Conditional expectation from a quantum perspective — a brief introduction

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Abstract We provide an introduction to conditional expectation and its formulation in terms of operator algebras that is portable to the quantum setting. We review the basics as well as recent results relating conditional expectation to categorical, order-theoretic, and logical settings.

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1 Introduction

Conditional probability $P(A \mid B)$ is a basic element of undergraduate courses. Still, there is opportunity to push the topic in new directions, see for example the work of Nguyen and Walker on an algebra of conditional events [16] and in a different vein, the role of conditional probability in a quantum setting [8].

In this note we consider the familiar notion of conditional expectation. We review the basics as it is used in classical probability theory. We then pass to its formulation in terms of function spaces and operator algebras as developed by Umegaki [24] and others roughly in the 1960's. We then move to discuss more recent developments of the past 15 years related to categorical treatments, as well as order-theoretic ties that link to classical and quantum logic.

This note is a gentle introduction to several current streams of research that are linked to conditional expectation in the quantum setting. The interested reader can pursue the topic in greater details through the following sources: [2, 13, 24] for

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the operator algebra perspective, [4, 11, 17, 18] for the categorical perspective, and [6, 7] for the order-theoretic and logical perspective.

2 Classical conditional expectation

The results in this section are standard features of probability theory. We direct the reader to [15, Sec. 27] for a more detailed account.

Assume $(\Omega,\mathscr{F},P)$ is a probability space. For events $A,B\in\mathscr{F}$ with the probability $P(B)\neq 0$, the conditional probability of A given B is

$$P(A\,|\,B\,)\;=\;\frac{P(A\cap B)}{P(B)}.$$

Clearly $P(\,\cdot\mid B\,)$ is a probability measure on $(\Omega,\mathscr{F})$. Suppose that $X:\Omega\to\mathbb{R}$ is an essentially bounded (e.b.) random variable. The expected value of X with respect to the measure $P(\,\cdot\,|\,B\,)$ is called the conditional expectation of X with respect to the event B and written $E[X\,|\,B]$. Our restriction to e.b. random variables is for simplicity and because this is the setting that will be employed later in this note. There is a generalization of this notion of conditional expectation.

Definition 1. Let $(\Omega,\mathscr{F},P)$ be a probability space, $X:\Omega\to\mathbb{R}$ be an e.b. random variable, and $\mathscr{G}\leq\mathscr{F}$ be a sub σ-algebra. A conditional expectation of X with respect to $\mathscr{G}$ is an e.b. random variable $Y:\Omega\to\mathbb{R}$ that is $\mathscr{G}$-measurable and for all $B\in\mathscr{G}$

$$E[X\,|\,B]\;=\;E[Y\,|\,B].$$

It is known [23, p. 210] that there is at least one such conditional expectation Y and that any two agree almost surely (a.s.). It is usual to denote a particular such, or the equivalence class of such, by $E[X\,|\,\mathscr{G}]$.

Example 1. If $\mathscr{G}$ is the σ-algebra $\{\emptyset,\Omega\}$, then $E[X\,|\,\mathscr{G}]$ is the constant function that takes value $E[X]$. More generally, if the event B belongs to $\mathscr{F}$, then for the σ-algebra $\mathscr{G}=\{\emptyset,B,B^c,\Omega\}$ we have $E[X\,|\,\mathscr{G}]$ is the function taking constant value $E[X\,|\,B]$ on B and constant value $E[X\,|\,B^c]$ on B^c. So conditional expectation with respect to a σ-algebra generalizes conditional expectation with respect to an event.

Often, we call applying a process such as conditional expectation with respect to a sub σ-algebra, "conditioning" on that σ-algebra.

Example 2. Suppose $\mathscr{G}$ and $\mathscr{G}'$ are independent sub σ-algebras of $\mathscr{F}$, meaning that each event in $\mathscr{G}$ is independent of each event in $\mathscr{G}'$. If X is an e.b. $\mathscr{G}'$-measurable random variable, then for any $B\in\mathscr{G}$ we have that X is independent of B, and it follows (see [3, 4.1.4]) that $E[X\,|\,B]$ is $E[X]$. Thus, $E[X\,|\,\mathscr{G}]$ is the constant function $E[X]$. In other words, conditioning on $\mathscr{G}$ provides no information for a $\mathscr{G}'$-random variable.

Given a sub σ-algebra $\mathscr{G} \leq \mathscr{F}$, we consider $E[\,\cdot\,|\mathscr{G}\,]$ as a mapping from the e.b. random variables on Ω that are measurable with respect to $\mathscr{F}$ to the e.b. random variables that are measurable with respect to $\mathscr{G}$. This conditional expectation has a number of properties, among which are the following:

Proposition 1. *Conditional expectation $E[\,\cdot\,|\mathscr{G}\,]$ has the following properties:*

1. *It is linear;*
2. *It is unital, meaning that it takes the constant function 1 to itself;*
3. *It is positive, meaning that it takes positive random variables to positive ones;*
4. *It is the identity when applied to $\mathscr{G}$-measurable random variables;*
5. *If Y is e.b. and $\mathscr{G}$-measurable, then $E[XY\,|\mathscr{G}\,] = YE[X\,|\mathscr{G}\,]$.*

Conditional expectations are adapted to conditioning with respect to a random variable $Y : (\Omega,\mathscr{F}) \to (\Omega',\mathscr{F}')$ by taking the σ-algebra $\mathscr{G}_Y$ of pre-images of events in $\mathscr{F}'$. One then writes $E[X\,|\,Y\,]$ in place of $E[X\,|\mathscr{G}_Y\,]$.

