' \to K$ are *equivalent*, and write $\alpha \equiv \alpha'$, if there is an isomorphism $\mu : L \to L'$ such that $\alpha' \star \mu = \alpha$. Let $\mathsf{RE}(K)$ be all equivalence classes of Raney extensions with target K.

We remark that any Raney extension $\alpha : L \to K$ is equivalent to one where the underlying set of L is a subset of K and $e : L \to K$ is the identical embedding. Thus, we can canonically construct representatives of each equivalence class and use these to form a set instead.

Definition 6.6. Let $\alpha : L \to K$ and $\alpha' : L' \to K$ be two Raney extensions. We say that α' is *less than or equal to* α and write $\alpha' \leq \alpha$ provided there is a proximity morphism $\mu : L' \to L$ such that $\alpha \star \mu = \alpha'$.

We next show that $\leq$ is compatible with equivalence of Raney extensions. For this we require the following result.

Lemma 6.7. *Each one-to-one proximity morphism is a monomorphism.*

Proof. Suppose $\alpha : L \to K$ and $\beta, \gamma : M \to L$ are proximity morphisms such that α is one-to-one and $\alpha \star \beta = \alpha \star \gamma$. We need to show that $\beta = \gamma$. Let $a \in M$. It is enough to show that $\beta(a) \leq \gamma(a)$ since the reverse inequality can be proved similarly. Since $\beta(a) = \bigvee\{\beta(b) : b \prec a\}$ it is sufficient to show that $\beta(b) \leq \gamma(a)$ for each $b \prec a$. We have $\alpha\beta(b) \leq (\alpha \star \beta)(a) = (\alpha \star \gamma)(a) \leq \alpha\gamma(a)$. Since α preserves finite meets, $\alpha\beta(b) = \alpha\beta(b) \wedge \alpha\gamma(a) = \alpha(\beta b \wedge \gamma a)$. Then, since α is one-to-one, we obtain $\beta b = \beta b \wedge \gamma a$, so $\beta b \leq \gamma a$, concluding the proof. $\square$

Proposition 6.8. *For α, α' Raney extensions with target K, $\alpha \equiv \alpha'$ iff $\alpha \leq \alpha'$ and $\alpha \leq \alpha'$.*

Proof. If $\alpha \equiv \alpha'$ there is an isomorphism μ with $\alpha' \star \mu = \alpha$, and hence $\alpha \star \mu^{-1} = \alpha'$. Thus, $\alpha \leq \alpha'$ and $\alpha' \leq \alpha$. Conversely, if $\alpha' \leq \alpha$ and $\alpha \leq \alpha'$, then there are μ, μ' with $\alpha \star \mu = \alpha'$ and $\alpha' \star \mu' = \alpha$. Therefore, $\alpha \star \mu \star \mu' = \alpha' \star \mu' = \alpha$. Since α is one-one by the definition of a Raney extension, Lemma 6.7 gives that α is a monomorphism. So $\alpha \star \mu \star \mu' = \alpha \star 1_L$ yields that $\mu \star \mu' = 1_L$. That $\mu' \star \mu = 1_{L'}$ follows by symmetry. So μ is an isomorphism, and this gives that $\alpha \equiv \alpha'$. $\square$

Clearly $\leq$ is reflexive and transitive, so $\leq$ is a quasi-order on the class of Raney extensions with target K. The above result shows that the equivalence associated with this quasi-order is that of Definition 6.5. Thus, $\leq$ gives a partial ordering on $\mathsf{RE}(K)$. We next look at the corresponding dual notion. For this, we make use of the notion of equivalence of stable compactifications of a T_0-space from Section 2.

Definition 6.9. For a Raney lattice K, let $\mathsf{StComp}(X_K)$ be all equivalence classes of stable compactifications of spaces (X_K, τ) where $\tau \in \mathsf{Top}(X_K)$.

In Section 2 a quasi-ordering of the class of stable compactifications of a T_0-space X was given. We extend this to stable compactifications of the space X_K where the topologies are allowed to vary over the members of $\mathsf{Top}(X_K)$.

Definition 6.10. Let $e : (X_K, \tau) \to Y$ and $e' : (X_K, \tau') \to Y'$ be two stable compactifications where $\tau, \tau' \in \mathsf{Top}(X_K)$. We say that e' is *less than or equal to* e and write $e' \leq e$ provided $\tau' \subseteq \tau$ and there is a proper map $g : Y \to Y'$ such that $g \circ e = e'$.

