Lattice Theory Lecture 4

Non-distributive lattices

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Introduction

Here we mostly consider modular lattices, but also make some comments on free lattices.

Understanding free lattices is key to many aspects of lattice theory including projectives.

We won't have time for these other aspects, but will describe the free lattice $\mathcal{F}_L(X)$ on a set X and show that the class *Lat* of all lattices has decidable equational theory.

Free Lattices

As with free distributive lattices and free Boolean algebras, the free lattice $\mathcal{F}_L(X)$ is given by equivalence classes of terms $\mathcal{T}(X)/\equiv$. Here equivalence

$$(x \wedge y) \wedge (x \vee x) \;\equiv\; x \wedge y$$

means that the terms evaluate to the same in every lattice.

Unlike the distributive and Boolean case there is no test simple algebra such as 2, so no analog of the method of truth tables. We need a different way to describe $\equiv$.

Free Lattices

Definition Let $\sqsubseteq$ be the smallest relation on the set of lattice terms $\mathcal{T}(X)$ using variables from X such that

1. $x \sqsubseteq x$ for each $x \in X$
2. if $p \sqsubseteq s$ then $p \wedge q \sqsubseteq s$
3. if $q \sqsubseteq s$ then $p \wedge q \sqsubseteq s$
4. if $p \sqsubseteq s$ then $p \sqsubseteq s \vee t$
5. if $p \sqsubseteq t$ then $p \sqsubseteq s \vee t$

These are called Whitman's conditions.

Note, one can construct $\sqsubseteq$ recursively on the complexity of terms, so can effectively tell whether $p \sqsubseteq q$.

Free Lattices

Proposition The relation $\sqsubseteq$ is reflexive and transitive, hence is a quasi-order on $\mathcal{T}(X)$.

Pf A bit of an onerous inductive proof. Try it!

From Lecture 1, a quasi-order $\sqsubseteq$ on a set P has an associated equivalence relation θ on P given by $p\,\theta q$ iff $p \sqsubseteq q$ and $q \sqsubseteq p$, and then a partial order $\leq$ on P/θ given by $p/\theta \leq q/\theta$ iff $p \sqsubseteq q$.

Definition Let θ and $\leq$ be the equivalence relation and partial ordering for $\mathcal{T}(X)$ given by $\sqsubseteq$.

Free Lattices

Theorem (Whitman) Lattice terms s, t evaluate to the same in every lattice iff $s\theta t$ in the relation θ given by the quasi-order $\sqsubseteq$.

Pf "$\Leftarrow$" The conditions for $\sqsubseteq$ are very sparse, so if $s \sqsubseteq t$ then in any lattice L we have $s^L \leq t^L$.

"$\Rightarrow$" We need that if $s \not\theta\, t$ then $s^L \neq t^L$ in some lattice L. To do this we show that $\mathcal{T}(X)/\theta$ is a lattice! This is an even nastier inductive proof to show that if $p, q \sqsubseteq s$, then $p \vee q \sqsubseteq s$.

Corollary The free lattice on a set X is $\mathcal{T}(X)/\theta$

Free Lattices

Remarks

- The free lattice on 2 generators is a 4-element lattice.
- The free lattice on 3 generators is infinite
- The free lattice on 3 generators contains a sublattice isomorphic to the free lattice on countably many generators.
- The free lattice $\mathcal{F}_L(X)$ has no uncountable chains

Exercise Is N_5 a sublattice of $\mathcal{F}_L(3)$? Much harder — is M_5?

Modular Lattices

Here we try to explain why people would be interested in modular lattices.

We begin with an example, but first need a definition.

Definition An element c in a complete lattice L is compact if whenever $c \leq \bigvee S$ then $c \leq \bigvee S'$ for some finite $S' \subseteq S$. L is algebraic if each element is the join of compact ones.

Proposition A closure system $\mathcal{C}$ forms an algebraic lattice if it is also closed under unions of non-empty chains.

Vector Spaces

Definition For V a vector space over a field F, let $\mathcal{S}(V)$ be the collection of subspaces of V partially ordered by set inclusion.

