Logical Aspects of Quantum Structures

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Introduction

We describe some traditional logical questions related to the structures that arise in quantum mechanics such as the lattice of closed subspaces of Hilbert space.

This is rather different, although somewhat related, to quantum logic, which is devoted to obtaining a new type of logic to use in quantum mechanics.

After this, we discuss the first steps in a new direction motivated by this study.

First some background ...

Definition For a Hilbert space H we let $\mathcal{C}(H)$ be the collection of closed subspaces of H.

This forms a type of structure known as an orthomodular lattice.

Definition $(L, \wedge, \vee, \perp, 0, 1)$ is an orthomodular lattice if

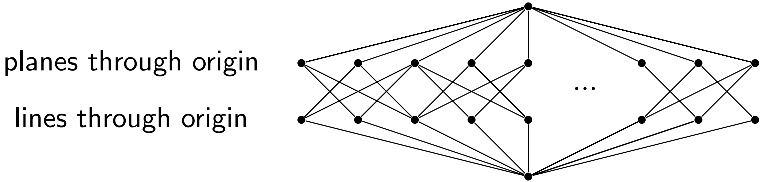
1. L is a bounded lattice
2. $x \vee y$ is the least upper bound of x, y
3. $x \wedge y$ is the greatest lower bound of x, y
4. L has a least element 0 and largest element 1
5. $x \leq y \Rightarrow y^{\perp} \leq x^{\perp}$
6. $x^{\perp\perp} = x$
7. $x \wedge x^{\perp} = 0$ and $x \vee x^{\perp} = 1$
8. $x \leq y \Rightarrow x \vee (x^{\perp} \wedge y) = x \vee y$

It is an ortholattice if (1) - (7) hold.

In $\mathcal{C}(H)$ the elements are closed subspaces of H where

- $A \leq B$ if $A \subseteq B$
- $A \wedge B = A \cap B$
- $A \vee B$ is the closure of the span $A + B$
- $A^{\perp}$ is the subspace of all vectors orthogonal to those in A

Example



This is a crude picture of $\mathcal{C}(\mathbb{C}^3)$.

Orthomodular posets

This is essentially the same definition as an orthomodular lattice except that we only require least upper bounds of elements x, y when they are orthogonal — meaning that $x \leq y'$.

History

Birkhoff and von Neumann introduced the lattice $\mathcal{C}(H)$ as a prominent object of study in 1936.

$\mathcal{C}(H)$ is really the basic ingredient of quantum mechanics.

Husimi identified general orthomodular lattices in 1956.

Mackey brought orthomodular posets to prominence in the 1960s via an axiomatic argument for their role in quantum mechanics.

Our objective

Our objective is to study decidability and axiomatizability issues of $\mathcal{C}(H)$ and related structures.

Definition Let L be an ortholattice. By $\mathrm{QL}(L)$ we mean the set of all equations that are valid in L.

We extend this notation in obvious ways, using $\mathrm{QL}(OML)$ for the equations valid in all orthomodular lattices, and so forth.

Our objective

Some natural questions we wish to consider are the following ...

Question Is there an algorithm to determine if an equation belongs to QL(L), that is, to determine if a given equation holds in L"

Question Is there a finite set of equations true in L from which every other equation true in L can be obtained?

The first question asks about the decidability of QL(L), and the second about the finite axiomatizability of QL(L).

Our Objective

To illustrate, consider the familiar Hilbert space $\mathbb{C}^3$. Is there an algorithm to determine if any given equation such as

$$(A \cap B) + (A \cap B^{\perp}) = A$$

holds in $\mathcal{C}(\mathbb{C}^3)$? i.e. is $\mathrm{QL}(\mathbb{C}^3)$ decidable?

Is it finitely axiomatizable? i.e. are there finitely many equations in $\mathcal{C}(\mathbb{C}^3)$ that imply all others?

Some basic results

Theorem QL(BA) is both decidable and finitely axiomatizable.

Note This says that the equations true in Boolean algebras are decidable and finitely axiomatizable.

Decidability is via truth tables for classical propositional logic.

Finite axiomatizability is given even by one equation!.

Theorem QL(OL) is both decidable and finitely axiomatizable.

Note Decidability is similar to lattices, it was established by my advisor Günter Bruns.

Results ...

Let's turn to the topic of most direct interest for quantum mechanics, $\mathcal{C}(H)$

The following omnibus result collects results of several people including Dunn, Hagge, Moss, Wang, Herrmann, and me.

