

Lattice Theory Lecture 3

Duality

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Introduction

Today we consider alternative ways to view distributive lattices and Boolean algebras. These are examples of what are known as dualities.

Dualities in category theory relate one type of mathematical structure to another. They can be key to forming bridges between areas as different as algebra and topology.

Stone duality is a primary example. It predates category theory, and is a terrific guide for further study of other dualities.

Various Dualities in Lattice Theory

Birkhoff Duality: finite distributive lattices \leftrightarrow finite posets

Stone Duality: Boolean algebras \leftrightarrow certain topological spaces

Priestley Duality: distributive lattices \leftrightarrow certain ordered top spaces

Esakia Duality: Heyting algebras \leftrightarrow certain ordered top spaces

We consider the first two. In both cases, prime ideals provide our key tool.

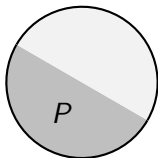
Prime Ideals

Recap ...

Definition For D a distributive lattice, $P \subseteq D$ is a prime ideal if

1. $x \leq y$ and $y \in P \Rightarrow x \in P$
2. $x, y \in P \Rightarrow x \vee y \in P$
3. $x \wedge y \in P \Rightarrow x \in P$ or $y \in P$ or both.

P is called trivial if it is equal to either \emptyset or all of D .



Prime Ideals for Finite Distributive Lattices

Proposition In a finite lattice L , the non-empty ideals of L are exactly the principal ideals $\downarrow a = \{x : x \leq a\}$ where $a \in L$.

Pf Since ideals are closed under binary joins, if I is a non-empty ideal in a finite lattice, the join of all of its elements again lies in I , and is clearly the largest element of I .

Which elements a have $\downarrow a$ a prime ideal?

Prime Ideals for Finite Distributive Lattices

Definition Let a be an element of a lattice D .

1. a is meet irreducible if $x \wedge y = a \Rightarrow x = a$ or $y = a$
2. a is meet prime if $x \wedge y \leq a \Rightarrow x \leq a$ or $y \leq a$

Proposition In a distributive lattice, these are equivalent.

1. a is meet irreducible
2. a is meet prime
3. $\downarrow a$ is a prime ideal

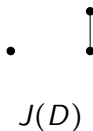
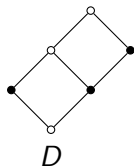
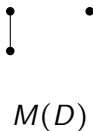
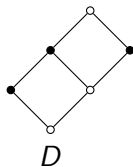
Pf Exercise.

Prime Ideals for Finite Distributive Lattices

Definition For a lattice D set

1. $M(D) = \{a : a \text{ is meet irreducible and } a \neq 1\}$
2. $J(D) = \{a : a \text{ is join irreducible and } a \neq 0\}$

We consider both as posets partially ordered by the order of D .

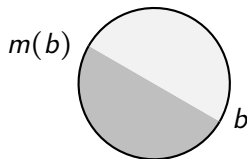
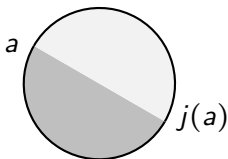


Prime Ideals for Finite Distributive Lattices

Proposition There are mutually inverse order isomorphisms

$$M(D) \begin{array}{c} \xrightarrow{j} \\ \xleftarrow{m} \end{array} J(D)$$

Pf Obvious from the connection between prime ideals and prime filters.



Object Level Birkhoff Duality

Definition For a poset P , a set $S \subseteq P$ is a downset if

$$a \leq b \text{ and } b \in S \Rightarrow a \in S$$

Let $\mathcal{D}(P)$ be all downsets of P partially ordered by set inclusion.

Proposition For P a poset, its downsets $\mathcal{D}(P)$ are a complete, completely distributive lattice.

Pf It is easy to see that the intersection and union of an arbitrary collection of downsets is a downset. So $\mathcal{D}(P)$ is a subset of the power set $\mathcal{P}(P)$ closed under arbitrary intersections and unions.

Object Level Birkhoff Duality

Definition Let $FDist$ be the class of finite distributive lattices and $FPos$ be the class of finite posets. We then have maps

$$FDist \begin{array}{c} \xrightarrow{J} \\ \xleftarrow{\mathcal{D}} \end{array} FPos$$

Here $J(D)$ is the poset of join irreducibles of the finite distributive lattice D and $\mathcal{D}(P)$ is the distributive lattice of downsets of P .