Proposition For a vector space V, the poset $\mathcal{S}(V)$ is

1. a complete lattice
2. atomistic
3. algebraic
4. complemented
5. directly irreducible
6. modular

Call a lattice with these properties geomodular.

Pf 1. The intersection of subspaces is a subspace. So $\mathcal{S}(V)$ is a complete lattice where meets are intersections. The join of two subspaces is

$$S \vee T = \{s + t : s \in S, t \in T\}$$

2. The atoms are 1-dimensional subspaces. Each subspace is the union of these, hence its atomistic.

3. Algebraic since closed under unions of chains.

4. Complemented: Given a subspace S find a basis $\mathcal{X}$ of S. Extend this to a basis $\mathcal{X} \cup \mathcal{Y}$ of V. Then for $T = \text{span } \mathcal{Y}$ we have $S \cap T = \{0\}$ and $S \vee T = V$. So T is a complement of S.

5. If $\mathcal{S}(V) \simeq L_1 \times L_2$ then there would be an element $(1,0)$ in it that is not a bound and has exactly one complement. A subspace S that is not a bound always has more than one complement since its basis $\mathcal{X}$ can be extended to a basis of V in many ways.

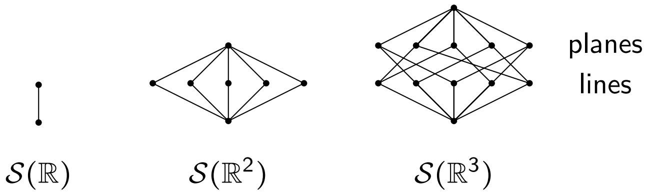
6. Let S, T, U be subspaces with $S \subseteq U$. We need

$$S \vee (T \cap U) = (S \vee T) \cap U$$

That $\subseteq$ holds is trivial. For $\supseteq$ suppose $x \in \mathrm{RHS}$. Then $x \in U$ and $x = s + t$ for some $s \in S, t \in T$. Then $t = x - s$ and since $x, s \in U$ we have $t \in U$. So $x = s + t$ where $s \in S$, $t \in T \cap U$ gives $x \in \mathrm{LHS}$.

Vector spaces

Example Lets consider subspaces of $\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{R}^3$.



Both $\mathcal{S}(\mathbb{R}^2)$ and $\mathcal{S}(\mathbb{R}^3)$ are infinite, we show only part.

$\mathcal{S}(\mathbb{R}^2)$: $\{0\}$, lines through the origin, $\mathbb{R}^2$

$\mathcal{S}(\mathbb{R}^3)$: $\{0\}$, lines through the origin, planes through the origin, $\mathbb{R}^3$

Dimension

Subspaces of a vector space and modular lattices share several key properties. Finite-dimensional vector spaces have a dimension function. There is an analog for modular lattices of finite height.

Proposition Let M be a modular lattice where every chain is finite. Then for each $x \in M$ every maximal chain in $[0, x]$ has the same number of elements. We define $h : M \to \mathbb{N}$ by

$$h(x) = \text{the length of a maximal chain in } [0, x] - 1$$

Proposition In the appropriate setting

1. $\dim S + \dim T = \dim (S \vee T) - \dim (S \wedge T)$
2. $h(x) + h(y) = h(x \vee y) - h(x \wedge y)$

Links to Geometry

Suppose we have a 3-dimensional vector space V. If we think of the 1-dimensional subspaces as points of a geometry, and the 2-dimensional subspaces as lines of a geometry, we have

- any two points lie on a unique line
- any two lines intersect in a unique point

The second item is because

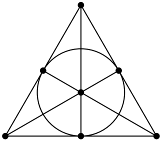
$$\dim\,(S \wedge T) = \dim\,S + \dim\,T - \dim\,(S \vee T) = 2 + 2 - 3$$

This is the idea behind a projective plane. The exact definition of a projective geometry will account for higher dimensions as well.

Projective Geometry

Example Consider the subspaces of $(\mathbb{Z}_2)^3$, a three-dimensional vector space over the 2-element field. We draw 1-dimensional subspaces as points, and 2-dimensional ones as lines.