Conditional probability

The general definition of conditional expectation can be used to treat conditional probability. In the following we use 1_A for the indicator function of a set $A \subseteq \Omega$.

Definition 2. Let $(\Omega,\mathscr{F},P)$ be a probability space and $\mathscr{G} \leq \mathscr{F}$ be a sub σ-algebra. Then the conditional probability of an event $A \in \mathscr{A}$ with respect to $\mathscr{G}$ is given by

$$P^{\mathscr{G}}(A) \;=\; E[\,1_A\,|\mathscr{G}\,].$$

Note that for a given event A, we have that $P^{\mathscr{G}}(A)$ is a function from Ω to $[0,1]$. If we fix $\omega \in \Omega$, we obtain a function $P^{\mathscr{G}}_{\omega} : \mathscr{F} \to [0,1]$. One might hope that each such $P^{\mathscr{G}}_{\omega}$ is a probability measure. This is not generally the case. Recalling that $E[\,1_A\,|\mathscr{G}\,]$ is defined only up to a.s. equivalence, the issue is whether choices can be made in a uniform manner to allow the $P^{\mathscr{G}}_{\omega}$ to be probability measures. It turns out that this is the case when $(\Omega,\mathscr{F},P)$ is a standard Borel probability space.

Definition 3. Given measurable spaces $(\Omega,\mathscr{F})$ and $(\Omega',\mathscr{F}')$, a Markov kernel is a mapping $k : \Omega \times \mathscr{F}' \to [0,1]$ that satisfies

1. $k(\omega,\,\cdot\,)$ is a probability measure k_ω for each $\omega \in \Omega$;
2. $k(\,\cdot\,,A'\,)$ is measurable for each $A' \in \mathscr{F}'$.

A conditional probability is called regular if it is a Markov kernel.

Proposition 2. *If $(\Omega,\mathscr{F},P)$ is a standard probability space and $\mathscr{G} \leq \mathscr{F}$, then the conditional probability $P^{\mathscr{G}}$ is regular.*

The notion of conditional probability can be extended to the setting of a random variable. In the following, suppose $(\Omega,\mathscr{F},P)$ is a probability space, $\mathscr{G} \leq \mathscr{F}$ is a sub σ-algebra and $X : (\Omega,\mathscr{F}) \to (\Omega',\mathscr{F}')$ is a random variable.

Definition 4. The conditional probability of X with respect to $\mathscr{G}$ is the function $\mu_{X\,|\,\mathscr{G}} : \Omega \times \mathscr{F}' \to [0,1]$ given by setting for each $A' \in \mathscr{F}'$

$$\mu_{X\,|\,\mathscr{G}}(\omega,A') \;=\; P^{\mathscr{G}}_{\omega}(X^{-1}(A')).$$

If $\mu_{X\,|\,\mathscr{G}}$ is a Markov kernel, then by definition for each $\omega \in \Omega$ we have $\mu_{X\,|\,\mathscr{G}}(\omega,\cdot)$ is a probability measure that we can integrate over. We often write $d\mu_{X\,|\,\mathscr{G}}$ leaving the value of ω implied.

Proposition 3. *If the spaces involved are standard Borel spaces, then $\mu_{X\,|\,\mathscr{G}}$ is a Markov kernel. Further, if X is real-valued, then*

$$E[X\,|\,\mathscr{G}] \;=\; \int_{\mathbb{R}} x\, d\mu_{X\,|\,\mathscr{G}} \qquad a.s.$$

So Markov kernels arise from random variables between standard Borel measurable spaces. Further, all such arise this way. To see this, it is convenient to further extend our terminology. Suppose $(\Omega,\mathscr{F},P)$ is a probability space and $X:(\Omega,\mathscr{F}) \to (\Omega',\mathscr{F}')$ and $Y:(\Omega,\mathscr{F}) \to (\Omega'',\mathscr{F}'')$ are random variables. We write $\mu_{X\,|\,Y}$ for the conditional probability $\mu_{X\,|\,\mathscr{G}}$ where $\mathscr{G}$ is the sub σ-algebra of $\mathscr{F}$ induced by Y.