Object Level Birkhoff Duality

Theorem For a finite distributive lattice D , and finite poset P there are isomorphisms $\phi : D \rightarrow \mathcal{D}(J(D))$ and $\psi : P \rightarrow J(\mathcal{D}(P))$ where

1. $\phi(x) = \{a \in J(D) : a \leq x\}$
2. $\psi(a) = \downarrow a$

Pf (1) This is surely a downset. Since each element is the join of the join irreducibles beneath it, $x \leq y \Leftrightarrow \phi(x) \subseteq \phi(y)$. To see it is onto, for a downset A let $x = \bigvee A$. Since members of $J(D)$ are join prime, if $a \in \phi(x)$ then $a \leq b$ for some $b \in A$. So $\phi(x) = A$.

Object Level Birkhoff Duality

Theorem For a finite distributive lattice D , and finite poset P there are isomorphisms $\phi : D \rightarrow \mathcal{D}(J(D))$ and $\psi : P \rightarrow J(\mathcal{D}(P))$ where

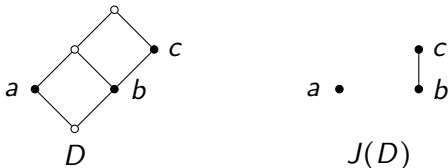
1. $\phi(x) = \{a \in J(D) : a \leq x\}$
2. $\psi(a) = \downarrow a$

Pf (2) Surely $\downarrow a$ is a downset, and since a union of downsets cannot contain $\downarrow a$ without one of them having a in it, $\downarrow a$ is join irreducible in $\mathcal{D}(P)$. Its easy to see that these are the only join irreducibles. Then $a \leq b \Leftrightarrow \downarrow a \subseteq \downarrow b$ gives the rest.

Object Level Birkhoff Duality

So there is a complete correspondence between finite distributive lattices and finite posets.

Example



Then $J(D)$ has six downsets.

$$\emptyset \quad \{a\} \quad \{b\} \quad \{a, b\} \quad \{b, c\} \quad \{a, b, c\}$$

And $\mathcal{D}(J(D)) \simeq D$

Full Birkhoff Duality

Definition A category \mathcal{C} consists of

1. a collection of objects A, B, C, \dots
2. a collection of morphisms or arrows $f : A \rightarrow B$ between objects
3. a rule to compose arrows $g \circ f$ when $f : A \rightarrow B$ and $g : B \rightarrow C$
4. for each object A , an identity morphism $1_A : A \rightarrow A$

We require that composition is associative when defined, and that if $f : A \rightarrow B$ then $1_B \circ f = f = f \circ 1_A$.

Full Birkhoff Duality

Definition Consider the categories

1. $FDist$ = finite distributive lattices + bounded lattice homo's
2. $FPos$ = finite posets + order preserving maps

We extend the correspondence between finite distributive lattices and finite posets to include also the morphisms between objects of the same kind. This is full Birkhoff duality.

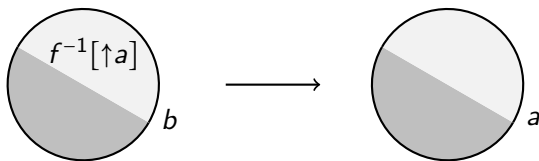
Full Birkhoff Duality

Proposition Let $f : D \rightarrow E$ be a homomorphism between finite distributive lattices and $a \in J(E)$

1. $f^{-1}[\uparrow a]$ is a prime filter of D
2. there is a least element $b \in D$ mapped above a and $b \in J(D)$

This gives a map $J(f) : J(E) \rightarrow J(D)$

Pf



(1) is an exercise and (2) follows from (1).

Full Birkhoff Duality

Proposition Given an order preserving map $h : P \rightarrow Q$ between finite posets, there is a distributive lattice homomorphism $\mathcal{D}(h) : \mathcal{D}(Q) \rightarrow \mathcal{D}(P)$ given by

$$\mathcal{D}(h)(A) = h^{-1}[A]$$

Pf The preimage of a downset is a downset. Preimages preserve unions and intersections, and these are the lattice operations in the lattices $\mathcal{D}(P)$ and $\mathcal{D}(Q)$ of downsets.

Full Birkhoff Duality

We extended our correspondence on objects to morphisms.

$$FDist \begin{array}{c} \xrightarrow{J} \\ \xleftarrow{\mathcal{D}} \end{array} FPos$$

If $f : D \rightarrow E$ in $FDist$ then $J(f) : J(E) \rightarrow J(D)$ in $FPos$

If $h : P \rightarrow Q$ in $FPos$ then $\mathcal{D}(h) : \mathcal{D}(Q) \rightarrow \mathcal{D}(P)$ in $FDist$

Theorem J and \mathcal{D} give a dual equivalence, or duality, between the categories $FDist$ and $FPos$.