Remark 6.11. In other words, Definition 6.10 provides that $e' \le e$ iff there is a proper map $g : Y \to Y'$ with $(1_{X_K}, g)$ a morphism from $e : (X_K, \tau) \to Y$ to $e' : (X_K, \tau') \to Y'$ in StComp.

Clearly $\le$ is a quasi-ordering and it restricts to the quasi-ordering of Section 2 on the class of stable compactifications of the same space. Therefore, if e and e' are equivalent stable compactifications, then $e \le e'$ and $e' \le e$. But if $e \le e'$ and $e' \le e$, then they are stable compactifications of the same space, and hence are equivalent. Thus, $\le$ induces a partial ordering on $\mathsf{StComp}(X_K)$. We call this *the poset of stable compactifications affiliated with* X_K.

Theorem 6.12. *For a Raney lattice K, the functors $(-)_*$ and $(-)^*$ of Theorem 5.8 induce mutually inverse order-isomorphisms between* $\mathsf{RE}(K)$ *and* $\mathsf{StComp}(X_K)$.

Proof. Let $\alpha : L \to K$ be a Raney extension and $e : (X_K, \tau) \to Y$ a stable compactification where $\tau \in \mathsf{Top}(X_K)$. To avoid cumbersome notation, we denote by α_* the stable compactification $e_\alpha : (X_K, \tau_\alpha) \to Y_L$ and by e^* the Raney extension $e^{-1} : \mathcal{RO}(Y) \to \mathcal{U}(X_K)$. By Proposition 6.4, $\tau_\alpha \in \mathsf{Top}(X_K)$. Using the isomorphism $\gamma_K : K \to \mathcal{U}(X_K)$, we have that $\gamma_K^{-1} \circ e^* : \mathcal{RO}(Y) \to K$ is a Raney extension by Proposition 4.7. To establish our result, it is enough to show the following.

(1) if $\alpha \le \alpha'$, then $\alpha_* \le (\alpha')_*$
(2) if $e \le e'$, then $\gamma_K^{-1} \circ e^* \le \gamma_K^{-1} \circ (e')^*$
(3) $\alpha \equiv \gamma_K^{-1} \circ (\alpha_*)^*$
(4) $e \equiv (\gamma_K^{-1} \circ e^*)_*$.

Items (1) and (2) follow from the definitions of the orderings and that $(-)_*$ and $(-)^*$ are functors. Since the second diagram in the proof of Theorem 5.8 commutes, we have $\zeta_L : L \to \mathcal{RO}(Y_L)$ is an isomorphism with $\alpha = \gamma_K^{-1} \star (\alpha_*)^* \star \zeta_L = (\gamma_K^{-1} \circ (\alpha_*)^*) \star \zeta_L$, so $\alpha \equiv \gamma_K^{-1} \circ (\alpha_*)^*$. This yields (3).

To see (4), we specialize the first diagram in the proof of Theorem 5.8 to the current situation.

$$\begin{array}{ccc} X_K & \xrightarrow{\quad e \quad} & Y \\ \Big\downarrow{\scriptstyle \varepsilon_{X_K}} & & \Big\downarrow{\scriptstyle \eta_Y} \\ X_{\mathcal{U}(X_K)} & \xrightarrow[(e^*)_*]{\quad\quad} & Y_{\mathcal{RO}(Y)} \end{array}$$

Since the diagram commutes and η_Y is a homeomorphism, e is equivalent to $(e^*)_* \circ \varepsilon_{X_K}$. Therefore, it remains to show that $(e^*)_* \circ \varepsilon_{X_K} = \delta_*$ where $\delta = \gamma_K^{-1} \circ e^*$. Let $x \in X_K$ and $U \in \mathcal{RO}(Y)$. By the calculation following the first diagram in the proof of Theorem 5.8, $U \in ((e^*)_* \circ \varepsilon_{X_K})(x)$ iff there is $V \prec U$ such that $e(x) \in V$. By Definition 5.1, $\delta_*(x) = \Uparrow\delta^{-1}[\uparrow x]$ where $\uparrow x$ is the upset of x in K. So $U \in \delta_*(x)$ iff there is $V \prec U$ such that $V \in \delta^{-1}[\uparrow x]$. This last condition is equivalent to $x \le \delta(V) = \gamma_K^{-1} e^{-1}(V)$. Since $\gamma_K : K \to \mathcal{U}(X_k)$ is an isomorphism, this means that $\gamma_K(x) \subseteq e^{-1}(V)$. But $\gamma_K(x) = \{y \in X_K : y \le x\}$ is the principal upset of x in $(X_K, \sqsubseteq)$. Since $e^{-1}(V)$ is an upset of $(X_K, \sqsubseteq)$, we have that $\gamma_K(x) \subseteq e^{-1}(V)$ iff $x \in e^{-1}(V)$. Thus, $(e^*)_* \circ \varepsilon_{X_K} = \delta_*$. □

We now return attention to the poset of stable compactifications of a T_0-space X, and consider this from a dual perspective. This will be a matter of specializing Theorem 6.12.

