Lattice Theory Lecture 5

Completions

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Completions

Definition A completion of a poset P is a pair (C, e) where C is a complete lattice and $e : P \to C$ satisfies $x \leq y \Leftrightarrow e(x) \leq e(y)$.

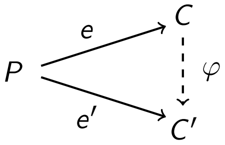
Often we are interested in completions that preserve some set of existing joins and meets. A primary instance is when P is a lattice, when we nearly always desire to preserve finite joins and meets.

Completions

Abstract Characterizations

Often we give a concrete way to make a completion (C, e) of P, then give an abstract characterization of it.

This means giving properties of (C, e), then showing if (C', e') is another completion with these properties, then there is a unique isomorphism $\varphi : C \to C'$ making the following commute.



Completions

Example of a completion and its abstract characterization

The first example of a completion was completing the rationals $\mathbb{Q}$ to the reals $\mathbb{R}$, really the extended reals.

This was done first by Dedekind around 1870 using "cuts" and a few years later using Cauchy sequences by Cantor.

Few people view real numbers as either cuts or Cauchy sequences of rationals, but rather via their abstract characterization in terms of the rationals.

The MacNeille Completion

Theorem For a poset P, there is a completion (C, e) of P that satisfies

1. each element of C is a join of elements of $e[P]$
2. each element of C is a meet of elements of $e[P]$

Further, any two such completions are isomorphic up to unique isomorphism. They are called MacNeille completions.

Notes Item (1) is called "join dense" and (2) is "meet dense".

The MacNeille Completion

Example Here is a join and meet dense completion

Example And one that is neither join or meet dense.

Constructing the MacNeille Completion

Definition For a poset P and $A \subseteq P$ set

$$U(A) = \{u : u \text{ is an upper bound of } A\}$$
$$L(A) = \{v : v \text{ is a lower bound of } A\}$$

The sets A with $A = LU(A)$ are called normal ideals of P.

Theorem The set of all normal ideals $M(P)$ is a complete lattice and $e : P \to M(P)$ where $e(a) = \downarrow a$ is a join and meet dense completion of P, thus a MacNeille completion of P.

The MacNeille Completion

Pf Each principal ideal $\downarrow x$ is a normal ideal, so $e(x) = \downarrow x$ is well defined, and clearly $x \le y$ iff $e(x) \subseteq e(y)$.

One shows that A is normal iff it is the intersection of principle ideals $\downarrow x$, namely the $x \in U(A)$. Therefore $M(P)$ is closed under intersections, so is a complete lattice, and we have meet density.

Since a normal ideal is a downset, it is the union of the principal ideals it contains, hence is the join of the principal ideals it contains. So this completion is join dense.

The MacNeille Completion

Proposition For a completion $e : P \to C$ of a poset P

1. meet dense $\Rightarrow$ preserves existing joins
2. join dense $\Rightarrow$ preserves existing meets

Pf 1. Suppose $A \subseteq P$ and $\bigvee A = x$. Since $e(a) \leq e(x)$ for all $a \in A$ we have $\bigvee e[A] \leq e(x)$.

To show that $e(x) \leq \bigvee e[A]$ we use meet density. It is enough to show that any element $e(y)$ above $\bigvee e[A]$ is above $e(x)$.

But $\bigvee e[A] \leq e(y)$ gives $e(a) \leq e(y)$ for each $a \in A$, so $a \leq y$ for each $a \in A$, so $x \leq y$, and hence $e(x) \leq e(y)$.

MacNeille Completions

Corollary MacNeille completions are join and meet dense, so preserve all existing joins and meets.

Proposition Join and meet dense completions of P are unique up to isomorphism.

Pf Suppose (C, e) is a join and meet dense completion of P. For each $c \in C$ set $A = \{x : e(x) \leq c\}$ and $B = \{y : c \leq e(y)\}$.