Proposition 4. *If $k:(\Omega',\mathscr{F}') \to (\Omega'',\mathscr{F}'')$ is a Markov kernel between standard Borel spaces, then there is a standard Borel space $(\Omega,\mathscr{F},P)$ and random variables $X:(\Omega,\mathscr{F}) \to (\Omega',\mathscr{F}')$ and $Y:(\Omega,\mathscr{F}) \to (\Omega'',\mathscr{F}'')$ such that $k = \mu_{X\,|\,Y}$.*

This result is established by taking $(\Omega,\mathscr{F})$ to be the product of the measurable spaces $(\Omega',\mathscr{F}')$ and $(\Omega'',\mathscr{F}'')$, choosing a probability measure P' on $(\Omega',\mathscr{F}')$, and letting P be the probability measure on $(\Omega,\mathscr{F})$ defined on rectangles $A' \times A''$ by

$$P(A' \times A'') \;=\; \int_{A'} k(\omega',A'')\,dP'.$$

Then the natural projections $\pi' : \Omega' \times \Omega'' \to \Omega'$ and $\pi'' : \Omega' \times \Omega'' \to \Omega''$ are random variables and a.s. $P^X(Y^{-1}(A'')(\omega) = k(\pi'(\omega),A'')$. In effect, Markov kernels arise as conditionals of one projection with respect to another.

Remark 1. We can view a Markov kernel $k:(\Omega,\mathscr{F}) \to (\Omega',\mathscr{F}')$ as type of non-deterministic mapping from Ω to Ω' where $k(\omega,A')$ is the probability that ω is mapped by k within A'. A measurable function $f:(\Omega,\mathscr{F}) \to (\Omega',\mathscr{F}')$ is viewed as a deterministic mapping. Set $\delta_{\omega'}$ to be the point mass at ω' for each $\omega' \in \Omega'$. We can describe f by the Markov kernel k_f where $k_f(\omega,A') = \delta_{f(\omega)}(A')$. The situation is somewhat akin to that of functions and relations between sets. A function is viewed as a deterministic map between X and Y while a relation is seen as a sort of possibilistic or many-valued map from X to Y. We will return to this topic later.

3 An operator algebra viewpoint

We view the contents of the previous section from the viewpoint of commutative operator algebras, in particular von Neumann (vN) algebras. For this discussion, we won't need the details of the theory of vN algebras, but we remind the reader of a few basic facts. For a complete account, see any standard book on operator algebras such as [14].

Example 3. For a Hilbert space $\mathscr{H}$, its collection $\mathscr{B}(\mathscr{H})$ of bounded operators is a motivating example of a vN algebra. It is in particular a complex vector space endowed with a means to multiply AB (compose) elements; a unit 1 for this multiplication, namely the identity operator; and a unary operation A^* of adjunction.

A vN algebra can be defined as a subalgebra of $\mathscr{B}(\mathscr{H})$ that is closed in what is known as the weak operator topology. Our attention in this section will be directed towards commutative, or abelian, vN algebras, ones whose multiplication is commutative, and these have an explicit description. We begin with a simple example.

Example 4. For the Hilbert space $\mathbb{C}^2$, its bounded operators can be described as the collection M_2 of complex 2×2 matrices. This is a non-commutative vN algebra. Its subalgebra D_2 of diagonal matrices is a commutative vN algebra. Note that D_2 can be described as the collection of all maps from a 2-element set into the complex numbers by listing a 2×2 diagonal matrix by its diagonal entries.

Definition 5. Let $L^\infty(\Omega, \mathscr{F}, \mu)$ be all a.e. equivalence classes of essentially bounded measurable complex-valued functions on the measure space $(\Omega, \mathscr{F}, \mu)$.

Note that $L^\infty(\Omega, \mathscr{F}, \mu)$ has a vector space structure through pointwise addition and scalar multiplication of functions, a multiplication of pointwise multiplication of functions, a unit 1 for multiplication given by the constant function of value 1, an adjoint of taking pointwise complex conjugate of a complex-valued function, and a norm given by the essential supremum $\|f\|_\infty$ of a function. The following is well known, see e.g. [14].

Theorem 1. *Each $L^\infty(\Omega, \mathscr{F}, \mu)$ for a measurable space $(\Omega, \mathscr{F}, \mu)$ is an abelian vN algebra and every abelian vN algebra is isomorphic to one of this form.*

Remark 2. Because of this theorem, the study of vN algebras is often called non-commutative measure theory.

Definition 6. An element a of a vN algebra is self-adjoint if $a = a^*$, a projection if it is self-adjoint and $a = a^2$, and positive if it is equal to bb^* for some b. A map between vN algebras is positive if it takes positive elements to positive ones.

Example 5. In the vN algebra $L^\infty(\Omega, \mathscr{F}, \mu)$ self-adjoint elements are the real-valued functions. Projections are the functions taking values $0, 1$, and these are given by measurable sets.

Definition 7. A state on a vN algebra $\mathscr{A}$ is a positive linear functional $\phi : \mathscr{A} \to \mathbb{C}$ with $\phi(1) = 1$. A state is normal if it is continuous with respect to the σ-weak topology and is faithful if the only positive element mapped to 0 is 0.

Example 6. If $\mathscr{A} = L^\infty(\Omega, \mathscr{F}, P)$ is the vN algebra associated with the probability space $(\Omega, \mathscr{F}, P)$, then integration $f \mapsto \int_\Omega f\,dP$ is a normal faithful state on $\mathscr{A}$. There may be other normal states on $\mathscr{A}$, and these will be given by probability measures Q on $(\Omega, \mathscr{F})$ that are absolutely continuous $Q << P$ with respect to P.