Definition 6.13. For a T_0-space (X, τ), let $\mathsf{StComp}(X, \tau)$ be the poset of stable compactifications of (X, τ). For a Raney lattice K and a dense subframe S of K, let $\mathsf{RE}(K, S)$ be the poset of Raney extensions $\alpha : L \to K$ where the subframe generated by the image $\alpha[L]$ is S.

To consider the stable compactifications of a T_0-space X, we can identify X with X_K where K is the Raney lattice of upsets of the specialization order of X. Then by Theorem 6.3 the topology on X corresponds to a dense subframe S of K. Specializing Theorem 6.12 gives the following.

Corollary 6.14. *Let (X, τ) be a T_0-space, K the Raney lattice of upsets of X under its specialization order, and S the dense subframe of K consisting of open subsets of (X, τ). Then the functors $(-)_*$ and $(-)^*$ of Theorem 5.8 induce mutually inverse order-isomorphisms between $\mathsf{RE}(K, S)$ and $\mathsf{StComp}(X, \tau)$.*

Proof. Let $\alpha : L \to K$ be a Raney extension and $\alpha_* : (X_K, \tau_\alpha) \to Y_L$ its dual stable compactification. We have that $\alpha \in \mathsf{RE}(K, S)$ iff S is generated by $\alpha[L]$. By Lemma 5.4(1), τ_α is equal to $\gamma_K[S]$. So $\alpha \in \mathsf{RE}(K, S)$ iff $\alpha_* \in \mathsf{StComp}(X_K, \tau_\alpha)$. Therefore, the inverse order-isomorphisms of Theorem 5.8 between $\mathsf{RE}(K)$ and $\mathsf{StComp}(X_K)$ restrict to inverse order-isomorphisms between $\mathsf{RE}(K, S)$ and $\mathsf{StComp}(X_K, \tau_\alpha)$. Since ε_X is a homeomorphism from (X, τ) to (X_K, τ_α), we conclude that there are inverse order-isomorphisms between $\mathsf{RE}(K, S)$ and $\mathsf{StComp}(X, \tau)$. □

Given a T_0-space (X, τ), Corollary 6.14 provides a characterization of the poset of stable compactifications $\mathsf{StComp}(X, \tau)$ that is an alternate to Smyth's characterization [17, Thm. 2]. Theorem 6.12 considers a more general situation. It takes as primitive the specialization order of X, and the resulting Raney lattice K of upsets of this specialization order. It then provides a characterization of the poset $\mathsf{StComp}(X_K)$. This poset is a disjoint union of the posets $\mathsf{StComp}(X, \tau)$ where $\tau \in \mathsf{Top}(X_K)$, but there are additional comparabilities as we see in the following example.

Example 6.15. For a countable set X, let τ be the discrete topology and τ' the cofinite topology on X. Clearly the specialization order of τ is the identity relation $=$ on X. Therefore, the Raney lattice K of the upsets of this specialization order is the powerset of X. Since τ' is a T_1-topology, its specialization order is also the identity relation $=$. Identifying X with X_K, we have that $\tau, \tau' \in \mathsf{Top}(X_K)$ and $\tau \neq \tau'$. Let $e : (X, \tau) \to (Y, \pi)$ be the standard one-point compactification of (X, τ) where $Y = X \cup \{\infty\}$. Let also π' be the topology of cofinite subsets of Y containing ∞. Then $e \in \mathsf{Comp}(X, \tau) \subseteq \mathsf{StComp}(X, \tau)$. It is also not difficult to check that $e' \in \mathsf{StComp}(X, \tau')$, and that π is the patch topology of both (Y, π) and (Y, π'). Thus, the identity map from (Y, π) to (Y, π') is a proper map showing that $e' \leq e$.

7. Duality for T_0-spaces

In this section, we obtain several equivalences and dual equivalences for T_0-spaces. We begin with the following category that will play a key role.