Fano plane



Exercise If the corners of the triangle are the 1-dimensional subspaces $\langle(1,0,0)\rangle$, $\langle(0,1,0)\rangle$, $\langle(0,0,1)\rangle$ label the rest. Hint, the middle is $\langle(1,1,1)\rangle$ and the line at bottom is the plane $y = 0$.

Projective Geometry

Definition A projective geometry $\mathcal{G} = (\mathbb{P}, \mathbb{L}, \mathbb{I})$ consists of a set $\mathbb{P}$ of points, a set $\mathbb{L}$ of lines, and a relation $\mathbb{I} \subseteq \mathbb{P} \times \mathbb{L}$ where

1. any two distinct points lie on a unique line
2. $pq'r$, $p'qr$ collinear $\Rightarrow$ exists r' with pqr', $p'q'r'$ collinear

Item 2 says that coplanar lines intersect.

Definition If $\mathcal{G}$ is a projective geometry, then a subset $S \subseteq \mathbb{P}$ is a subspace of $\mathcal{G}$ if $p, q \in S$ and pqr collinear $\Rightarrow r \in S$.

The Connections

Proposition Let V be a vector space over a division ring D. Set

1. $\mathbb{P}$ = the one-dimensional subspaces of V
2. $\mathbb{L}$ = the two-dimensional subspaces of V
3. $p \, \mathbb{I} \, L \Leftrightarrow p$ is contained in L

Then $\mathcal{G} = (\mathbb{P}, \mathbb{L}, \mathbb{I})$ is a projective geometry.

Pf Two points (1-dim subspaces) span a line (2-dim subspace). Let S, T be lines (2-dim subspaces). Having them coplanar means $\dim (S \vee T) = 3$. Then

$$\dim (S \cap T) = \dim S + \dim T - \dim (S \vee T) = 2 + 2 - 3 = 1$$

So coplanar lines meet in a point.

The Connections

Proposition For a geomodular lattice M, let

1. $\mathbb{P}$ = atoms p of M
2. $\mathbb{L}$ = the elements L that cover atoms (height 2)
3. $p \mathbb{I} L \Leftrightarrow p \leq L$

Then $\mathcal{G} = (\mathbb{P}, \mathbb{L}, \mathbb{I})$ is a projective geometry.

Pf This is easy using the height function h on the elements of M of finite height.

The Connections

Proposition For a projective geometry $\mathcal{G}$, its subspaces $\mathcal{S}(\mathcal{G})$, partially ordered by set inclusion form a geomodular lattice.

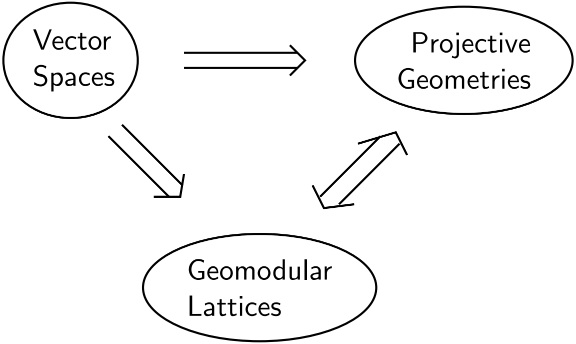
Pf The intersection of subspaces is a subspace and the union of a chain of subspaces is a subspace. This shows it is complete and algebraic. The atoms are the singletons $\{p\}$ for a point p, so it is atomistic. The other items require a description of the join of two subspaces S, T when neither contains the other and are similar to the proofs for $\mathcal{S}(V)$.

$$S \vee T = \{r : \text{ there are } s \in S \text{ and } t \in T \text{ with } rst \text{ collinear}\}$$

Exercise complete the proof.

The Connections

So far we have shown the following relationships.



The Connections

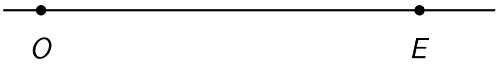
Completing the equivalences doesn't quite work. We require an additional geometric condition on the projective geometry known as Desargue's law (1600's!). Its correspondent for geomodular lattices becomes an equation known as the Arguesian equation.

