Lattice Theory Lecture 2

Distributive lattices

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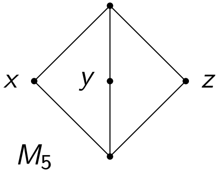
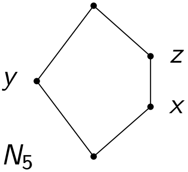
Distributive lattices

Distributive law	for all x, y, z	$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$
Modular law	if $x \leq z$ then	$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$

Definition The lattices M_5 and N_5 are as follows:





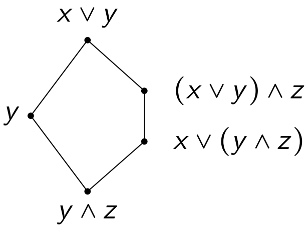
Note M_5 is **M**odular, not distributive, and N_5 is **N**on-modular.
Both have 5 elements.

Recognizing distributive lattices

Theorem Let L be a lattice.

1. L is modular iff N_5 is not a sublattice of L
2. L is distributive iff neither M_5, N_5 is a sublattice of L

Proof The "⇒" direction of each is obvious. For 1 "⇐" if L is not modular, there are $x < z$ with $x \vee (y \wedge z) < (x \vee y) \wedge (x \vee z)$ (why?) Then the following is a sublattice of L.



Exercise

Give the details that the figure on the previous page is a sublattice.

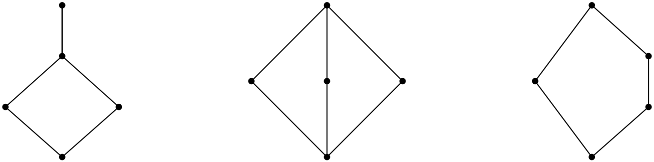
Do the 2 "$\Leftarrow$" direction.

The lattice N_5 is "projective" in lattices, meaning that if L is a lattice and $f : L \to N_5$ is an onto lattice homomorphism, then there is a one-one lattice homomorphism $g : N_5 \to L$ with $f \circ g = id$.

Complements

Definition Elements x, y of a bounded lattice L are complements if $x \wedge y = 0$ and $x \vee y = 1$.

In general, an element might have no complements, or many.



Complements

Theorem In a bounded distributive lattice, an element has at most one complement.

Pf Suppose y, z are complements of x. Then

$$y = y \wedge (x \vee z) = (y \wedge x) \vee (y \wedge z) = y \wedge z$$

$$z = z \wedge (x \vee y) = (z \wedge x) \vee (z \wedge y) = y \wedge z$$

Surprisingly, a finite lattice where each element has exactly one complement is distributive! But not so in the infinite case.

Boolean algebras

Definition A Boolean algebra $(B, \wedge, \vee, ', 0, 1)$ is an algebra of type $2, 2, 1, 0, 0$ where

1. B is abounded distributive lattice
2. x' is a complement of x for each $x \in B$

Note The difference between a complemented distributive lattice and a Boolean algebra is what we consider to be a subalgebra. A subalgebra of a Boolean algebra must include complements.

Properties of Boolean algebras

Proposition In any Boolean algebra

1. $(x \wedge y)' = x' \vee y'$
2. $(x \vee y)' = x' \wedge y'$
3. $x'' = x$

Note These are called De Morgan's laws.

Exercise Prove these. For (1) show $x \wedge y$ and $x' \vee y'$ are complements.

Complements

Definition For L a lattice and $a, b \in L$ with $a \leq b$ the interval $[a, b]$ is the sublattice of L given by

$$[a, b] = \{x : a \leq x \leq b\}$$

Proposition Each interval $[a, b]$ in a complemented distributive lattice L is complemented with the complement of x being the element $x^{\#}$ given by

$$x^{\#} = (x' \wedge b) \vee a$$

We say that L is relatively complemented when its intervals are complemented. The above result works for modular lattices too.

Complements

Definition For a bounded distributive lattice L, let its center $C(L)$ be the set of all complemented elements of L.

Proposition The center of L is a sublattice of L.

Proposition $c \in C(L)$ gives an isomorphism $\varphi : L \to [0, c] \times [0, c']$ where

$$\varphi(x) = (x \wedge c, x \wedge c')$$

Further, each direct product decomposition of L arises this way.

Pf Define $\psi : [0, c] \times [0, c'] \to L$ by $\psi(p, q) = p \vee q$ and show it is inverse to ϕ. For the further comment, if $L = A \times B$, then $c = (1, 0)$ is in $C(L)$.

