VARIETIES OF ORTHOLATTICES CONTAINING OML

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CONTAINING THE ORTHOMODULAR LATTICES

By

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ABSTRACT

This thesis considers certain classes of ortholattices defined by implications which are weaker forms of the orthomodular law.

All classes considered are shown to be varieties, and equational characterizations are given. The relationships between these classes are also determined.

Furthermore, in the lattice of ortholattice varieties, an isomorphic copy of the lattice of self-dual lattice varieties is constructed between the smallest of our classes and the orthomodular lattices.

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INTRODUCTION

Dealing with orthocomplemented lattices, the variety of orthomodular lattices is defined by the equation $x \vee (x' \wedge (x \vee y)) = x \vee y$. For an orthomodular lattice (OML) L, define the relation C by aCb if $(a \wedge b) \vee (a \wedge b') = a$ and define the function $\gamma(a,b) = (a \vee b) \wedge (a \vee b') \wedge (a' \vee b) \wedge (a' \vee b')$. It is shown (Chapter 1, [2]) that the following seven statements are equivalent.

1. L is an OML.
2. For all $a,b \in L$ if aCb then bCa.
3. For all $a,b \in L$ if aCb then a'Cb.
4. For all $a,b \in L$ aCb iff $a \vee (a' \wedge b) = a \vee b$.
5. For all $a,b \in L$ aCb iff $\gamma(a,b) = 0$.
6. The ortholattice 0_6 (Figure 1) is not a subalgebra of L.
7. For all $a,b \in L$ if $a \leq b$ then $a \vee (a' \wedge b) = b$.

We define the following predicates P1(a,b) through P7(a,b).

P1(a,b) iff aCb

P2(a,b) iff bCa

P3(a,b) iff $a \vee (a' \wedge b) = a \vee b$

P4(a,b) iff $b \vee (b' \wedge a) = b \vee a$

P5(a,b) iff $a \wedge (a' \vee b) = a \wedge b$

P6(a,b) iff $b \wedge (b' \vee a) = a \wedge b$

P7(a,b) iff $\gamma(a,b) = 0$.

Using the above predicates, define $K_{i,j}$ to be the class of all ortholattices L in which $P_i(a,b)$ implies $P_j(a,b)$ for all $a,b \in L$.

We next give equational characterizations of all classes $K_{i,j}$, which is a somewhat surprising result, as classes which are defined by implications are not in general varieties (page 219, [1]).

SECTION 1

In this section, explicit equational characterizations of the classes $K_{i,j}$ are given. For the convenience of the reader, the results have been summarized in the following table.

$K_{i,j}$

i \ j	1	2	3	4	5	6	7
1	OL	OML	OML	OL	E1	OML	E2
2	OML	OL	OL	OML	OML	E1	E2
3	OML	OML	OL	OML	OML	E3	E4
4	OML	OML	OML	OL	E3	OML	E4
5	OML	OML	OML	E3	OL	OML	E4
6	OML	OML	E3	OML	OML	OL	E4
7	OML	OML	OML	OML	OML	OML	OL

For this table OL represents the variety of ortholattices, OML represents the variety of orthomodular lattices, and En represents the variety of ortholattices which universally satisfy the equation En, where

E1: $((a \wedge b) \vee (a \wedge b')) \wedge (((a' \vee b') \wedge (a' \vee b)) \vee b) =$
$((a \wedge b) \vee (a \wedge b')) \wedge b$

E2: $\gamma((a \wedge b) \vee (a \wedge b'), b) = 0$

E3: $b \wedge (a \vee (a' \wedge b)) \wedge (b' \vee a) = b \wedge a$

E4: $\gamma(a, b \wedge (a \vee (a' \wedge b))) = 0$

Justification of these results for each class $K_{i,j}$ is given below. One should note that each class contains OML, and that the classes $K_{i,i}$ are trivially the class of all ortholattices (OL).

$\underline{K_{1,2} = \text{OML}}$

Proof. The statement $aCb \rightarrow bCa$ is equivalent to the orthomodular law.

$\underline{K_{1,3} = \text{OML}}$

Proof. The statement $a \leq b \rightarrow a \vee (a' \wedge b) = b$ is equivalent to the orthomodular law. For $L \in K_{1,3}$ and $a, b \in L$ with $a \leq b$, we have aCb. So $a \vee (a' \wedge b) = a \vee b = b$, and $L \in \text{OML}$.