Assuming this, we can construct from a Desarguesian projective geometry $\mathcal{G}$ a division ring D and vector space V over D with $\mathcal{G}$ the projective geometry built from V.

The ideas are 1000's of years old. We briefly describe them.

Constructing a Vector Space from $\mathcal{G}$

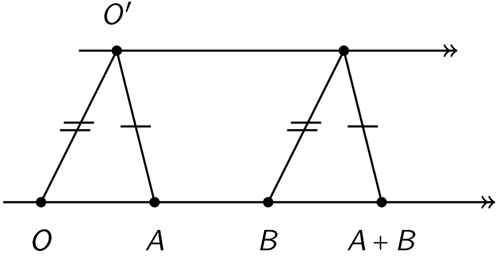
Step 1 Choose two distinct points O and E and let D be the points on the line OE.



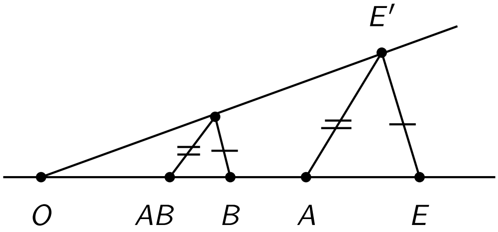
Eventually D will be the division ring and O, E will be its $0, 1$.

There are no parallel lines in a projective plane, but if we pick a distinguished line and call it the "line at infinity L_∞", then we call lines parallel if their intersection is on L_∞.

Step 2 We add points A, B as follows.



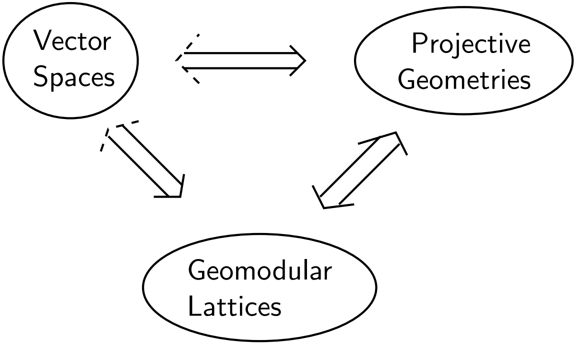
Step 3 We multiply points as follows.



There is a lot to show that this all works, but it does if we have Desargue's law. Multiplication being commutative is equivalent to Pappus' law, another geometric condition.

The Connections

We wind up with nearly an object level equivalence. If you want to look further at moving towards a full categorical treatment, see the book by Faure and Frölicher.



Modular Lattices

Modular lattices play important roles in algebra, geometry, and combinatorics. It would be good to have a nice theory of them as with distributive lattices. But there are problems.

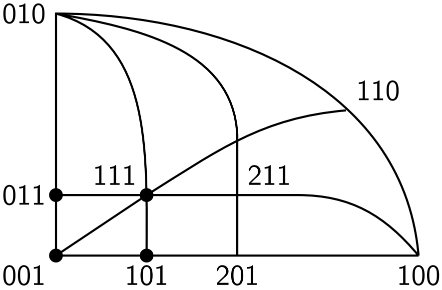
Theorem The free modular lattice on 3 generators has 28 elements, the free one on 4 or more generators is infinite.

Theorem The equational theory of modular lattices is undecidable.

Lets see how a 4-generated modular lattice can be infinite. It goes a long way to making the connection between vector spaces, modular lattices, and geometry concrete.

Proposition The lattice $\mathcal{S}(\mathbb{Q}^3)$ is generated by 4 elements.

Pf



Get $(001 \vee 101) \wedge (011 \vee 111) = 100$, then $010, 110$ similar

Get $(101 \vee 110) \wedge (011 \vee 100) = 211$

Exercise Describe how you get $(-1, 1/3, 1)$

Past Modular Lattices

In a somewhat mysterious way, modularity + complementation is much stronger than just modularity.

Theorem (Frink) Every complemented modular lattice M can be embedded into a possibly reducible geomodular one.

Pf Difficult. One constructs a geometry from the filters of M.

Corollary There is a modular lattice that cannot be embedded into a complemented modular lattice.

Pf Take a non-Desarguesian irreducible modular lattice with height > 3. Every geomodular lattice of height > 3 is Desarguesian.