Conditional expectation

Having seen how some basic concepts in probability translate to the setting of abelian vN algebras, we turn our attention to conditional expectation. These results are standard, and can be found in [2, p. 132ff]. For a probability space $(\Omega, \mathscr{F}, P)$ and a sub σ-algebra $\mathscr{G} \leq \mathscr{F}$, we have an abelian vN algebra $\mathscr{A} = L^\infty(\Omega, \mathscr{F}, P)$ and a vN subalgebra of it $\mathscr{B} = L^\infty(\Omega, \mathscr{G}, P)$. An e.b comples random variable on $(\Omega, \mathscr{F}, P)$ is simply an element $f \in \mathscr{A}$. It follows that conditional expectations $E[\,\cdot\,|\mathscr{G}\,]$ are given by mappings

$$E[\,\cdot\,|\mathscr{G}\,] : \mathscr{A} \to \mathscr{B}.$$

Basic properties of conditional expectation give that this map is linear, positive, and restricts to the identity on $\mathscr{B}$, hence is idempotent and unital. For $Y = E[\,f\,|\mathscr{G}\,]$, the definition of conditional expectation gives $E[\,f\,|\Omega] = E[\,Y\,|\,\Omega\,]$, hence

$$\int_\Omega f\,dP \;=\; \int_\Omega E[\,f\,|\mathscr{G}\,]\,dP.$$

Thus, if ϕ is the state on $\mathscr{A}$ given by the probability measure P, for each $f \in \mathscr{A}$ we have $\phi(f) = \phi(E[\,f\,|\mathscr{G}\,])$. For the following, see e.g. [2, p. 132].

Definition 8. Let $\mathscr{A}$ be an abelian vN algebra with normal unital faithful state ϕ and $\mathscr{B}$ be a vN subalgebra of $\mathscr{A}$. A conditional expectation of $\mathscr{A}$ onto $\mathscr{B}$ with respect to ϕ is a an idempotent onto mapping $E : \mathscr{A} \to \mathscr{B}$ of norm 1 with $\phi = \phi \circ E$.

Every abelian vN algebra with a normal unital faithful state ϕ can be realized as $L^\infty(\Omega, \mathscr{G}, P)$ for some probability space $(\Omega, \mathscr{F}, P)$, and every vN subalgebra $\mathscr{B}$ of it can be realized by taking some sub σ-algebra $\mathscr{G} \leq \mathscr{F}$. Then there is a unique conditional expectation of $\mathscr{A}$ onto $\mathscr{B}$ with respect to ϕ and it is given by the classical conditional expectation $E[\,\cdot\,|\mathscr{G}\,]$. See [2, p. 135].

4 A categorical viewpoint

We next view our discussion of conditional expectation and conditional probability via Markov kernels through the lens of category theory. The intent here is to sketch an outline in broad strokes, focusing on the overall picture, with the understanding that the reader may not have much prior exposure to category theory. With this in mind, we begin with a brief review.

Very often in mathematics, one considers a type of structure of a certain sort, a set, or topological space, or group, etc. These structures usually come equipped with a notion of a map, or process, that relates one of these things to another, for example functions between sets, continuous maps between topological spaces, or group homomorphisms between groups. Category theory takes the view that processes between the particular kinds of objects are of primary importance, and essentially forgets the details of the internal structure of the things under consideration. For an account of the following, and of category theory in general, see e.g. [21].

Definition 9. A category C is a collection of two sorts of things; objects, written with capitals $A,B,C,\ldots$, and morphisms $f:A\to B$ between objects; together with a rule to compose a morphism $f:A\to B$ with a morphism $g:B\to C$, to produce a morphism $g\circ f:A\to C$. It is required that this composition is associative when defined, and that for each object A there is an identity morphism $1_A:A\to A$ that acts as a identity on both sides for appropriate compositions.

Example 7. As mentioned, there is a category Set of sets whose objects are sets, whose morphisms are functions between sets, and whose composition is the usual composition of functions. There is also a category Rel whose objects are sets, whose morphisms are the binary relations between sets, and whose composition is usual composition of relations.

There are a number of categories of measurable spaces and of vN algebras that are relevant here. For the following, we note that a homomorphism between vN algebras is a linear map that preserves multiplication and the involution $*$.

Definition 10. Let Prob be the category of probability spaces and measurable maps between them, and let AbvN be the category of abelian vN algebras and the normal unital vN algebra homomorphisms between them.

There is a close connection between these categories. As we have seen, every probability space $(\Omega,\mathscr{F},P)$ gives rise to an abelian vN algebra $L^\infty(\Omega,\mathscr{F},P)$ as well as a unital normal faithful state ϕ on it, and each abelian vN algebra with such a state arises in this way. Further, each measurable function $g:(\Omega,\mathscr{F},P)\to(\Omega',\mathscr{F}',P')$ gives a normal unital vN algebra homomorphism $\hat{g}:L^\infty(\Omega',\mathscr{F}',P')\to L^\infty(\Omega,\mathscr{F},P)$ where $\hat{g}$ takes an essentially bounded $f:(\Omega',\mathscr{F}',P')\to\mathbb{C}$ to the essentially bounded map $\hat{g}(f):(\Omega,\mathscr{F},P)\to\mathbb{C}$ given by $f\circ g$. Moreover, each such normal unital vN algebra homomorphism arises in this way.