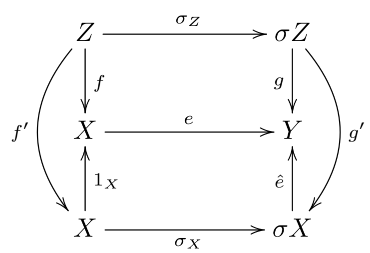
Definition 7.1. We let Smyth be the full subcategory of StComp consisting of Smyth compactifications.

By Corollary 3.6, the Smyth compactification yields a reflector $\sigma : \mathsf{Top}_0 \to \mathsf{StKSp}$. This is the essential ingredient in the following.

Proposition 7.2. Smyth *is a coreflective subcategory of* StComp *that is equivalent to* Top_0.

Proof. Suppose $\sigma_Z : Z \to \sigma Z$ is a Smyth compactification and $e : X \to Y$ is a stable compactification. Since σ is a reflector, for each continuous map $f : Z \to X$ there is a unique proper map $g : \sigma Z \to Y$ with $g \circ \sigma_Z = e \circ f$. Thus, there is a one-one correspondence between continuous maps $f : Z \to X$ and morphisms (f, g) in StComp from σ_Z to e.

For a stable compactification $e : X \to Y$, let $\hat{e} : \sigma X \to Y$ be such that $(1_X, \hat{e})$ is a morphism from σ_X to e. To show that this yields a coreflector from StComp to Smyth, by the dual statement to [1, Thm. I.18.2] it is enough to show that for any Smyth compactification $\sigma_Z : Z \to \sigma Z$ and morphism (f, g) in StComp from σ_Z to e, there is a unique morphism (f', g') in Smyth from σ_Z to σ_X with $(1_X, \hat{e}) \circ (f', g') = (f, g)$.



Surely there is a unique continuous map $f' : Z \to X$ with $1_X \circ f' = f$, namely $f' = f$. By the first paragraph, morphisms $f : Z \to X$ are in bijective correspondence with morphisms (f, g') in StComp from σ_Z to σ_X. Thus, there is a unique morphism (f, g') from σ_Z to σ_X. It remains to show that $\hat{e} \circ g' = g$. We have

$$\hat{e} \circ g' \circ \sigma_Z = \hat{e} \circ \sigma_X \circ f = e \circ f.$$

But g is the unique morphism with $g \circ \sigma_Z = e \circ f$, so $\hat{e} \circ g' = g$. This gives that Smyth is a coreflective subcategory of StComp.

Associating to each T_0-space X its Smyth compactification is functorial since σ is a reflector. Combining this with the forgetful functor that takes a Smyth compactification $\sigma_X : X \to \sigma X$ to X then provides an equivalence between Top_0 and Smyth. □

We next introduce the full subcategory of RE that is dual to Smyth. Let $\alpha : L \to K$ be a Raney extension and S the subframe of K generated by $\alpha[L]$. We call α *maximal* provided $\alpha' \le \alpha$ for any $\alpha' \in \mathsf{RE}(K, S)$. As an immediate consequence of Corollary 6.14 we have:

Theorem 7.3. *A Raney extension $\alpha : L \to K$ is maximal iff the stable compactification $e_\alpha : X_K \to Y_L$ is the Smyth compactification.*

Let MRE be the full subcategory of RE consisting of maximal Raney extensions. The above theorem together with Theorem 5.8 and Proposition 7.2 yield:

Theorem 7.4. MRE *is a reflective subcategory of* RE *that is dually equivalent to* Smyth*, and hence dually equivalent to* Top_0*.*

In [6] we obtained a duality between Top_0 and the category of Raney algebras. This duality is closely related to the duality of Theorem 7.4. We recall that a *Raney algebra* is a pair $(K, \Box)$ where K is a Raney lattice and $\Box$ is an interior operator on K such that its fixpoints form a dense subframe of K. Since the interior operator is determined by its fixpoints, Raney algebras can alternately be described as pairs (K, S) where S is a dense subframe of K. A morphism between Raney algebras (K, S) and (K', S') is a complete lattice homomorphism $\psi : K \to K'$ such that $\psi[S] \subseteq S'$. Let RAlg be the category of Raney algebras and their morphisms. The situation is shown in Figure 2 where $\leftrightarrow$ indicates equivalence, $\leftrightarrow$ with d on top dual equivalence, and $\subseteq$ full reflective subcategory.

$$\mathsf{RAlg} \quad \overset{\mathsf{d}}{\longleftrightarrow} \quad \mathsf{Top}_0 \quad \longleftrightarrow \quad \mathsf{Smyth} \quad \overset{\mathsf{d}}{\longleftrightarrow} \quad \mathsf{MRE} \quad \subseteq \quad \mathsf{RE}$$

FIGURE 2. Equivalences, dual equivalences, and containments

As an immediate consequence we obtain:

Theorem 7.5. RAlg *is equivalent to* MRE.