Past Modular Lattices

Exercise Show that every lattice can be embedded into a complemented lattice. Hint: if it is bounded, you only need to include one more element!

Exercise Show that every distributive lattice can be embedded into a Boolean algebra. Hint: Use theorems we learned.

Past Modular Lattices

Subspace lattices $\mathcal{S}(V)$ for a vector space V, are complemented. But often there is even more — a distinguished complement.

Definition An real inner product on a real vector space V is a map $\langle\cdot,\cdot\rangle : V^2 \to \mathbb{R}$ where

1. $\langle x, y\rangle = \langle y, x\rangle$
2. $\langle \lambda x, y\rangle = \lambda \langle x, t\rangle$
3. $\langle x + y, z\rangle = \langle x, z\rangle + \langle y + z\rangle$
4. $\langle x, x\rangle \geq 0$ with equality iff $x = 0$.

All we will say works for complex inner products as well, we use real ones for simplicity here.

Past Modular Lattices

Example On $\mathbb{R}^n$ the familiar "dot product" is an inner product given by

$$(x_1, \ldots, x_n) \cdot (y_1, \ldots, y_n) = x_1 y_1 + \cdots + x_n y_n$$

Example On the vector space $L^2(X, \mu)$ of a.e. classes of square integrable functions on a measure space (X, μ), there is an inner product given by

$$\langle f, g \rangle = \int f(x) g(x) \, d\mu$$

This is an example of a Hilbert space, a primary ingredient in modern physics and differential equations.

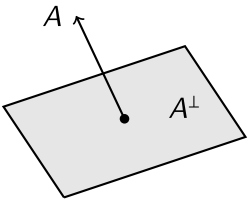
Past Modular Lattices

Definition For a subspace A of an inner product space V define

$$A^{\perp} = \{v : \langle a, v\rangle = 0 \text{ for all } a \in A\}$$

It is easy to see that $A^{\perp}$ is a subspace of V. So $\perp$ is a unary operation on $\mathcal{S}(V)$. We call $A^{\perp}$ the orthogonal subspace of A.

Example



In $\mathbb{R}^3$, the orthogonal subspace to a line through the origin is the plane through the original normal to it.

Ortholattices

Definition An ortholattice is an algebra $(L, \wedge, \vee, 0, 1, \perp)$ where

1. $(L, \wedge, \vee, 0, 1)$ is a bounded lattice
2. $x \wedge x^{\perp} = 0$
3. $x \vee x^{\perp} = 1$
4. $x = x^{\perp\perp}$
5. $x \leq y \Rightarrow y^{\perp} \leq x^{\perp}$

Example Every Boolean algebra is an ortholattice. In fact, Boolean algebras are exactly the distributive ortholattices.

Ortholattices

Proposition For any finite dimensional inner product space V, the lattice of subspaces $(\mathcal{S}(V), \perp)$ is a modular ortholattice.

Pf Its easy to see $A^\perp$ is a subspace and $A \subseteq B \Rightarrow B^\perp \subseteq A^\perp$. We also have $A \cap A^\perp = \{0\}$ since v in both gives $\langle v, v \rangle = 0$, so $v = 0$.

The crucial point is that $A \vee A^\perp = V$. It follows from the fact that any vector v is given by the sum of the projections

$$v = v_A + v_{A^\perp}$$

That $A^{\perp\perp} = A$ is then an exercise.

Ortholattices

For an infinite dimensional inner product space, it is not the case that for each subspace A we have

$$v = v_A + v_{A^\perp}$$

Indeed, we can't even define the projection v_A for arbitrary A.

A Hilbert space such as $L^2(X, \mu)$ is a complete metric space under the topology given by the norm and we can define the projection v_A when A is a closed subspace. It is the vector in A closest to v.

Key Lemma If A is a closed subspace of a Hilbert space $\mathcal{H}$, then $A^\perp$ is a closed subspace and for each v we have $v = v_A + v_{A^\perp}$.