Use meet density to show that $A = L(B)$ so is a normal ideal of P and join density to get a bijection between C and the normal ideals.

MacNeille Completions

MacNeille completions have great order theoretic properties, but are poorly behaved with respect to preserving equations.

Theorem The varieties of lattices that are closed under MacNeille completions are

1. the variety of 1-element lattices
2. the variety of all lattices

So the MacNeille completion of a distributive lattice need not be distributive, and that of a modular lattice need not be modular.

Ideal Completions

Theorem The embedding $e : L \to \mathcal{I}(L)$ into its ideal lattice given by $e(a) = {\downarrow}a$ is a completion and $\mathcal{I}L$ satisfies the same equations as L.

Pf We discussed this in Lecture 1.

Proposition For a lattice L, the completion $(\mathcal{I}(L), e)$ satisfies

1. it is join dense
2. if $e(a) \leq \bigvee S$ then $e(a) \leq \bigvee S'$ for some finite $S' \subseteq S$

Further, these conditions characterize the ideal completion.

Ideal Completions

Pf (1) Each ideal is the union of the principal ideals it contains, hence is their join. So it is join dense.

(2) Let S be a set of ideals. Then $\bigvee S$ is the ideal generated by them. So if $\downarrow a \subseteq \bigvee S$ it follows that a belongs to the join of finitely many of these ideals.

Let (C, e) be another completion with these properties. For $c \in C$ let $A = \{a : e(a) \leq c\}$. Then A is an ideal of L, and the second condition shows that every ideal of L arises this way. Join density gives that different elements of C give different ideals.

Ideal Completions

Since it is join and meet dense, the MacNeille completion preserves existing joins and meets.

The ideal completion is join dense so preserves existing meets, and condition (2) says that it destroys all but essentially finite joins.

We construct one more completion that destroys all but essentially finite joins and meets. It cannot be either join or meet dense.

Canonical Completions

We use $\subseteq_\omega$ to mean "is a finite subset of"

Theorem For a bounded lattice L there is a completion (C, e) that preserves finite joins and meets where

1. each $c \in C$ is a join of meets of elements of $e[L]$
2. each $c \in C$ is a meet of joins of elements of $e[L]$
3. $\bigwedge e[S] \leq \bigvee e[T] \Rightarrow \exists\, S' \subseteq_\omega S, T' \subseteq_\omega T$ with $\bigwedge e[S'] \leq \bigvee e[T']$

These conditions characterize the completion up to isomorphism. It is called the canonical completion of L and written L^σ.

Recap

MacNeille completion $M(L)$

Join and meet dense, preserves all existing joins and meets

Ideal completion $\mathcal{I}(L)$

Join dense, not meet dense. Preserves existing meets and finite joins. Destroys all non-finite joins. Preserves all lattice equations.

Canonical completion L^σ

Not join or meet dense. Elements are joins of meets and meets of joins. Preserves finite joins and meets, destroys all non-finite ones.

Canonical Completions

We outline how to build the canonical completion L^σ. We need some ideas of independent interest.

Definition For a relation R from X to Y, its polars are maps

$$\mathcal{P}(X) \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} \mathcal{P}(Y)$$

given by

$$\Phi(A) = \{y : a\,R\,y \text{ for all } a \in A\}$$
$$\Psi(B) = \{x : x\,R\,b \text{ for all } b \in B\}$$

Note Polars are an example of a Galois connection, maps Φ, Ψ between power sets with $A \subseteq \Psi(B) \Leftrightarrow B \subseteq \Phi(A)$.

Galois Connections

- Galois connections are miniature versions of adjunctions in category theory
- The A with $A = \Psi\Phi(A)$ are the Galois closed subsets of X and the B with $B = \Phi\Psi(B)$ are the Galois closed subsets of Y
- The Galois closed sets $\mathcal{G}(X)$ and $\mathcal{G}(Y)$ are complete lattices and Φ, Ψ are anti-isomorphisms between them
- The MacNeille completion was constructed as the Galois closed sets of the polars of the relation $\leq$ from a poset P to P.