Dualities

We won't give the formal definition of a dual equivalence or duality here. Basically, it means that there is a correspondence for objects, and also a correspondence for morphisms between objects. It is a dual equivalence because we switch the direction of the arrows.

In practice, it means that problems you can formulate about finite distributive lattices and the homomorphisms between them can be reformulated (and solved!) as problems about finite posets and order preserving maps. And vice versa.

Example Is there a finite distributive lattice whose automorphism group is \mathbb{Z}_5 ? Its probably easier to solve this for a finite poset.

Stone Duality

Definition Let Boo be the category of Boolean algebras and the homomorphisms between them.

Stone duality is between Boo and a category $BSpace$ of certain topological spaces and the continuous maps between them.

Review of Topology

Definition A topological space (X, τ) is a set X with a collection τ of subsets of X that we call open sets such that

1. \emptyset, X are open
2. the intersection of two open sets is open
3. the union of arbitrarily many open sets is open

A set C whose complement is open is called closed, and a set that is both open and closed is called clopen.

Proposition The clopen sets $\text{Clopen}(X)$ of a topological space X form a Boolean subalgebra of the power set $\mathcal{P}(X)$.

Review of Topology

Proposition Let X be a set and $\mathcal{R} \subseteq \mathcal{P}(X)$ be closed under finite unions and intersections. Then there is a topology on X given by

$$\tau = \{A : A \text{ is an arbitrary union of members of } \mathcal{R}\}$$

Recall, for a Boolean algebra B , and $a \in B$

$$\beta(B) = \{P : P \text{ is a non-trivial prime ideal}\}$$

$$\beta(a) = \{P : a \notin P\}$$

Definition For a Boolean algebra B its Stone space is (X, τ) where $X = \beta(B)$ and τ is determined by $\mathcal{R} = \{\beta(a) : a \in B\}$.

Outline of Stone Duality

We have means to associate to a Boolean algebra a topological space and conversely.

- Given B form its Stone space (X, τ)
- Given (X, τ) take $\text{Clopen}(X)$

For an arbitrary topological space (X, τ) , such as the real line, we won't realize it as the Stone space of $\text{Clopen}(X)$. So we must narrow the class of topological spaces we consider.

Stone Spaces

Definition For a top space (X, τ) a subset $K \subseteq X$ is compact if whenever K is contained in the union $\bigcup_i A_i$ of a family of open sets, the union of finitely many of these open sets contains K .

Definition A top space (X, τ) is a Boolean space if

1. X is compact
2. Each open set is a union of clopen sets
3. For any $x \neq y$ in X there is a clopen A with $x \in A, y \notin A$

If you know the words, (X, τ) is a compact zero-dimensional Hausdorff space.

Stone Spaces

Proposition For a Boolean algebra B , its Stone space (X, τ) is a Boolean space.

Proof Since $X \setminus \beta(a) = \beta(a')$ each $\beta(a)$ is clopen, and since τ is generated by such sets, each open set is a union of clopen ones. So X is zero-dimensional.

Elements of X are prime ideals. If $P \neq Q \in X$ there is a with $a \in Q$ and $a \notin P$. Then $P \in \beta(a)$ and $Q \notin \beta(a)$. So its Hausdorff.

Stone Spaces

Pf For compactness, suppose $X = \bigcup_I A_i$ is a union of open sets. Since each open is a union of ones from \mathcal{R} , we may assume consider just the case when $A_i \in \mathcal{R}$, so that $X = \bigcup_I \beta(a_i)$. Let

$J =$ the ideal generated by the a_i where $i \in I$

If the ideal J does not contain 1, the prime ideal theorem applied to J and $\{1\}$ says there is a non-trivial prime ideal P containing J . Then $P \not\subseteq \beta(a_i)$ for each $i \in I$ a contradiction.

Since $1 \in J$ there are a_{i_1}, \dots, a_{i_n} with $1 = a_{i_1} \vee \dots \vee a_{i_n}$. Then

$$X = \beta(1) = \beta(a_{i_1}) \cup \dots \cup \beta(a_{i_n})$$

Stone Spaces

Proposition For a Boolean algebra B with Stone space (X, τ) , B is isomorphic to $Clopen(X)$.