Theorem

$\mathrm{QL}(\mathbb{C}) \supset \mathrm{QL}(\mathbb{C}^2) \supset \cdots \supset \bigcap\{\mathrm{QL}(\mathbb{C}^{\mathrm{n}}) : \mathrm{n} \geq 1\} = \mathrm{QL}(\mathrm{CG}(\mathbb{C})) = \mathrm{QL}(\mathcal{R})$ for each type II_1 factor $\mathcal{R}$. Each of these containments is strict. Each of these equational theories is decidable, and the first order theory of each $\mathcal{C}(\mathbb{C}^n)$ for $n \geq 1$ is decidable.

Note In many ways, $\mathcal{R}$ is really a closer analog to an infinite dimensional version of $\mathcal{C}(\mathbb{C}^n)$ than is $\mathcal{C}(H)$ with dim $H = \infty$.

There is a lot of information in this theorem ...

(1) $\mathrm{QL}(\mathbb{C}^n) \subset \mathrm{QL}(\mathbb{C}^{n+1})$

Each equation true in $\mathcal{C}(\mathbb{C}^{n+1})$ is true in $\mathcal{C}(\mathbb{C}^n)$ and there are equations true for n and not for $n+1$.

(2) $\bigcap\{\mathrm{QL}(\mathbb{C}^\mathrm{n}) : \mathrm{n} \geq 1\} = \mathrm{QL}(\mathrm{CG}(\mathbb{C})) = \mathrm{QL}(\mathcal{R})$

The equations true in all $\mathcal{C}(\mathbb{C}^n)$ are exactly those that are true in von Neumann's continuous geometry and these are those true in each type II_1 factor.

(3) $\mathcal{C}(\mathbb{C}^n)$ has decidable first order theory

This means we can also decide when quantified statements hold:

$$\forall A\, \exists B\, ((A \cap B) + (A \cap B^\perp) = A)$$

Results

The situation for aximatizability is rather clear ...

Theorem

The first order theory of $\mathcal{C}(\mathbb{C}^n)$ is finitely axiomatizable iff $n = 1$.
The first order theory of $\mathcal{C}(\mathcal{H})$ with H infinite-dimensional is not finitely axiomatizable.

Decidability of $\mathcal{C}(H)$ for dim $H = \infty$

We would like to know whether QL(H) is decidable for H an infinite-dimensional Hilbert space. This is open, but ...

There is relatively recent progress by Fritz using deep results of Slofstra on combinatorial group theory.

Definition A quasi-equation is one of the form

$$\forall x_1, \ldots, x_n \; (s_1 \approx t_1 \;\&\; \cdots \;\&\; s_n \approx t_n \Rightarrow s \approx t)$$

Example

$$\forall A \, \forall B \; (A \cap B = A \Rightarrow (A \cap B) + (A \cap B^\perp) = A)$$

Quasi-equations

Theorem

There is no algorithm to determine which quasi-equations are true in $\mathcal{C}(H)$ for an infinite-dimensional Hilbert space H.

"Proof" The rough idea is to show that there is no algorithm to determine whether a given finite configuration can be embedded into $\mathcal{C}(H)$, and that this embedding is equivalent to determining whether a given quasi-equation is valid in $\mathcal{C}(H)$.

Note

This is similar, but not identical, to determining whether a given finite OMP can be embedded into $\mathcal{C}(H)$.

A different direction

From the undecidability result for quasi-equations in $\mathcal{C}(H)$ I've recently become interested in the following questions.

Question Which finite orthomodular posets can be embedded into $\mathcal{C}(H)$ for H an infinite-dimensional Hilbert space?

Question Which finite orthomodular lattices can be embedded into $\mathcal{C}(H)$ for H an infinite-dimensional Hilbert space?

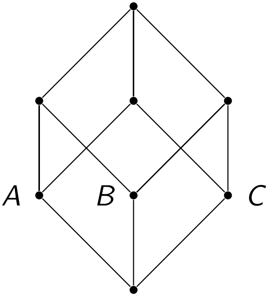
Note

This may seem difficult, but perhaps we can be less ambitious and see if we can just get a bit of a feel ...

Easy Every finite Boolean algebra can be embedded into $\mathcal{C}(H)$.