Ideals and Filters

Definition An ideal of a lattice L is a subset $I \subseteq L$ where

1. if $y \in I$ and $x \leq y$, then $x \in I$
2. if $x, y \in I$ then $x \vee y \in I$

Definition A filter of a lattice L is a subset $F \subseteq L$ where

1. if $x \in F$ and $x \leq y$, then $y \in F$
2. if $x, y \in F$ then $x \wedge y \in F$

Definition Let $\mathcal{I}(L)$ be the set of ideals of L partially ordered by $\subseteq$.

Ideals and filters

Definition For any L its ideal lattice $\mathcal{I}(L)$ is a complete lattice with meets given by intersections. The join of two ideals I and J is

$$I \vee J = \{x : x \leq a \vee b \text{ for some } a \in I, b \in J\}$$

Pf We check that the intersection of ideals is an ideal. Then the ideals are a closure system, hence a complete lattice. We check that the description above is an ideal, and then must be the smallest ideal containing I, J.

Exercise Do the details.

Ideals and Filters

Definition For a lattice L and $a \in L$, the principal ideal and principal filter generated by a are $\downarrow a = \{x : x \leq a\}$ and $\uparrow a = \{x : a \leq x\}$.

Proposition $\phi(a) = \downarrow a$ is a lattice embedding of L into $\mathcal{I}(L)$.

Pf Clearly $\downarrow a \cap \downarrow b = \downarrow(a \wedge b)$ and $\uparrow a \vee \uparrow b = \uparrow(a \vee b)$

Definition An ideal of L is trivial if it is either empty or all of L.

Ideals and Filters

Proposition If L is distributive, so is $\mathcal{I}(L)$.

Pf Let I, J, K be ideals. Always $I \vee (J \wedge K) \subseteq (I \vee J) \wedge (I \vee K)$.

- If $x \in$ RHS, then
- $x \in I \vee J$ and $x \in I \vee K$
- exist $a_1, a_2 \in I$, $b \in J$, $c \in K$ with $x \leq a_1 \vee b$ and $x \leq a_2 \vee c$
- $a = a_1 \vee a_2 \in I$
- $x \leq (a \vee b) \wedge (a \vee c) = a \vee (b \wedge c)$
- $x \in$ LHS.

Ideals and Filters

A fancier version of this shows ...

Theorem For any lattice L, the ideal lattice $\mathcal{I}(L)$ satisfies exactly the same equations as L.

There are several results constructing $\mathcal{I}(L)$ as a homomorphic image of a subalgebra of an ultrapower of L.

Note These results are for lattice equations. For a Boolean algebra B, its ideal lattice $\mathcal{I}(B)$ need not be complemented.

Ideals and Filters

For certain lattices, ideals play a role similar to that of normal subgroups for groups.

Proposition For I an ideal of a distributive lattice L, there is a congruence θ_I of L where

$$\theta_I = \{(a, b) : a \vee x = b \vee x \text{ for some } x \in I\}$$

If L is sectionally complemented, these are all of its congruences.

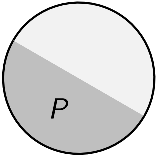
Exercise Prove the first statement using the definition of a congruence. The second is a bit harder.

Prime ideals

Definition An ideal P of a lattice L is prime if for any $a, b \in L$

$$a \wedge b \in P \quad \Rightarrow \quad a \in P \text{ or } b \in P$$

Proposition An ideal P is prime iff $L \setminus P$ is a filter.



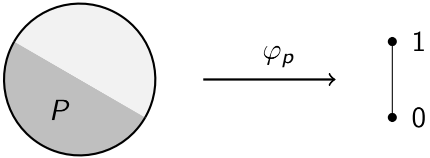
A prime ideal and its complementary filter split the lattice in two.

Prime ideals

Definition Let 2 be the 2-element lattice

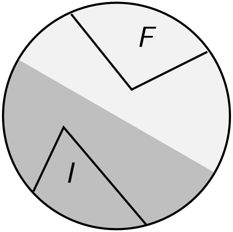
Proposition For P a prime ideal of a distributive lattice D, there is a homomorphism $\varphi_p : D \to 2$ where

$$\varphi_p(x) = \begin{cases} 0 & \text{if } x \in P \\ 1 & \text{if } x \notin P \end{cases}$$



The Prime Ideal Theorem

Theorem Let I be an ideal of a distributive lattice D, and F be a filter of D with $I \cap F = \varnothing$. Then there is a prime ideal P with $I \subseteq P$ and F disjoint from P.