Pf We have $\beta : B \rightarrow Clopen(X)$ preserves finite meets, finite joins, and complements, so is a Boolean algebra homomorphism. If $a \neq b$ then there is a prime ideal containing one of a, b and not the other, so β is one-one.

For onto, let $A \in Clopen(X)$. It is known that closed subspaces of a compact space are compact, so A is compact. Since A is open it is a union of sets of the form $\beta(a)$, hence by compactness is a finite union of such sets. Thus

$$A = \beta(a_1) \cup \dots \cup \beta(a_n) = \beta(a_1 \vee \dots \vee a_n)$$

Stone Spaces

Definition A map $f : X \rightarrow Y$ between topological spaces is continuous if the preimage of each open set in Y is open in X .

Definition A bijection $f : X \rightarrow Y$ between topological spaces is a homeomorphism if f and its inverse f^{-1} are continuous.

Stone Spaces

Proposition Let (X, τ) be a Boolean space and $x \in X$. Set

$$P_x = \{A \in Clopen(X) : x \notin A\}$$

Then P_x is a prime ideal of the Boolean algebra $Clopen(X)$ and the map $\Phi : X \rightarrow \beta(Clopen(X))$ with $\Phi(x) = P_x$ is a homeomorphism.

Pf Exercise.

Stone Duality

At this point we have an object level correspondence

$$Boo \begin{array}{c} \xrightarrow{\beta} \\ \xleftarrow{Clopen} \end{array} BSpace$$

We extend it to a full duality. Here *Boo* is the category of Boolean algebras and their homomorphisms, and *BSpace* is the category of Boolean spaces and the continuous maps between them.

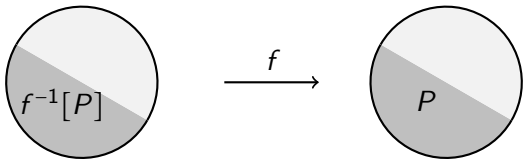
Stone Duality

Proposition Let $f : B \rightarrow C$ be a Boolean algebra homomorphism and P be a prime ideal of C . Then $f^{-1}[P]$ is a prime ideal of B . So there is a map

$$\beta(f) : \beta(C) \rightarrow \beta(B) \quad \text{given by} \quad \beta(f)(P) = f^{-1}[P]$$

This map $\beta(f)$ is continuous.

Pf



As an exercise, show $\beta(f)^{-1}[\beta(a)] = \beta(f(a))$ to see that Φ is continuous.

Stone Duality

Proposition If $h: X \rightarrow Y$ is a continuous map, then for each clopen $A \subseteq Y$ we have $h^{-1}[A]$ is clopen in X . So there is a map

$$\text{Clopen}(h) : \text{Clopen}(Y) \rightarrow \text{Clopen}(X) \quad \text{taking } A \text{ to } h^{-1}[A]$$

This map is a Boolean algebra homomorphism.

Pf That it is a Boolean algebra homomorphism follows since inverse image preserves intersection, union, and complementation.

Stone Duality

Theorem There is a dual equivalence

$$Boo \begin{array}{c} \xrightarrow{\beta} \\ \xleftarrow{Clopen} \end{array} BSpace$$

Pf We leave this to those who know the precise definition of a duality. But its not difficult with what we now have.

We spend the remainder of this lecture discussing aspects of Stone duality. Its importance in many branches of mathematics is hard to overstate. This includes modal logic, but also algebra and functional analysis.

Stone Duality

Recall, a coatom is an element that is covered by 1.

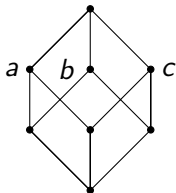
Proposition Let B be a finite Boolean algebra with n coatoms.

1. the prime ideals of B are the $\downarrow c$ where c is a coatom of B
2. the Stone space $\beta(B)$ is an n -element discrete space
3. B is isomorphic to the power set $\mathcal{P}(\{1, \dots, n\})$

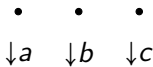
Pf (1) Each ideal of B is of the form $\downarrow c$ for some $c \in B$. Since the interval $[c, 1]$ of B is complemented, for c to be meet irreducible $[c, 1]$ must have 2 elements. (2) Every finite Hausdorff space is discrete. (3) Every subset of a discrete space is clopen.