Markov kernels and positive maps

We next describe a related categorical formulation that involves the conditional probabilities and Markov kernels discussed earlier.

Definition 11. If $k:(\Omega,\mathscr{F},P)\to(\Omega',\mathscr{F}',P')$ and $\ell:(\Omega',\mathscr{F}',P')\to(\Omega'',\mathscr{F}'',P'')$ are Markov kernels, then their composite $\ell\circ k:\Omega\times\mathscr{F}''\to[0,1]$ is given by

$$\ell\circ k(\omega,A'') \;=\; \int_{\Omega'}\ell(\omega',A'')\,dk_\omega$$

It is known that $\ell \circ k$ is a Markov kernel, that this composition of Markov kernels is associative when defined, and that there is a Markov kernel acting as the identity on a space $(\Omega, \mathscr{F}, P)$, namely $k(x,A) = 1_A(x)$. Thus, we may define

Definition 12. Mark is the category whose objects are probability spaces and whose morphisms are Markov processes.

Recall from the previous section the definitions of a normal, positive, unital linear map between abelian vN algebras. These are called channels. It is easily seen that the composition of channels is a channel and the identity map is a channel. Thus, we may define

Definition 13. Let AbChan be the category whose objects are abelian von Neumann algebras that have a normal unital state and whose morphisms are channels.

Again, there are ties between the categories Mark and AbChan. We have already discussed the relationship between their objects, we discuss the relationship between their morphisms.

Definition 14. For $k : (\Omega, \mathscr{F}, P) \to (\Omega', \mathscr{F}', P')$ a Markov kernel and $f : \Omega' \to \mathbb{C}$ an e.b. map, define

$$T_k(f)(\omega) \;=\; \int_\Omega f(\omega')\, dk_\omega.$$

This yields an e.b. map from $\Omega \to \mathbb{C}$. However, in general we can have f and g agree a.s. with respect to P but $T_k(f)$ and $T_k(g)$ not agree almost surely since the measures k_ω need not be related to P'. However, if each $k_\omega << P'$, this does not occur and we can view T_k as a mapping $T_k : L^\infty(\Omega', \mathscr{F}', P') \to L^\infty(\Omega, \mathscr{F}, P)$. Further, in this case T_k is a channel.

Conversely, under sufficient conditions on the probability spaces, one can show that a normal unital positive map $T : L^\infty(\Omega', \mathscr{F}', P') \to L^\infty(\Omega, \mathscr{F}, P)$ arises in this way from a Markov kernel k defined by setting $k(\omega, A) = T[1_A](\omega)$.

Remark 3. There is a great deal more that can be said about categories formed from measurable spaces and Markov kernels. We direct the reader to [17, 4] and the other works cited in these.

5 Passage to the quantum setting

Here we paint in broad strokes an outline of the formulation of quantum mechanics in terms of vN algebras for those unfamiliar with the subject. In many ways, it is a sort of non-commutative version of probability theory. To begin, to each physical system, we associate a Hilbert space $\mathscr{H}$.

Definition 15. Let $\mathscr{P}(\mathscr{H})$ be the collection of projection operators of $\mathscr{H}$.

Informally, the Hilbert space plays somewhat the role of a sample space and the projections, which correspond to closed subspaces, play the role of a σ-algebra of subsets of the sample space. The projections $\mathscr{P}(\mathscr{H})$ carry many properties of a σ-algebra, and we investigate these in the following section.

Observables

Observables of a physical system are self-adjoint operators on $\mathscr{H}$. While observables need not be bounded operators, they can be described via the spectral theorem through bounded operators. It is not required that all self-adjoint operators are observables of the system, and generally the observables are assumed to be the self-adjoint operators that are affiliated with a vN subalgebra $\mathscr{V}$ of the algebra $\mathscr{B}(\mathscr{H})$ of all bounded operators. Thus, to a physical system is associated a vN algebra.

States

States of a system are intended as a complete as possible description of the behavior of the system. They are given by normal unital positive maps $\phi : \mathscr{V} \to [0,1]$ on the vN algebra attached to the system. Every vN subalgebra of $\mathscr{B}(\mathscr{H})$ is generated as a vN algebra by its projections, and it follows that every state is determined by its restriction to the projections $\phi : \mathscr{P}(\mathscr{V}) \to [0,1]$. This restriction has many properties of a probability measure, including being σ-additive, and under mild conditions, each such σ-additive measure on $\mathscr{P}(\mathscr{V})$ yields a state on $\mathscr{V}$. So informally, states on $\mathscr{V}$ correspond to probability measures on a measurable space.

Compound systems

In classical probability, one combines two measurable spaces $(\Omega, \mathscr{F})$ and $(\Omega', \mathscr{F}')$ by taking the product $\Omega \times \Omega'$ of their underlying sets with the σ-algebra $\mathscr{F} \otimes \mathscr{F}'$ generated by all rectangles in the familiar way. Probability measures P and P' give a product probability measure on this product space, but there are many other probability measures on this product space that do not arise this way. For any probability measure on the product space, we can form its marginals in the familiar way.