"Proof" View H as ℓ^2 all square summable sequences. Let

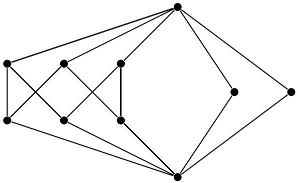
$$A = \{(a_1, 0, 0, a_2, 0, 0, a_3, 0, 0, \ldots) : a_1, a_2, \ldots \in \mathbb{C}\}$$
$$B = \{(0, b_1, 0, 0, b_2, 0, 0, b_3, 0, \ldots) : b_1, b_2, \ldots \in \mathbb{C}\}$$
$$C = \{(0, 0, c_1, 0, 0, c_2, 0, 0, c_3, \ldots) : c_1, c_2, \ldots \in \mathbb{C}\}$$



This is really first embedding into $\mathcal{C}(\mathbb{C}^3)$, then embedding $\mathbb{C}(\mathbb{C}^3)$ into $\mathcal{C}(\ell^2)$.

A different direction

Lets make things a bit tougher, but just a bit ...



This is an eight element Boolean algebra 2^3 and a 4-element Boolean algebra 2^2 joined at the top and bottom.

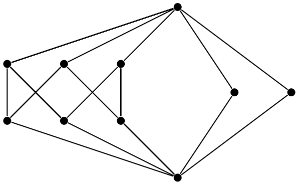
This is usually denoted $2^3 \oplus 2^2$.

We can ask whether $2^3 \oplus 2^2$ can be embedded into some $\mathcal{C}(H)$ as an orthomodular poset or as an orthomodular lattice.

For the first, we preserve orthogonal joins, for the second all joins.

Proposition $2^3 \oplus 2^2$ embeds as an OMP into $\mathbb{C}(\mathbb{C}^3)$ and hence also embeds into $\mathcal{C}(H)$ for infinite-dimensional H.

Proof Consider the x-axis, y-axis, and z-axis as the atoms of the left side, and the line L through the origin in direction (1,1,1) and the plane orthogonal to L as the elements on the right side.



Note The OMP $2^3 \oplus 2^2$ even embeds into a Boolean algebra!

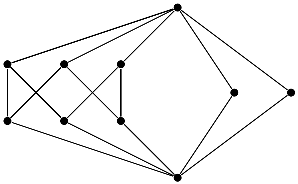
Embedding as an OML is a different story!

Proposition $2^3 \oplus 2^2$ cannot be embedded as an OML into $\mathcal{C}(\mathbb{C}^n)$.

Proof A simple dimensionality argument works.

Recall dim A + dim B = dim $(A + B)$ - dim $(A \cap B)$

The sum of the dimensions of any element on the left and any element on the right must equal n. Impossible!



Proposition The OML $2^3 \oplus 2^2$ embeds into $\mathcal{C}(H)$ if dim $H = \infty$.

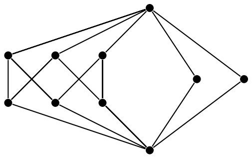
Notes

- In a 1969 paper of Greechie, it is remarked that Ramsey had shown this, but I haven't found Ramsey's result in print.
- I don't know of general literature that addresses such questions.
- One can show that something of substance must be used to prove this result.

Proposition The OML $2^3 \oplus 2^2$ embeds into $\mathcal{C}(H)$ if dim $H = \infty$.

Proof (sketch) Let $\mathcal{H} = L^2(\mathbb{R})$ and $\mathcal{F}$ be the Fourier transform.

$$A = \{f : f \text{ vanishes on } (-\infty, -1)\}$$
$$B = \{f : f \text{ vanishes on } (-1, 1)\}$$
$$C = \{f : f \text{ vanishes on } (1, \infty)\}$$
$$D = \{f : \mathcal{F}f \text{ vanishes on } (-\infty, 0)\}$$
$$E = \{f : \mathcal{F}f \text{ vanishes on } (0, \infty)\}$$



The result follows by showing that $(A + B) \cap D = 0$ etc.

If $f \in D$, by Titchmarsh there exists a holomorphic F on the upper halfplane taking value f on the boundary. If such $f \in A + B$ it is zero on a set of positive measure, by a result of Luzin it is zero a.e.

Questions

Question 1 Is there a simpler proof that $2^3 \oplus 2^2$ embeds in $\mathcal{C}(H)$ without all the complex analysis and measure theory?

Question 2 Can the OML $2^3 \oplus 2^3$ be embedded into $\mathcal{C}(H)$?

Question 3 Is it possible that so little is known about $\mathcal{C}(H)$?

Note These questions shed light on the behavior of complementary observables in quantum mechanics.

Question 4 Which finite lattices can be embedded into $\mathcal{C}(H)$?

Question 5 Does $\mathcal{C}(H)$ satisfy any non-trivial lattice equations?

Thanks for listening.

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