Remark 7.6. Since MRE is a reflective subcategory of RE, there is a functor RE → MRE, and hence a functor RE → RAlg. It is easy to describe this functor directly. For $\alpha : L \to K$ a Raney extension with S the subframe of K generated by $\alpha[L]$, this functor takes α to the Raney algebra (K, S). For a morphism (ϕ, ψ) from $\alpha : L \to K$ to $\alpha' : L' \to K'$, this functor takes (ϕ, ψ) to ψ. The functor RAlg → MRE can be described as a composite of existing functors. It can also be given directly by taking (K, S) to the largest member of $\mathsf{RE}(K, S)$.

8. Further duality results

In this final section, we specialize our results to obtain several further equivalences and dual equivalences. To begin, we recall that the category Comp of Hausdorff compactifications is a full subcategory of StComp. It is well known (see, e.g., [10, Thm. 3.5.1]) that a space X has a Hausdorff compactification iff it is completely regular. We next place the results of [7] in the context of Raney extensions.

Definition 8.1. A *de Vries extension* is a Raney extension $\alpha : L \to K$ where L is a de Vries algebra and K is a complete atomic Boolean algebra. The category DeVe is the full subcategory of RE consisting of de Vries extensions.

Remark 8.2. In Definition 8.1, we treat the category of de Vries algebras as a full subcategory of the category of regular proximity frames. While this is not obvious from the usual definition of de Vries algebras and de Vries morphisms, it is shown in [4, Prop 7.4].

Remark 8.3. In Definition 8.1, the requirement that K is a complete atomic Boolean algebra is redundant, and follows already from the assumption that $\alpha : L \to K$ is a Raney extension with L a de Vries algebra. Indeed, taking the dual stable compactification $e_\alpha : X_K \to Y_L$, the space Y_L is Hausdorff because L is a de Vries algebra (see [9, Thm. I.3.6]). Therefore, X_K is also Hausdorff, hence its specialization order is identity. Thus, $\mathcal{U}(X_K)$ is the powerset of X_K, and hence K is a complete atomic Boolean algebra.

The following shows that our results extend those obtained in [7].

Theorem 8.4. *The duality between* StComp *and* RE *restricts to a duality between* Comp *and* DeVe.

Proof. By Theorem 5.8, it is sufficient to observe that $e \in \mathsf{Comp}$ implies $e^{-1} \in \mathsf{DeVe}$ and $\alpha \in \mathsf{DeVe}$ implies $e_\alpha \in \mathsf{Comp}$. If $e : X \to Y$ in Comp, then the patch topology on Y coincides with the original topology, and hence $\mathcal{RO}(Y)$ is the de Vries algebra of regular opens of Y. Moreover, since Y is Hausdorff, so is X. Therefore, the specialization order on X is identity, and hence $\mathcal{U}(X) = \mathcal{P}(X)$. Thus, $e^{-1} \in \mathsf{DeVe}$. If $\alpha : L \to K$ is a de Vries extension, then L is a de Vries algebra, so Y_L is Hausdorff, and hence $e \in \mathsf{Comp}$. □

Remark 8.5. In [7], a duality was obtained between the category CReg of completely regular spaces and the category of maximal de Vries extensions. This was established by taking the Stone-Čech compactification of a completely regular space. This is analogous to the process used here to establish a duality between T_0-spaces and maximal Raney extensions using Smyth compactifications. We note that the Smyth compactification of a completely regular space need not be Hausdorff, so Raney extensions are not a convenient tool to work with CReg. However, the notion of Raney algebras applies in a very simple way in this setting. Completely regular spaces correspond exactly to the Raney algebras (K, S) where K is a complete atomic Boolean algebra and S is a completely regular subframe of K that is dense. We note that in this case S is spatial since it is a subframe of a spatial frame. So this amounts essentially to the duality between completely regular spaces and spatial completely regular frames.