$\underline{K_{1,4} = OL}$

Proof. For an arbitrary $L \in OL$ and $a,b \in L$ with aCb, by definition of C $(a \wedge b) \vee (a \wedge b') = a$. So $a \vee b = b \vee (a \wedge b) \vee (a \wedge b') = b \vee (b' \wedge a)$, which gives $L \in K_{1,4}$.

$\underline{K_{1,5}}$ is the variety of ortholattices generated by the equation $c \wedge (c' \vee b) = c \wedge b$, for $c = (a \wedge b) \vee (a \wedge b')$. Furthermore, $K_{1,5}$ is not equal to OL or OML.

Proof. For $L \in OL$, if $c \wedge (c' \vee b) = c \wedge b$ for all $a,b \in L$, then aCb would imply $a \wedge (a' \vee b) = a \wedge b$, as aCb gives $a = c$, by definition. So $L \in K_{1,5}$. Conversely, assume $L \in K_{1,5}$. Then as $c \leq a$, and $c \geq a \wedge b, a \wedge b'$, we have $c \wedge b = a \wedge b$ and $c \wedge b' = a \wedge b'$. So $(c \wedge b) \vee (c \wedge b') = c$, and cCb. But, $L \in K_{1,5}$, so $c \wedge (c' \vee b) = c \wedge b$.

The non-orthomodular lattice of figure 1 is an element of $K_{1,5}$, and the ortholattice of figure 2 is not included in $K_{1,5}$.

$\underline{K_{1,6} = OML}$

Proof. The statement $a \leq b \rightarrow a \vee (a' \wedge b) = b$ is equivalent to the orthomodular law. For $L \in K_{1,6}$ and $a,b \in L$ with $a \leq b$, we have $b' \leq a'$ and therefore b'Ca'. As $L \in K_{1,6}$ $a' \wedge (a \vee b') = b'$, and also $a \vee (a' \wedge b) = b$, so $L \in OML$.

$\underline{K}_{1,7}$ is the variety of ortholattices generated by the equation $\gamma(c,b) = 0$, where $c = (a \wedge b) \vee (a \wedge b')$. Furthermore, $K_{1,7}$ is not equal to OL or OML.

Proof. Assume L is an ortholattice, and $\gamma(c,b) = 0$ for all $a,b \in L$. If aCb, then $c = a$, and $\gamma(a,b) = 0$, giving $L \in K_{1,7}$. Assuming $L \in K_{1,7}$, as cCb (shown in the proof for $K_{1,5}$), $\gamma(c,b) = 0$ for all $a,b \in L$.

The non-orthomodular lattice of figure 1 is an element of $K_{1,7}$, and the ortholattice of figure 3 is not an element of $K_{1,7}$.

$\underline{K_{2,1} = \text{OML}}$

Proof. By symmetry with $K_{1,2}$.

$\underline{K_{2,3} = \text{OML}}$

Proof. By symmetry with $K_{1,4}$.

$\underline{K_{2,4} = \text{OML}}$

Proof. By symmetry with $K_{1,3}$.

$\underline{K_{2,5} = \text{OML}}$

Proof. By symmetry with $K_{1,6}$.

$\underline{K_{2,6} = K_{1,5}}$

Proof. By symmetry.

$\underline{K_{2,7} = K_{1,7}}$

Proof. By symmetry.

$\underline{K_{3,1} = OML}$

Proof. The statement $a \leq b \to a \vee (a' \wedge b) = b$ is equivalent to the orthomodular law. For $L \in K_{3,1}$ with $a,b \in L$ and $a \leq b$, $b \vee (b' \wedge a) = a \vee b$, so bCa. Using the proof of $K_{1,4}$, we then have $a \vee (a' \wedge b) = b$, and therefore $L \in OML$.

$\underline{K_{3,2} = OML}$

Proof. The statement $a \leq b \to a \vee (a' \wedge b) = b$ is equivalent to the orthomodular law. For $L \in K_{3,2}$, with $a,b \in L$ and $a \leq b$, $a' \vee (a \wedge b) = a' \vee b$, so bCa′ and bCa. Using the proof of $K_{1,4}$ we then have $a \vee (a' \wedge b) = b$, and then $L \in OML$.