Construction of the Canonical Completion

Definition For a bounded lattice L let

$$\mathcal{I} = \{ I : I \text{ is an ideal of } L\}$$
$$\mathcal{F} = \{ F : F \text{ is a filter of } L\}$$

Then let R be the relation from $\mathcal{I}$ to $\mathcal{F}$ with $I \, R \, F \Leftrightarrow I \cap F \neq \varnothing$.

Theorem Let $\mathcal{G}$ be the Galois closed elements of the polars of the relation R and $e : L \to \mathcal{G}$ be given by $e(a) = \Phi\Psi(\{\downarrow a\})$. Then e is an embedding and $(\mathcal{G}, e)$ is a canonical completion of L.

Pf This one is more involved and we skip it.

A Template for Completions

One can choose other setss of downsets $\mathcal{I}'$ and upsets $\mathcal{F}'$ of L rather than the set of all ideals and filters of L and proceed as in the construction of the canonical completion.

To get an embedding, one wants that the downsets in $\mathcal{I}'$ and the upsets in $\mathcal{F}'$ separate points.

One will get a completion $\mathcal{G}(\mathcal{I}', \mathcal{F}')$ that preserves those existing joins in L under which each member of $\mathcal{I}'$ is closed, and preserve those existing meets in L under which each member of $\mathcal{F}'$ is closed.

A Template for Completions

Many common completions come about this way.

- $M(L)$ has $\mathcal{I}'$ and $\mathcal{F}'$ all principal ideals and filters
- $\mathcal{I}(L)$ has $\mathcal{I}'$ all ideals and $\mathcal{F}'$ all principal ideals

If we take $\mathcal{I}'$ and $\mathcal{F}'$ to be all ideals and filters closed under existing countable joins and meets, we get a completion that preserves all existing countable joins and meets and destroys all others.

Overview

At this point it is difficult to see the reason for the canonical completion L^σ

When restricted to lattices, both MacNeille and ideal completions have better order-theoretic properties than L^σ and the ideal completion is as good as can be at preserving equations.

Question why a canonical completion? Because we want to complete lattices with additional operations.

Extending Operations

Let $f : L \to L$ be order preserving and (C, e) be a completion of L. There are several common ways to extend f to C. To simplify notation, assume L is a sublattice of C, so $e = id$.

Definition Define extensions of f to C as follows

1. $f^{-}(c) = \bigvee \{ f(x) : x \leq c \}$
2. $f^{+}(c) = \bigwedge \{ f(x) : c \leq x \}$
3. $f^{\sigma}(c) = \bigvee \{ \bigwedge \{ f(x) : x \in K \} : K \subseteq L \text{ and } \bigwedge K \leq c \}$
4. $f^{\pi}(c) = \bigwedge \{ \bigvee \{ f(x) : x \in U \} : U \subseteq L \text{ and } c \leq \bigvee U \}$

The f^{-} extension will work best with join dense completions, f^{+} with meet dense. The f^{σ} and f^{π} work with joins of meets and meets of joins, so are intended for canonical completions.

Extending Operations

These definitions extend to *n*-ary operations that preserve order in some coordinates and reverse it in others. Call these monotone operations.

Example Heyting negation $\to$ is order reversing in the first coordinate, preserving in the second.

$$(\to)^{-}(c,d) = \bigvee\{(x,y) : c \leq x \text{ and } y \leq d\}$$

Terminology The MacNeille completion with lower extension f^{-} is the lower MacNeille completion, and so forth.

Completions and Extra Operations

We want completions of lattices with additional operations that preserve structure, usually equations.

Primary examples are

- Boolean algebras $(B,')$
- Heyting algebras $(H, \rightarrow)$
- modal algebras $(B, \diamond)$
- Boolean algebras with additional operations $(B, (f_i)_I)$

We try to give a bit of a feel for this large subject.