Stone Duality

Example



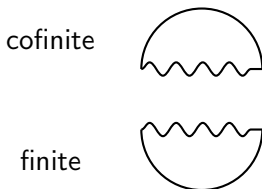
B



$\beta(B)$

Stone Duality

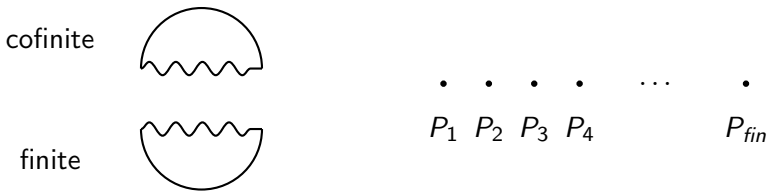
Example A subset A of \mathbb{N} is cofinite if its complement $\mathbb{N} \setminus A$ is finite. Let B be the set of all $A \subseteq \mathbb{N}$ that are finite or cofinite.



It has a coatom $\mathbb{N} \setminus \{n\}$ for each n . Each coatom again gives a prime ideal P_n . There is one more prime ideal, the whole of the bottom part “finite” is a prime ideal P_{fin} .

Stone Duality

What is the Stone space of the finite/cofinite Boolean algebra?



The clopen sets $\beta(A)$ of the Stone space are as follows:

$$\beta(A) = \begin{cases} \{P_n : n \in A\} & \text{if } A \text{ is finite} \\ \{P_n : n \notin A\} \cup \{P_{fin}\} & \text{if } A \text{ is cofinite} \end{cases}$$

This is the one-point compactification of \mathbb{N} !

Stone Duality

A natural question is to describe the Stone space of $B = \mathcal{P}(\mathbb{N})$. It turns out that this is the Stone Cech compactification of \mathbb{N} with the discrete topology.

As far as actually describing this Stone space in a concrete way, this is one of the more difficult problems you can imagine!

A Dictionary to Translate

Atoms Atoms $a \in B \leftrightarrow$ clopen singletons $\{x\}$ in Stone space

The only prime ideal P with $a \notin P$ is $\downarrow a'$, so $\beta(a) = \{\downarrow a'\}$

Ideals Ideals $I \subseteq B \leftrightarrow$ open sets $A \subseteq X$

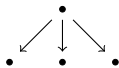
Given I then $A = \bigcup\{\beta(a) : a \in I\}$ is open.

Filters Filters $F \subseteq B \leftrightarrow$ closed sets $K \subseteq X$

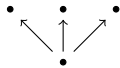
Given F then $\bigcap\{\beta(a) : a \in F\}$ is closed.

A Dictionary to Translate

Since duality turns arrows around, products in one category turn to coproducts in the other.



Product



Coproduct

Products in Boolean algebras are easy, and finite coproducts Boolean spaces are easy — they are disjoint unions.

Example The Stone space of 2 is the one-element space. The Stone space of 2^n is the disjoint union of n one-element spaces, hence an n -element space.

Example $2^{\mathbb{N}}$ is the power set $\mathcal{P}(\mathbb{N})$. Its Stone space is the Stone-Cech compactification of the union of \mathbb{N} 1-element spaces.

A Dictionary to Translate

Products in Boolean spaces are the usual product of topological spaces. Coproducts in Boolean algebras pose a challenge.

Free algebras

The free Boolean algebra $\mathcal{F}_B(X)$ on a set X of generators is the coproduct of X copies of the free Boolean algebra $\mathcal{F}_B(1)$ on one generator.

We approach this using Stone spaces ...

A Dictionary to Translate

- Note $\mathcal{F}_B(1)$ is a 4-element Boolean algebra $\{0, a, a', 1\}$ so its dual space is a 2-element discrete space.
- So the dual space of $\mathcal{F}_B(X)$ is the space 2^X .
- In the finite case, $\mathcal{F}_B(n)$ is 2^{2^n} from our discussion of truth tables. Its dual space is 2^n as it should be.
- In the infinite case, $\mathcal{F}_B(\mathbb{N})$ can be approached algebraically, but dually its Stone space is $2^{\mathbb{N}}$ — the Cantor space!

A Dictionary to Translate

Completeness

We can consider completeness of a Boolean algebra B in terms of its Stone space X .

Ideals I of B correspond to open sets A of X . Having B complete means each ideal I has a least upper bound a . This corresponds to having the closure of an open set A of X be a clopen set $\beta(a)$.

We return to completeness in Lecture 5.

Thanks for listening.

Papers at www.math.nmsu.edu/~jharding