$\underline{K_{3,4} = OML}$

Proof. The statement $a \leq b \to a \vee (a' \wedge b) = b$ is equivalent to the orthomodular law. For $L \in K_{3,4}$, with $a,b \in L$ and $a \leq b$, $b \vee (b' \wedge a) = b \vee a$, so $a \vee (a' \wedge b) = a \vee b$, and so $L \in OML$.

$\underline{K_{3,5} = OML}$

Proof. The statement $a \leq b \to a \vee (a' \wedge b) = b$ is equivalent to the orthomodular law. For $L \in K_{3,5}$, with $a,b \in L$ and $a \leq b$, $a' \vee (a \wedge b') = a' \vee b'$, so $a' \wedge (a \vee b') = a' \wedge b'$ and also $a \vee (a' \wedge b) = a \vee b$. Therefore $L \in OML$.

$\underline{K}_{3,6}$ is the variety of ortholattices generated by the equation $b \wedge e \wedge (b' \vee a) = b \wedge a$, where $e = a \vee (a' \wedge b)$. Furthermore $K_{3,6}$ is not equal to OL or OML.

Proof. Assume L is an ortholattice, and for all $a,b \in L$, $b \wedge e \wedge (b' \vee a) = b \wedge a$. If $a \vee (a' \wedge b) = a \vee b$, then $e = a \vee b$, and $b \wedge e \wedge (b' \vee a) = b \wedge (a \vee b) \wedge (b' \vee a) = b \wedge (b' \vee a)$. Then, $b \wedge (b' \vee a) = b \wedge a$, and therefore $L \in K_{3,6}$. Assume $L \in K_{3,6}$. As $a' \wedge b \leq e$, $a \vee (a' \wedge b \wedge e) = e$, so $e \leq a \vee (a' \wedge b \wedge e) \leq a \vee (b \wedge e) \leq e$, and therefore $a \vee (a' \wedge (b \wedge e)) = a \vee (b \wedge e)$. Then, as $L \in K_{3,6}$, we obtain $b \wedge e \wedge (b' \vee e' \vee a) = b \wedge e \wedge a$, and therefore $b \wedge e \wedge (b' \vee a) = b \wedge a$. The non-orthomodular lattice of figure 1 is an element of $K_{3,6}$, and the ortholattice of figure 2 is not contained in $K_{3,6}$.

$\underline{K}_{3,7}$ is the variety of ortholattices generated by the equation $\gamma(a, b \wedge e) = 0$, where $e = a \vee (a' \wedge b)$. Furthermore, $K_{3,7}$ is not equal to OL or OML.

Proof. Assume L is an ortholattice, and for all $a,b \in L$, $\gamma(a, b \wedge e) = 0$. If $a \vee (a' \wedge b) = a \vee b$, then $e = a \vee b$, and $b \wedge e = b$. So $L \in K_{3,7}$. Assume $L \in K_{3,7}$. As in the proof of $K_{3,6}$, $a \vee (a' \wedge (b \wedge e)) = a \vee (b \wedge e)$. Then, as $L \in K_{3,7}$ we obtain $\gamma(a, b \wedge e) = 0$. The non-orthomodular lattice of figure 1 is contained in $K_{3,7}$, and the ortholattice of figure 2 is not contained in $K_{3,7}$.

$\underline{K_{4,1} = \text{OML}}$

Proof. By symmetry with $K_{3,2}$.

$\underline{K_{4,2} = \text{OML}}$

Proof. By symmetry with $K_{3,1}$.

$\underline{K_{4,3} = \text{OML}}$

Proof. By symmetry with $K_{3,4}$.

$\underline{K_{4,5} = K_{3,6}}$

Proof. By symmetry.

$\underline{K_{4,6} = \text{OML}}$

Proof. By symmetry with $K_{3,5}$.

$\underline{K_{4,7} = K_{3,7}}$

Proof. As $\gamma(a,b) = \gamma(b,a)$, and symmetry.

$\underline{K_{5,1} = OML}$

Proof. The statement $a \leq b \rightarrow a \vee (a' \wedge b) = b$ is equivalent to the orthomodular law. For $L \in K_{5,1}$, and $a,b \in L$ with $a \leq b$, we have $b \wedge (b' \vee a') = b \wedge a'$, and therefore bCa′ and bCa. Then, as in the proof of $K_{1,4}$, $a \vee (a' \wedge b) = b$, and $L \in OML$.