Ideal Completions

Again, ideal completions are great if you have only order preserving operations.

Proposition Let $(L, (f_i)_I)$ be a lattice with additional order preserving operations. Then its lower ideal completion satisfies all equations it satisfies.

However, the following is fatal for use of ideal completions past the order preserving setting.

Proposition The ideal completion of a Boolean algebra $(B,')$ is Boolean iff B is finite.

MacNeille Completions

Theorem Let $(B,')$ be a Boolean algebra with Stone space X.

1. Its lower and upper MacNeille completions agree
2. They are Boolean
3. They are isomorphic to $Reg(X)$

Here $Reg(X)$ is the family of regular open subsets of X. These are the sets that are equal to the interior of their closures.

MacNeille Completions

Theorem Let $(H, \to)$ be a Heyting algebra with Stone space X.

1. Its upper MacNeille completion is a Heyting algebra
2. The lower one is not

Theorem The only varieties of Heyting algebras that are closed under MacNeille completions are the 1-element Heyting algebras, Boolean algebras, and the variety of all Heyting algebras.

MacNeille Completions

There are scattered positive results about MacNeille completions and preservation of equations.

- closure algebras
- ortholattices
- a variety generated by a finite orthomodular lattice

But its mostly hit and miss. The few systematic results come from Monk, Givant and Venema, and Crown, Harding and Janowitz.

Having little structure to preserve is one path. The other is having lots of structure in the way of operations that are part of Galois connections, or a strong decomposition theory for your algebras.

Canonical Completions

These arose with Jónsson and Tarski in the 50's.

Proposition For a Boolean algebra $(B,')$ with Stone space X

1. Its σ and π canonical completions agree
2. They are Boolean
3. They are isomorphic to the power set $\mathcal{P}(X)$

Exercise Show that $Clopen(X) \to \mathcal{P}(X)$ is a canonical extension. Use Hausdorff to show $\{x\}$ is the intersection of clopens and view compactness of X in terms of an intersection of closed sets being contained in the union of opens.

Canonical Completions

In the distributive setting, including Heyting algebras, we have ...

Proposition For a bounded distributive lattice D with Priestley space X, its canonical completion is the set of upsets of X.

The reason for canonical extensions comes from Jónsson and Tarski's work on Boolean algebras with operators and complex algebras. This is now known under Kripke frames for modal logic.

Complex Algebras

Definition A binary relational structure $\mathfrak{X} = (X, R)$ is a set X with a binary relation R.

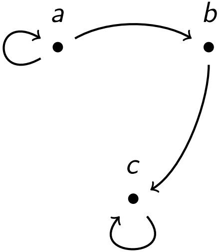
Definition The complex algebra $\mathfrak{X}^+$ is the Boolean algebra $\mathcal{P}(X)$ with unary operation f where

$$f(A) = \{x : a\,R\,x \text{ for some } a \in A\}$$

This has extension to structures $\mathfrak{X} = (X, (R_i)_I)$ with more than one relation, or with relations of higher arity. Here, an $n+1$-ary relation produces an n-ary operation via relational image.

Complex Algebras

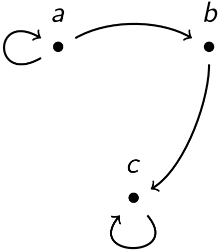
Example When R is binary, we can represent $\mathfrak{X} = (X, R)$ as a digraph!