We next turn our attention to a result of Smyth [17, Prop. 20] that provides a characterization of the spectral compactifications of a T_0-space X in terms of lattice bases of its topology. We recall that a *lattice basis* of X is a basis of X that is a bounded sublattice of the lattice of opens. A *spectral space* is a sober space whose compact open sets are a lattice basis. It is well known, and easy to see, that each spectral space is stably compact.

Definition 8.6. A *spectral compactification* is a stable compactification $e : X \to Y$ such that Y is a spectral space.

Smyth provides the following.

Theorem 8.7. [17, Prop. 20] *For a T_0-space X, the equivalence classes of spectral compactifications of X are in bijective correspondence with the lattice bases of X.*

We now place this result in a categorical setting. Let SpComp be the full subcategory of StComp consisting of spectral compactifications. We recall (see [4, Def. 8.2]) that an element r of a regular proximity frame L is *reflexive* if $r \prec r$, and that L is *coherent* if $a \prec b$ implies there is a reflexive

$r \in L$ such that $a \prec r \prec b$. We call a Raney extension $\alpha : L \to K$ *coherent* if L is coherent. Let CohRE be the full subcategory of RE consisting of coherent Raney extensions.

Theorem 8.8. *The duality between* StComp *and* RE *restricts to a duality between* SpComp *and* CohRE.

Proof. In view of [4, Thm. 8.6], Y a spectral space implies that $\mathcal{RO}(Y)$ is a coherent regular proximity frame, and L a coherent regular proximity frame implies that Y_L is a spectral space. Thus, $e : X \to Y$ a spectral compactification implies that $e^{-1} : \mathcal{RO}(Y) \to \mathcal{U}(X)$ is a coherent Raney extension, and $\alpha : L \to K$ a coherent Raney extension implies that $e_\alpha : X_K \to Y_L$ is a spectral compactification. Now apply Theorem 5.8. □

The category CohRE can be equivalently given in simpler terms. By [4, Thm 8.14] there is an equivalence between the category of coherent regular proximity frames and their proximity morphisms and the category of bounded distributive lattices and bounded lattice homomorphisms. This equivalence takes a coherent regular proximity frame L to its distributive lattice of reflexive elements, and a proximity morphism $\varphi : L \to L'$ to its restriction to reflexive elements.

Definition 8.9. A *Raney lattice extension* is a bounded lattice embedding $\lambda : D \to K$ of a bounded distributive lattice D into a Raney lattice K such that the image $\lambda[D]$ is dense in K. A morphism between Raney lattice extensions $\lambda : D \to K$ and $\lambda' : D' \to K'$ is a pair (μ, ψ) such that $\mu : D \to D'$ is a bounded lattice homomorphism, $\psi : K \to K'$ is a complete lattice homomorphism, and $\psi \circ \lambda = \lambda' \circ \mu$. We let RLE be the category of Raney lattice extensions and their morphisms.

$$\begin{array}{ccc} D & \xrightarrow{\ \lambda\ } & K \\ {\scriptstyle \mu}\downarrow & & \downarrow{\scriptstyle \psi} \\ D' & \xrightarrow[\ \lambda'\ ]{} & K' \end{array}$$

Corollary 8.10. RLE *is equivalent to* CohRE *and dually equivalent to* SpComp.

Proof. That RLE is equivalent to CohRE follows from the equivalence between the categories of coherent regular proximity frames and bounded distributive lattices. The second statement then follows from Theorem 8.8. □

Each Raney lattice extension $\lambda : D \to K$ is isomorphic to one where D is a bounded sublattice of K and the embedding is the identical embedding.

Therefore, Raney lattice extensions are given, up to isomorphism, by pairs (K, D) where K is a Raney lattice and D is a dense bounded sublattice of K. Viewed in this way, the category RAlg is a full subcategory of RLE. The situation is shown in Figure 3 where $\leftrightarrow$ indicates equivalence, and $\leftrightarrow$ with a d indicates dual equivalence.

$$\begin{array}{ccccc}
\mathsf{RAlg} & \longleftrightarrow & \mathsf{MRE} & \overset{\mathsf{d}}{\longleftrightarrow} & \mathsf{Smyth} \\
\cap\!| & & \cap\!| & & \cap\!| \\
\mathsf{RLE} & \longleftrightarrow & \mathsf{CohRE} & \overset{\mathsf{d}}{\longleftrightarrow} & \mathsf{SpComp}
\end{array}$$

FIGURE 3. Equivalences, dual equivalences, and containments

Thus, Raney lattice extensions provide a natural generalization of the Raney algebras introduced in [6].

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