$\underline{K_{5,2} = OML}$

Proof. The statement $a \leq b \rightarrow a \vee (a' \wedge b) = b$ is equivalent to the orthomodular law. For $L \in K_{5,2}$ and $a,b \in L$ with $a \leq b$, we have $a \wedge (a' \vee b) = a \wedge b$, and therefore bCa. Then, as in the proof of $K_{1,4}$, $a \vee (a' \wedge b) = b$, and $L \in OML$.

$\underline{K_{5,3} = OML}$

Proof. By duality with $K_{3,5}$.

$\underline{K_{5,4} = K_{3,6}}$

Proof. By duality.

$\underline{K_{5,6} = OML}$

Proof. By duality with $K_{3,4}$.

$\underline{K_{5,7} = K_{3,7}}$

Proof. As $\gamma(a,b) = \gamma(a',b')$, and duality.

$\underline{K_{6,1} = OML}$

Proof. By symmetry with $K_{5,2}$.

$\underline{K_{6,2} = OML}$

Proof. By symmetry with $K_{5,1}$.

$\underline{K_{6,3} = K_{3,6}}$

Proof. By symmetry with $K_{5,4}$.

$\underline{K_{6,4} = OML}$

Proof. By symmetry with $K_{5,3}$.

$\underline{K_{6,5} = OML}$

Proof. By symmetry with $K_{5,6}$.

$\underline{K_{6,7} = K_{3,7}}$

Proof. As $\gamma(a,b) = \gamma(b,a)$, and symmetry with $K_{5,7}$.

$\underline{K_{7,1} = OML}$

Proof. The statement $a \leq b \rightarrow a \vee (a' \wedge b) = b$ is equivalent to the orthomodular law. For $L \in K_{1,7}$, and $a,b \in L$ with $a \leq b$, we have $\gamma(b,a) = 0$, and therefore, bCa. As in the proof of $K_{1,4}$, $a \vee (a' \wedge b) = b$, so $L \in OML$.

$\underline{K_{7,2} = OML}$

Proof. As $\gamma(a,b) = \gamma(b,a)$, and symmetry with $K_{7,1}$.

$\underline{K_{7,3} = OML}$

Proof. The statement $a \leq b \rightarrow a \vee (a' \wedge b) = b$ is equivalent to the orthomodular law. For $L \in K_{7,3}$ and $a,b \in L$ with $a \leq b$, we have $\gamma(a,b) = 0$, and therefore, $a \vee (a' \wedge b) = b$. Then $L \in OML$.

$\underline{K_{7,4} = OML}$

Proof. As $\gamma(a,b) = \gamma(b,a)$, and symmetry with $K_{7,3}$.

$\underline{K_{7,5} = OML}$

Proof. As $\gamma(a,b) = \gamma(a',b')$, and duality with $K_{7,3}$.

$\underline{K_{7,6} = OML}$

Proof. As $\gamma(a,b) = \gamma(a',b')$, and duality with $K_{7,4}$

SECTION 2

In this section we discuss the relation of the six varieties discussed above to the lattice of ortholattice varieties.

Proposition The class $K_{3,6}$ is properly contained in $K_{1,5}$.

Proof. Take $L \in K_{3,6}$, and $a,b \in L$ such that aCb. By definition of C, $(a \wedge b) \vee (a \wedge b') = a$, so $b \vee (b' \wedge a) = b \vee a$. As $L \in K_{3,6}$, we then have $a \wedge (a' \vee b) = a \wedge b$, which gives $L \in K_{1,5}$. Figure 4 gives an example of an ortholattice which is an element of $K_{1,5}$, but not of $K_{3,6}$.

Proposition. The class $K_{3,7}$ is properly contained in $K_{1,7}$.

Proof. Take $L \in K_{3,7}$, and $a,b \in L$ such that aCb. Then, as in the proof of $K_{1,4}$ above, we have $b \vee (b' \wedge a) = b \vee a$, and as $L \in K_{3,7}$, $\gamma(a,b) = 0$. So $L \in K_{1,7}$. Figure 5 gives an example of an ortholattice which is an element of $K_{1,7}$ but not of $K_{3,7}$.

Proposition. Both of the classes $K_{3,6}$ and $K_{1,5}$ are incomparable to each of $K_{3,7}$ and $K_{1,7}$.