Ortholattices

Theorem Let $\mathcal{H}$ be a Hilbert space. Then its collection $\mathcal{C}(\mathcal{H})$ of closed subspaces has the following properties

1. it is complete as a lattice
2. it is an ortholattice
3. it satisfies $A \subseteq B \Rightarrow A \vee (A^{\perp} \cap B) = B$

Note, $\mathcal{C}(\mathcal{H})$ is modular iff $\mathcal{H}$ is finite-dimensional. This is because joins are more complex, they are the closure of the span.

Orthomodular Lattices

Definition An ortholattice L is an orthomodular lattice if it satisfies

$$x \leq y \;\Rightarrow\; x \vee (x^{\perp} \wedge y) = y$$

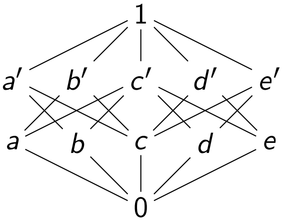
Examples of orthomodular lattices include

- any Boolean algebra
- any modular ortholattice
- the closed subspaces $\mathcal{C}(\mathcal{H})$ Hilbert space
- the projections of a von Neumann algebra (from analysis)

Note modular + ortholattice $\Rightarrow$ orthomodular lattice, not $\Leftarrow$

Orthomodular Lattices

Example The following is an orthomodular lattice

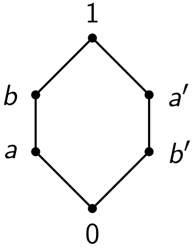


Note that the part at the left is an 8-element Boolean algebra, as is the part at right. Every orthomodular lattice is built by "gluing together" Boolean algebras in a sense we can make precise.

Orthomodular Lattices

Proposition For an ortholattice L, these are equivalent.

1. L is orthomodular
2. L does not have a subalgebra isomorphic to Benzene below
3. its order $\leq$ is the union of those of if its Boolean subalgebras



Note that in Benzene, $a \leq b$, but a, b do not belong to a Boolean subalgebra.

Orthomodular Lattices

Remark The study of orthomodular lattices has two parts. One is the study of Boolean algebras and classical issues related to them.

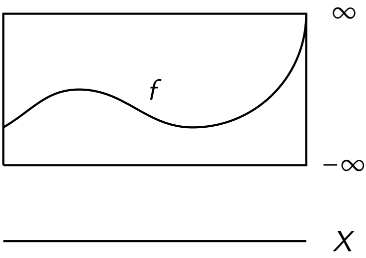
The second is in how these Boolean algebras are "glued together" to form an orthomodular lattice. This second part contains the geometric content that can be very challenging.

Example The subspaces $\mathcal{S}(\mathbb{R}^3)$ are built from 8-element Boolean algebras "glued together". From the way they are glued, we can reconstruct $\mathbb{R}^3$. This gluing contains the full geometric content!

Orthomodular Lattices

For a Hilbert space $\mathcal{H}$, the Boolean subalgebras of $\mathcal{C}(\mathcal{H})$ are key to spectral representations.

Spectral Theorem For a self-adjoint operator A on $\mathcal{H}$ there is a Boolean subalgebra B of $\mathcal{C}(\mathcal{H})$ and, for $X = \beta(B)$ its Stone space, a representation of A by a continuous function $f : X \to [-\infty, \infty]$.



Orthomodular Lattices

This result was discovered in part by Marshall Stone, an analyst. It was the reason for his work on Boolean algebras and Stone spaces.

There is much more to all this.

Theorem (Takeuti) The self-adjoint operators affiliated with a maximal Boolean subalgebra B of $\mathcal{C}(\mathcal{H})$ correspond to the real numbers in a B-valued model of set theory.

Final Comment

A growing trend is study of "quantum" or "non-commuative" versions of classical topics, such as non-commutative geometry.

Something to look for with these is the idea of versions of a classical structure being "glued together" with the gluing providing geometric content. Orthomodular lattices are a simple example, being a quantum, or non-commutative version of Boolean algebras.

Analogy

$$\begin{array}{rcl} \text{distributive lattices} & :: & \text{modular lattices} \\ \text{diagonal matrices} & :: & \text{matrices} \end{array}$$

Thanks for listening.

Papers at www.math.nmsu.edu/~jharding