Given two physical systems represented by vN algebras $\mathscr{V}$ and $\mathscr{V}'$, we can consider the pair of systems as a physical system in its own right and attach to it the vN algebra $\mathscr{V} \otimes \mathscr{V}'$ formed by taking the tensor product. A pair of states ϕ and ϕ' on these vN algebras yields a product state $\phi \otimes \phi'$ on the coupled system, but there are states on the coupled system that are not of this form — these are called entangled states. States on the compound system have marginals on each component system.

Processes

The simplest processes between quantum systems represented by $\mathscr{V}$ and $\mathscr{V}'$ are given by unitary operators $U : \mathscr{V} \to \mathscr{V}'$ between them. More general processes are given by what are called quantum channels $C : \mathscr{V} \to \mathscr{V}'$. These are linear maps that are unital, normal, and completely positive. Positivity for a vN algebra is reference to elements of the for aa^*, which are called positive elements. Completely positivity of C means that the tensor product $C \otimes 1 : \mathscr{V} \otimes \mathscr{W} \to \mathscr{V}' \otimes \mathscr{W}$ is positive for the identity map $1 : \mathscr{W} \to \mathscr{W}$ for any vN algebra $\mathscr{W}$. Every positive map between abelian vN algebras is completely positive, so quantum channels generalize the channels between abelian vN algebras discussed earlier and which correspond to Markov processes between their associated probability spaces.

The moral of the story

The category of vN algebras with channels between them and tensor products to treat compound systems can be viewed as a sort of quantum analog of the category of probability spaces and Markov kernels, using product spaces for compound systems. This is one of many instances of a process known as mathematical quantization that relates features in an area of mathematics involving non-commutative structures to ones in a classical area of mathematics, such as the analogy between vN algebras and measurable spaces. See [25] for more examples of mathematical quantization.

6 The logical viewpoint

We next return to our original consideration of conditional expectations and view them from a quantum perspective. In the 1950's, Tomiyama [22] introduced the notion of conditional expectation for vN algebras, which we put in the form below.

Definition 16. A conditional expectation for a vN algebra $\mathscr{V}$ is a normal, unital, positive linear mapping $E : \mathscr{V} \to \mathscr{S}$ onto a subalgebra of $\mathscr{W}$ that satisfies for all $a_1, a_2 \in \mathscr{V}$ and $b \in \mathscr{W}$

$$E(a_1 b a_2) \;=\; a_1 E(b) a_2.$$

These conditions imply that a conditional expectation is completely positive, hence a channel, idempotent, and contractive, meaning $\|E(a)\| \leq \|a\|$. The matter of existence of conditional expectations for vN algebras is delicate. We consider only the following setting [13, Theorem 7, Proposition 15] and note that a tracial state is a state with $\tau(ab) = \tau(ba)$ for all a, b.

Theorem 2. *If $\mathscr{V}$ is a vN algebra with a faithful tracial state and $\mathscr{S} \leq \mathscr{V}$ is a vN subalgebra, then there is a unique conditional expectation $E : \mathscr{V} \to \mathscr{S}$ that preserves this state.*

The vN algebra M_n of $n \times n$ complex matrices is isomorphic to $\mathscr{B}(\mathbb{C}^n)$. It has a faithful tracial state given by the usual trace of a matrix. Another instance of a vN algebra with a faithful tracial state is that of a type II_1 factor, the vN algebras that are continuous geometries in the sense of von Neumann, see e.g. [14]. To further explore properties of conditional expectations we first consider properties of the structure of the projections $\mathscr{P}(\mathscr{V})$ of a vN algebra $\mathscr{V}$.

Orthomodular lattices

We recall that a lattice is a partially ordered set with ordering $\leq$ where any two elements have a least upper bound $a \vee b$ and a greatest lower bound $a \wedge b$. A complete lattice is a lattice where every subset A has a least upper bound $\bigvee A$ and a greatest lower bound $\bigwedge A$. A bounded lattice has a least element 0 and a largest element 1.

Definition 17. A orthomodular lattice (OML) L is bounded lattice with a unary operation $\perp$ that satisfies

1. $x \le y \Rightarrow y^\perp \le x^\perp$;
2. $x^{\perp\perp} = x$;
3. $x \wedge x^\perp = 0$ and $x \vee x^\perp = 1$;
4. $x \le y \Rightarrow x \vee (x^\perp \wedge y) = y$.

Example 8. Every Boolean algebra (BA) is an OML. The motivating non-Boolean example is provided by the closed subspaces of a Hilbert space. These form a complete OML with greatest lower bounds given by intersections, least upper bounds by the closure of the span of the union, and the operation $\perp$ given by orthogonal complement. Since closed subspaces correspond to projections, the projections of $\mathscr{B}(\mathscr{H})$ form a complete OML. More generally, the projections $\mathscr{P}(\mathscr{V})$ of any vN algebra $\mathscr{V}$ form a complete OML.