Proof. Figure 2 gives an example of an ortholattice which is contained in $K_{3,7}$ but not in $K_{1,5}$, so $K_{3,7} \not\subseteq K_{1,5}$. Figure 3 gives an example of an ortholattice which is contained in $K_{3,6}$ but not in $K_{1,7}$.

For the remainder of this section, we demonstrate that an isomorphic copy of the lattice of self-dual lattice varieties can be embedded in the lattice of ortholattice varieties beneath $K_{3,6} \cap K_{3,7}$ having OML as a zero. Figure 6 summarizes our results.

<u>Definition.</u> An ortholattice L is said to be hyperbenzene if there exist disjoint non-trivial (i.e. having at least two elements) sublattices M, M' of L such that $M \cup M' = L\setminus\{0,1\}$. The unordered pair $\{M,M'\}$ is called an associator of L. We also let H represent the class of all hyperbenzene ortholattices.

<u>Proposition.</u> For $L \in H$, the associator of L is unique, and there exists a dual isomorphism between elements of the associator. Define $\mathcal{A}(L)$ as the associator of L.

Proof. Take $\{M,M'\}$ and $\{N,N'\}$ associators of L. For $a \in L\setminus\{0,1\}$, $a \in M$ iff $a' \in M'$ as $a,a' \in M$ would imply $0 \in M$. Assume $a \in M \cap N$ and $b \in M \cap N'$. Then we would have $b' \in N$, and $a \wedge b' \in N$ and hence $a \wedge b \in N'$. But $a' \in N$ and contrary to our definition $a \wedge b \wedge a' \in N'$. It follows that $M = N$ or $M = N'$, and orthocomplement is a dual isomorphism between M and M'.

<u>Proposition.</u> For any non-trivial lattice M, there exists $L \in H$ such that $M \in \mathcal{A}(L)$.

Proof. Given M, choose M' to be any lattice disjoint from M such that there exists a dual isomorphism α from M to M'. Define the ortholattice L by $L = (M \cup M' \cup \{\{M,M'\}, \{\{M,M'\}\}\}, \wedge, \vee, ', 0, 1)$ where $0 = \{M,M'\}$, $1 = \{\{M,M'\}\}$, and $\wedge, \vee$ are defined in the natural way from the partial order $\leq$ on L which is defined as the union of the partial orders on M, M', and $\{(0,0), (1,1), (0,a), (a,1) | a \in M \cup M'\}$. Orthocomplement is defined by $a' = \alpha(a)$ for $a \in M$, $a' = \alpha^{-1}(a)$ for $a \in M'$, $0' = 1$, and $1' = 0$. Then $L \in H$ and $M \in \mathcal{A}(L)$.

<u>Proposition</u> H is closed under ultraproducts.

Proof. Consider the first order sentence φ, which is the conjunction of the sentence saying there exist six distinct elements, with the sentence $\forall(x,y)((x = 0 \cup y = 0 \cup x = 1 \cup y = 1) \cup (x \wedge y \neq 0 \cap x \vee y \neq 1 \cap x \wedge y' = 0 \cap x \vee y' = 1) \cup (x \wedge y = 0 \cap x \vee y = 1 \cap x \wedge y' \neq 0 \cap x \vee y' \neq 1)$. Assume $L \in OL$ and $L \models \varphi$, then $|L| \geq 6$. Choose $x \in L\backslash\{0,1\}$, then define $S_x = \{y \in L | x \wedge y \neq 0, x \vee y \neq 1\}$, and $S'_x = \{y' | y \in S_x\}$. For $y \in L\backslash\{0,1\}$, if $y \notin S_x$ then $y \in S'_x$ as φ implies that $x \wedge y' \neq 0$ and $x \vee y' \neq 1$. If $y \in S_x \cap S'_x$ then $x \wedge y \neq 0$ and $x \wedge y' \neq 0$, a contradiction of the assumption $L \models \varphi$. We then have $S_x \cup S'_x = L\backslash\{0,1\}$ and $S_x \cap S'_x = \emptyset$. We claim that S_x is a sublattice of L. For $y,w \in S_x$, $x \vee (y \wedge w) \leq x \vee y < 1$. If $y \wedge w = 0$, then $(x \wedge y) \wedge (x \wedge w) = 0$, but as $L \models \varphi$, we would have $x \geq (x \wedge y) \vee (x \wedge w) = 1$, a contradiction. So, $y \wedge w \neq 0$ and $x \vee (y \wedge w) \neq 1$ implies, as $L \models \varphi$, $x \wedge (y \wedge w) \neq 0$, giving $y \wedge w \in S_x$. Similarily, for $y, w \in S_x$, $y \vee w \in S_x$. This proves our claim. By properties of orthocomplementation, S_x' is also a sublattice of L. Therefore, for an ortholattice L, $L \models \varphi$ iff $L \in H$. As first order sentences are preserved under ultraproducts (Łoś, page 210, [1]), our result follows.