The Prime Ideal Theorem

Pf Let $\mathfrak{X} = \{J : I \subseteq J \in \mathcal{I}(L) \text{ and } J \cap F = \varnothing\}$.

- $\mathfrak{X}$ is non-empty and closed under unions of chains
- By Zorn's Lemma $\mathfrak{X}$ has a maximal member P
- P is an ideal, $I \subseteq P$ and $P \cap F = \varnothing$
- Suppose $a, b \notin P$
- by maximality $\downarrow a \vee P, \downarrow b \vee P \notin \mathfrak{X}$
- exist $x_1, x_2 \in P$ with $a \vee x_1 \in F$ and $b \vee x_2 \in F$
- $x = x_1 \vee x_2 \in P$
- $(a \vee x) \wedge (b \vee x) = (a \wedge b) \vee x \in F$
- since $P \cap F = \varnothing$ then $a \wedge b \notin P$.

The Prime Ideal Theorem

If we consider the case of propositional statements ...

An ideal of statements is a collection that we can sensibly decide to assign value FALSE. A filter is a collection we can sensibly assign the value TRUE.

The prime ideal theorem says that we can do both together while assigning TRUE or FALSE to every proposition in a consistent way! This is a bit remarkable if you think of it.

Consequences of the Prime Ideal Theorem

Definition For a distributive lattice D, let $\beta(D)$ be the set of all non-trivial prime ideals of D.

Definition For D a distributive lattice and $a \in D$ set

$$\beta(a) = \{P \in \beta(D) : a \notin P\}$$

Proposition For D a distributive lattice and $a, b \in D$

1. $\beta(a \wedge b) = \beta(a) \cap \beta(b)$
2. $\beta(a \vee b) = \beta(a) \cup \beta(b)$

Exercise Prove this proposition.

Consequences of the Prime Ideal Theorem

Theorem Any distributive lattice D is isomorphic to a sublattice of the power set $\mathcal{P}(X)$ of the set $X = \beta(D)$.

Pf The map $\beta : D \to \mathcal{P}(X)$ preserves $\wedge$ and $\vee$. It remains to show it is one-one.

- Let $a \neq b$
- Either $a \nleq b$ or $b \nleq a$
- Assume $b \nleq a$
- Then $\downarrow a$ and $\uparrow b$ are a disjoint ideal and filter
- There is a prime ideal P with $\downarrow a \subseteq P$ and $\uparrow b \cap P = \varnothing$
- $a \in P$ and $b \notin P$
- $\beta(a) \neq \beta(b)$

Consequences of the Prime Ideal Theorem

Proposition For any set X, the power set $\mathcal{P}(X)$ is isomorphic to the power 2^X, all functions from X to 2 with pointwise operations.

Pf Let χ be the map from $\mathcal{P}(X)$ to 2^X sending a subset $A \subseteq X$ to its characteristic function χ_A where

$$\chi_A(x) = \begin{cases} 0 & \text{if } x \notin A \\ 1 & \text{if } 1 \in A \end{cases}$$

Theorem Any distributive lattice D is isomorphic to a sublattice of the power 2^X where $X = \beta(D)$.

Prime ideals for Boolean algebras

Exercises Let I be a non-trivial ideal of a Boolean algebra B. Show that the following are equivalent.

1. I is prime
2. For each $x \in B$ exactly one of x, x' belongs to I.
3. I is a maximal non-trivial ideal.

Thus $\beta(x') = \beta(B) \setminus \beta(x)$.

Pf Exercise.

For Boolean algebras of equivalence classes of logical statements, prime ideals correspond to consistent assignments of truth and falsehood to the statements.

Prime ideals for Boolean algebras

Corollary Let B be a Boolean algebra B and $X = \beta(B)$.

1. B is isomorphic to a subalgebra of the power set $\mathcal{P}(X)$.
2. B is isomorphic to a subalgebra of the power 2^X.

Remark This is much like Cayley's theorem for groups that says every group G is isomorphic to a subgroup of the group of permutations of a set.

Remark This theorem was proved independently by Stone and Birkhoff in the 1930's. Stone was an analyst and this result plays a key role in functional analysis.