Events and first-order logic

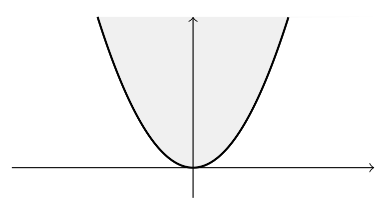
We switch perspective, and speak about matters related to logic, and in particular to first-order logic. Again, we give an overview in broad terms, assuming little prior familiarity with the topic except at a naive level.

Consider the situation where we can talk about the field of real numbers and its ordering. We consider formulas in at most the two variables x and y. Some examples of such formulas are

$$x < y \qquad x + y = x^2 \qquad (x < y) \text{ OR } (y^2 < x) \qquad \exists x (x < y).$$

To each such formula ϕ, we associate the subset $S(\phi)$ of $\mathbb{R}^2$ consisting of all those pairs (x,y) that make the formula true. The set associated to the first formula above is the half-plane below the diagonal, and to the second a parabola. Note that $S(\phi_1 \text{ OR } \phi_2)$ is the union of $S(\phi_1)$ and $S(\phi_2)$. Similarly, applying S to a conjunction ϕ_1 AND ϕ_2 is given by an intersection, and to a negation $\neg\phi$ by the set complement. Applying S to a quantified statement $\exists x\, \phi$ is performed by extending the set $S(\phi)$ horizontally to create a cylinder. For example, $S(x^2 < y)$ is all points lying above the parabola $y = x^2$ and its clyindrification $S(\exists x\, (x^2 < y))$ is the upper half-plane.

$S(x^2 < y)$



In this approach, we have a Boolean algebra of sets, namely the powerset $\mathscr{P}(\mathbb{R}^2)$ of the plane, to model events, and use the operations of union, intersection, and negation as is usual in probability theory. We also have new operations of cylindrification along the x and y axes to treat existential quantification $\exists x\, \phi$ and $\exists y\, \phi$. One treats universal quantification by treating $\exists x\, \phi$ as $\neg \forall x \neg \phi$.

Monadic and cylindric algebras

In the 1960's, Halmos [5] and independently Henkin and Tarski [9, 10], used these ideas to make an algebraic treatment of fist order logic formulated in terms of Boolean algebras with additional unary operations.

Definition 18. A quantifier $\exists$ on a Boolean algebra B is an operation that satisfies

1. $\exists 0 = 0$ and $\exists 1 = 1$,
2. $a \leq \exists a \ = \ \exists\exists a$,
3. $\exists(a \vee b) = \exists a \vee \exists b$,
4. $\exists\neg\exists\phi \ = \neg\exists\phi$.

In the example described above where we consider first-order formulas in the variables x, y for the ordered field of real numbers, we consider the Boolean algebra $\mathscr{P}(\mathbb{R}^2)$ with two quantifiers $\exists_x$ and $\exists_y$. Roughly, a monadic algebra is a Boolean algebra with one quantifier and a diagonal-free cylindric algebra is a Boolean algebra with a family of commuting quantifiers meaning that $\exists_i\exists_j a = \exists_j\exists_i a$. This reflects the fact that $\exists x\exists y\phi$ is logically equivalent to $\exists y\exists x\phi$. Diagonals on a cylindric algebra reflect the equality relation used in logical formulas and we do not discuss these here.

Definition 19. For a complete Boolean algebra B, a complete subalgebra of B is a subset $C \subseteq B$ that is closed under formation of arbitrary joins and meets in B as well as complementation in B.

This allows us to work with quantifiers in a very simple way.

Lemma 1. *For a complete Boolean algebra B, the range C of a quantifier $\exists$ on B is a complete subalgebra of B, and each complete Boolean subalgebra $C \leq B$ is the range of a unique quantifier on B.*

The idea is that for a complete subalgebra $C \leq B$, we obtain a quantifier by setting

$$\exists a \ = \ \text{the least element in } C \text{ that lies above } a.$$

Quantum monadic algebras

We define a quantifier on an OML by applying Definition 18 verbatim, but to an operator on an OML rather than to one on a Boolean algebra. By Lemma 1, quantifiers on complete OMLs are in bijective correspondence with complete subalgebras. For the following the reader should consult [6, 7].

Definition 20. A monadic OML consists of a an OML with a quantifier on it. A diagonal-free cylindric OML consists of an OML with a family of commuting quantifiers on it.

We give two examples to indicate the scope of these notions.

Example 9. A vN algebra $\mathscr{V}$ is a weakly closed subalgebra of the algebra $\mathscr{B}(\mathscr{H})$ of bounded operators on a Hilbert space $\mathscr{H}$. So the projections $\mathscr{P}(\mathscr{V})$ form a subalgebra of the complete OML of projections $\mathscr{P}(\mathscr{H})$, and since $\mathscr{V}$ is a vN subalgebra, it is known that $\mathscr{P}(\mathscr{V})$ is a complete subalgebra of $\mathscr{P}(\mathscr{H})$. So each vN subalgebra gives rise to a quantifier on $\mathscr{P}(\mathscr{H})$ that determines it. Not every complete subalgebra of $\mathscr{P}(\mathscr{H})$ is the projections of a vN subalgebra, so there are quantifiers on $\mathscr{P}(\mathscr{H})$ that do not arise this way. More generally, given any vN algebra $\mathscr{V}$ and any vN subalgebra $\mathscr{S} \leq \mathscr{V}$, we get a complete subalgebra $\mathscr{P}(\mathscr{S}) \leq \mathscr{P}(\mathscr{V})$, hence a quantifier on $\mathscr{P}(\mathscr{V})$.