Complex Algebras

The complex algebra of $\mathfrak{X} = (X, R)$ is the power set $\mathcal{P}(X)$ with

$$\Diamond A \;=\; \{\, x : a\,R\,x \;\text{ for some } a \in A \,\}$$



$$\Diamond\{a\} = \{a, b\}$$

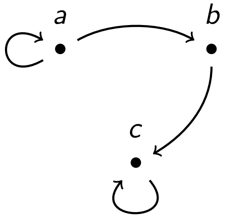
$$\Diamond\{b\} = \{c\}$$

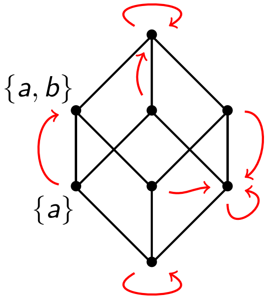
$$\Diamond\{c\} = \{c\}$$

$$\Diamond\{a, b\} = \{a, b, c\}$$

Complex Algebras

For a frame $\mathfrak{X} = (X, R)$ its complex algebra is $\mathfrak{X}^+ = (P(X), \Diamond)$.





An Interpretation

Points of X are worlds,

$$a \, R \, b \text{ means world } a \text{ is accessible from world } b$$

Given a valuation of the variables, a proposition p has a set of worlds P where it is true.

$$\begin{aligned} \Diamond P &= \text{all worlds } b \text{ where some } a \in P \text{ is accessible from them} \\ &= \text{all } b \text{ that have a world accessible from them where } p \text{ is true} \\ &= \text{all worlds } p \text{ is possible} \end{aligned}$$

Complex Algebras

This is a primary method to create examples of Boolean algebras with operators, meaning operations that preserve finite joins in each coordinate.

Example Modal algebras $(B, \Diamond)$ satisfying $\Diamond\Diamond a \leq \Diamond a$ are given by complex algebras $(X, R)^+$ where R is a transitive relation.

Example Relation algebras arise as complex algebras of groups. Relation algebras were Jónsson and Tarski's original motivation.

Complex Algebras and Canonical Extensions

Theorem Let (B, f) be a Boolean algebra with an operator f and let (B^σ, f^σ) be its canonical completion. Let X be the set of atoms of B^σ and define R on X by

$$x\,R\,y \Leftrightarrow y \leq f^\sigma(x)$$

Then (B^σ, f^σ) is isomorphic to $(X, R)^+$.

Complex Algebras and Canonical Completions

If a variety of Boolean algebras with operators is closed under canonical completions, each member of the variety is a subalgebra of a complex algebra $\mathfrak{X}^+$ for some relational structure $\mathfrak{X}$.

Such relational structures can often be easier to study than the algebras in the variety.

This property is sometimes known as strong Kripke completeness.

Canonical Completions

Fortunately, there are some systematic results about canonical completions.

Theorem In the setting of bounded lattices with monotone operations, the canonical completion is functorial and preserves subalgebras and quotients.

This means that if $h : L \to M$ is a homomorphism of lattices with additional operations, there is an extension to a homomorphism $h^\sigma : L^\sigma \to M^\sigma$ and this works in a way compatible with composition.

Canonical Completions

Theorem Let $\mathcal{K}$ be a class of bounded lattices with monotone operations. If $\mathcal{K}$ is closed under canonical completions and ultraproducts, then the variety generated by $\mathcal{K}$ is closed under canonical completions.

Corollary Any variety generated by a finite bounded lattice with monotone operations is closed under canonical completions.

This result has many other uses too. Linear Heyting algebras are those in the variety generated by the class $\mathcal{K}$ of chains. They are closed under canonical completions.

MacNeille and Canonical Completions

Theorem Let V be a variety of bounded lattices with monotone operations. If V is closed under lower or upper MacNeille completions, then it is closed under lower or upper canonical completions.

So if we are solely interested in a completion that preserves equations, the canonical completion is better than the MacNeille completion. Of course, we may also be interested in order theoretic properties where the MacNeille completion is likely superior.

Concluding Remarks

We close with some open questions about completions. All would be substantial results if solved.

1. Can every Heyting algebra H be embedded into a complete Heyting algebra that satisfies the same equations as H?
2. Can every orthomodular lattice be embedded into a complete orthomodular lattice?
3. If L is a complete lattice and each element of L has exactly one complement, must L be a Boolean algebra?

Thanks for listening.

Papers at www.math.nmsu.edu/~jharding