Consequences of the Prime Ideal Theorem

The result that each distributive lattice and each Boolean algebra is a subalgebra of $\mathcal{P}(X)$ has useful consequences.

Example To show that the following holds in each Boolean algebra

$$x \leq y \Leftrightarrow x \wedge y' = 0$$

its enough to verify it for $\mathcal{P}(X)$ where it is $S \subseteq T \Leftrightarrow S \cap T' = \varnothing$.

The result that each is a subalgebra of 2^X is also useful. We switch focus and look at familiar ideas from logic from this perspective.

Consequences of the Prime Ideal Theorem

Definition A term $t(x_1, \ldots, x_n)$ in the language of lattices is an expression built from $\wedge, \vee$ and the variables $x_1, \ldots, x_n$.

Example $t(x, y, z) = ((x \wedge y) \wedge x) \vee z$

Definition For a term $t(x_1, \ldots, x_n)$ and lattice A, the interpretation of t in A is the function $t^A : A^n \to A$ where $t^A(a_1, \ldots, a_n)$ is the value of the term when its inputs are $a_1, \ldots, a_n$.

Example $t^A(a, b, c) = ((a \wedge b) \wedge a) \vee c$

Truth tables

Definition A truth table is an interpretation of a term in 2.

Example For $t(x,y,z) = ((x \wedge y) \wedge x) \vee z$

x	y	z	$t(x,y,z)$
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

Decidability

Definition Terms s, t are equivalent if their interpretations s^A, t^A are equal in any distributive lattice A. We then write

$$s(x_1, \dots, x_n) \equiv t(x_1, \dots, x_n)$$

Theorem s, t are equivalent iff $s^2 = t^2$.

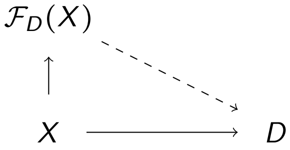
Pf "$\Rightarrow$" is vacuous. "$\Leftarrow$" For A a distributive lattice, $A \leq 2^X$. Evaluating s, t in A is componentwise. As $s = t$ in 2, $s = t$ in A.

Remark One says that the equational theory of distributive lattices is decidable. It all works for Boolean algebras too.

Free Algebras

Definition For a set X, a distributive lattice $\mathcal{F}_D(X)$ is called a free distributive lattice over X if

1. $\mathcal{F}_D(X)$ is generated by X
2. for any distributive lattice D and set mapping $f : X \to D$



there is a homomorphism $\overline{f} : \mathcal{F}_D(X) \to D$ extending f.

Free Algebras

This is a key notion in algebra, logic, algebraic topology, and computer science. Free groups and free Boolean algebras are defined similarly.

Constructing Free Algebras

Definition Let $\mathcal{T}(X)$ be all terms for distributive lattices whose variables are from X, and let $\equiv$ be the relation of equivalence of terms.

The following theorem from universal algebra holds with obvious modification for groups, rings, lattices, Boolean algebras, and so forth.

Theorem $\mathcal{F}_D(X)$ is equal to $\mathcal{T}(X)/\equiv$.

So $\mathcal{F}_D(X)$, and its Boolean counterpart $\mathcal{F}_B(X)$, are key in logic. They literally are logical propositions modulo logical equivalence.

Free Algebras

For the following, note that the elements of 2^{2^X} are truth tables!

Theorem Let X be a finite set.

1. $\mathcal{F}_D(X)$ is isomorphic to a sublattice of 2^{2^X}
2. $\mathcal{F}_B(X)$ is isomorphic to 2^{2^X}

Pf (1) Define $\varphi : \big(\mathcal{T}(X)/\equiv\big) \longrightarrow 2^{2^X}$ by $\varphi(t/\equiv) = t^2$

(2) Since every truth table can be realized by a Boolean algebra term we have that φ is onto in the Boolean case.

Free Algebras

Corollary A subalgebra of a distributive lattice or Boolean algebra generated by n elements is finite.

Pf Such a subalgebra is generated by n elements, and is therefore a homomorphic image of a free algebra on n generators.

Open problem Give a formula for the cardinality of $\mathcal{F}_D(n)$.

Exercise Give an infinite lattice that is generated by 3 elements.