We note, that this example applies in particular to subfactors of a vN factor, an area of considerable interest and the setting of the Jones index [12].

Example 10. Suppose that $\mathscr{H}$ and $\mathscr{K}$ are Hilbert spaces. Then for projections $P \in \mathscr{P}(\mathscr{H})$ and $Q \in \mathscr{P}(\mathscr{K})$ we have that $P \otimes Q$ is a projection of $\mathscr{H} \otimes \mathscr{K}$. The collection $\{1 \otimes Q \mid Q \in \mathscr{P}(\mathscr{K})\}$ is a complete subalgebra of $\mathscr{P}(\mathscr{H} \otimes \mathscr{K})$, hence gives a quantifier $\exists_{\mathscr{H}}$, and we similarly obtain a quantifier $\exists_{\mathscr{K}}$. For a family of Hilbert spaces $\mathscr{H}_1, \ldots, \mathscr{H}_n$, this extends in an obvious way to provide a family of n quantifiers $\exists_i$ on the projection lattice of $\mathscr{H}_1 \otimes \cdots \otimes \mathscr{H}_n$. It is shown in [6] that these quantifiers commute, thus providing a diagonal-free quantum cylindric set algebra. This example is closely related to Weaver's quantum set theory [25].

Ties between quantifiers and conditional expectations

Suppose $\mathscr{V}$ is a vN algebra. As we have seen, quantifiers on the complete OML $\mathscr{P}(\mathscr{V})$ correspond to complete subalgebras of $\mathscr{P}(\mathscr{H})$. If we assume that $\mathscr{V}$ has a unique faithful tracial state τ, as is the case with $M_n(\mathbb{C})$ or with a type II_1 factor, then by Theorem 2 each vN subalgebra $\mathscr{S} \leq \mathscr{V}$ has a unique conditional expectation that preserves τ. Thus, since a vN algebra is generated by its projections, we have

Theorem 3. *Let $\mathscr{V}$ be a vN algebra with a unique tracial state τ. Then the conditional expectations on $\mathscr{V}$ that preserve τ correspond to the complete subalgebras $S \leq \mathscr{P}(\mathscr{V})$ that are projection lattices of vN subalgebras of $\mathscr{V}$, and these in turn correspond to the quantifiers on $\mathscr{P}(\mathscr{V})$ whose range is such a subalgebra of $\mathscr{P}(\mathscr{V})$.*

There is another question of interest related to commuting quantifiers and cylindric OMLs. We first describe a similar notion in the vN algebra setting.

Definition 21. Let $\mathscr{V}$ be a vN algebra with a unique tracial state τ. A commuting square of vN subalgebras of $\mathscr{V}$ consists of subalgebra $\mathscr{R}, \mathscr{S}, \mathscr{T} \leq \mathscr{V}$ such that the conditional expectations $E_{\mathscr{S}}$ and $E_{\mathscr{T}}$ commute, $E_{\mathscr{S}} S_{\mathscr{T}} = E_{\mathscr{T}} E_{\mathscr{S}}$, and $\mathscr{R} = \mathscr{S} \cap \mathscr{T}$.

$$
\begin{array}{ccc}
\mathscr{S} & \leq & \mathscr{V} \\
\rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} \\
\mathscr{R} & \leq & \mathscr{T}
\end{array}
$$

This notion was introduced by Popa [19] in the case when the intersection $\mathscr{R}$ was trivial, and he then called $\mathscr{S}$ and $\mathscr{T}$ orthogonal subalgebras. Orthogonal subalgebras generalize from the abelian vN algebra setting the notion of independent σ-algebras from Example 2. Commuting squares in the form above were introduced by Popa in [20]. The following is found in [6].

Proposition 5. *Let $\mathscr{V}$ be a vN algebra with a unique faithful tracial state and $\mathscr{R},\mathscr{S},\mathscr{T},\mathscr{V}$ be a commuting square of subalgebras. Then the quantifiers associated to $\mathscr{S}$ and $\mathscr{T}$ commute.*

Commuting squares of subalgebras play a large role in subfactor theory. Even in what seems like the simplest setting, orthogonal maximal abelian subalgebras (MASAs) of a matrix algebra M_n, there is a considerable interest. Here, MASAs correspond to orthonormal bases (ONBs) and two MASAs are orthogonal when these ONBs are mutually unbiased, meaning the angles between two vectors, one from each, is constant. It is unknown for instance the maximal size of a collection of pairwise orthogonal MASAs in the matrix algebra of 6×6 matrices, see [1].

Remark 4. It would be of interest to consider the matter of commuting quantifiers in the general setting of OMLs. In particular, for an OML L, consider an associated graph whose vertices are the maximal Boolean subalgebras of L, usually called it blocks, with an edge between vertices iff the conditional expectations associated to these blocks commutes. What does this graph look like for $\mathscr{P}(\mathbb{C}^n)$?

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