Infinite Distributive Laws

The distributive laws say

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$
$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$$

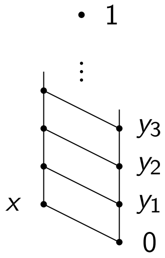
Definition A complete lattice D is infinitely meet distributive if it satisfies (1) and infinitely join distributive if it satisfies (2)

1. $x \wedge \bigvee_I y_i = \bigvee_I (x \wedge y_i)$
2. $x \vee \bigwedge_I y_i = \bigwedge_I (x \vee y_i)$

Exercise Show that any finite lattice, any complete chain, and any power set $\mathcal{P}(X)$ satisfies both.

Infinite Distributive Laws

Example The lattice below is complete and distributive, but does not satisfy the infinite meet distributive law.



Then $x \wedge \bigvee y_i = x \wedge 1 = x$ and $\bigvee (x \wedge y_i) = 0$.

Infinite Distributive Laws

Proposition The lattice $\mathcal{O}(X)$ of open sets of a topological space X satisfies the infinite meet distributive law but not necessarily the infinite join distributive law.

Pf In $\mathcal{O}(X)$ arbitrary joins are $\bigcup$ and finite meets are $\cap$, so the result follows from that for sets.

For the failure of infinite join continuity, consider the topological space $\mathbb{R}$. Set $A = \mathbb{R} \setminus \{0\}$ and $B_n = (-1/n, 1/n)$.

$$A \cup \bigwedge B_n = A \cup \varnothing = A \qquad \bigwedge (A \cup B_n) = \mathbb{R}$$

Here $\bigwedge B_n$ is the interior of its intersection $\{0\}$, which is empty.

Infinite Distributive Laws

There is a stronger version of distributivity involving both infinite joins and meets. To see its nature, lets return to when we were 7.

$$(x_{11} + x_{12}) \cdot (x_{21} + x_{22} + x_{23}) \cdot (x_{31} + x_{32} + x_{33} + x_{34}) \;=$$
$$(x_{11} \cdot x_{21} \cdot x_{31}) + (x_{11} \cdot x_{21} \cdot x_{32}) + \cdots + (x_{12} \cdot x_{23} \cdot x_{34})$$

There are $24 = 2 \times 3 \times 4$ terms here, one for each choice function.

Definition A complete lattice L is completely distributive if

$$\bigwedge_{I} \bigvee_{J_i} x_{ij} \;=\; \bigvee_{\alpha \in \prod J_i} \bigwedge_{I} x_{i,\alpha(i)}$$

Infinite Distributive Laws

Exercise Show that every finite distributive lattice, every complete chain, and every power set $\mathcal{P}(X)$ is completely distributive.

Exercise Show that the complete distributive law implies the infinite join and meet distributive laws.

Complete Boolean Algebras

Proposition In a complete Boolean algebra B

1. $\left(\bigvee x_i\right)' = \bigwedge x_i'$
2. $\left(\bigwedge x_i\right)' = \bigvee x_i'$

This works without completeness if we assume one side exists.

Pf (1) $x_i \leq y \Leftrightarrow y' \leq x_i'$. So if y is the least upper bound of the x_i, then y' is the greatest lower bound of the x_i'. (2) Similar.

Complete Boolean Algebras

Proposition Every complete Boolean algebra satisfies the infinite meet and join distributive laws.

Pf Exercise. Hint: for $\bigvee(x \wedge y_i) = x \wedge \bigvee y_i$ trivially $\textsc{Lhs} \leq \textsc{Rhs}$. For the other way, it is enough to show that $\textsc{Lhs}' \wedge \textsc{Rhs} = 0$. Then use $\textsc{Lhs}' \leq x' \vee y_j'$ for each j to obtain this.

In the final lecture, one more result will be of key importance.

Complete Boolean Algebras

Theorem For a complete Boolean algebra B, these are equivalent.

1. B is atomic
2. B is completely distributive
3. B is isomorphic to a power set $\mathcal{P}(X)$ for some set X.

Pf (Sketch) (1) ⇒ (3) ⇒ (2) are an exercise. To show (2) ⇒ (1) enumerate B as x_i and set $x_{i0} = x_i$ and $x_{i1} = x_i'$.

$$x_j < \bigwedge_I x_{i\alpha(i)} \Rightarrow \alpha(j) = 1 \Rightarrow x_j \leq x_j' \Rightarrow x_j = 0$$

So $\bigwedge_I x_{i\alpha(i)}$ is either 0 or an atom.

$$1 = \bigwedge_I \bigvee_2 x_{ij} = \bigvee_{\alpha \in 2^I} \bigwedge_I x_{i\alpha(i)}$$

Thanks for listening.

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