<u>Definition.</u> For $A \subseteq H$, define $A^* = \{M | M \in \mathcal{A}(L)$ for some $L \in A\}$.

<u>Lemma.</u>

1. For $A \subseteq H$ and $L \in P_u(A)$, $\mathcal{A}(L) \subseteq IP_u(A^*)$.
2. For $F \in H$ and $L \in S(\{F\})$, either L is Boolean, or $L \in H$ and $\mathcal{A}(L) \subseteq S(\{F\}^*)$.
3. For $F \in H$ and $L \in H(\{F\})$, either L is Boolean, or $L \in H$ and $\mathcal{A}(L) \subseteq H(\{F\}^*)$.
4. For $A \subseteq H$ and $M \in P_u(A^*)$, if $L \in H$ with $M \in \mathcal{A}(L)$ then $L \in IP_u(A)$.
5. For M a non-trivial subalgebra of N, with $N \in \mathcal{A}(G)$ and if $M \in \mathcal{A}(L)$ for some $L \in H$ then $L \in IS(\{G\})$.
6. For M a non-trivial homomorphic image of N, with $N \in \mathcal{A}(G)$ and if $M \in \mathcal{A}(L)$ for some $L \in H$ then $L \in H(\{G\})$.

Proofs. 1. For a family $\{L_i\}_I$ where $\mathcal{A}(L_i) = \{M_i, M_i'\}$, let L be the ultraproduct of $\{L_i\}_I$ by the ultrafilter $\mathfrak{A}$, and θ as the congruence defined by $\mathfrak{A}$. For $a \in L$, $a = [z]_\theta$ for some $z \in \Pi L_i$. We may assume $z \in \Pi M_i$ or $z \in \Pi M_i'$ or $z = 0$ or $z = 1$, as one of the sets $\{i|z(i) \in M_i\}$, $\{i|z(i) \in M_i'\}$, $\{i|z(i) = 0\}$, $\{i|z(i) = 1\}$ is an element of $\mathfrak{A}$ since $\mathfrak{A}$ is an ultrafilter. Define $N = \{[x]_\theta | x \in \Pi M_i\}$ and $N' = \{[x]_\theta | x \in \Pi M_i'\}$. N and N' are sublattices of L. If $0 \in N$, then $N' = \{1\}$, which contradicts the fact that $L \in H$. Similarily $N \cap N' = \emptyset$, as $a \in N \cap N'$ would imply $a = [x]_\theta = [y]_\theta$ for some $x \in \Pi M_i$, $y \in \Pi M_i'$, and therefore that $a = [x \wedge y]_\theta = [0]$. As $N \cup N' = L\backslash\{0,1\}$, $\mathcal{A}(L) = \{N,N'\}$. We can define a mapping $\alpha\colon N \to \Pi_{\mathfrak{A}} M_i$ by $\alpha([x]_\theta) = [x]_{\theta 1}$ where $\theta 1 = \theta \cap (\Pi M_i)^2$. As α is an isomorphism, our claim is established.

2. For $F \in H$, $\mathcal{A}(F) = \{N,N'\}$, and L a subalgebra of F, $L \cap N'$, $L \cap N$ are disjoint sublattices of L such that $(L \cap N) \cup (L \cap N') = L\backslash\{0,1\}$. If $L \cap N$ and $L \cap N'$ are non-trivial, then $L \in H$ and $\mathcal{A}(L) = \{L \cap N, L \cap N'\}$. Otherwise L is Boolean.

3. For $F \in H$, $\mathcal{A}(F) = \{N,N'\}$, and L the image of F under the homomorphism φ, $\varphi[N]$ and $\varphi[N']$ are sublattices of L. If $0 \in \varphi[N]$, then $\varphi[N'] = 1$ and $\varphi[N] = 0$, so L is Boolean. If $|\varphi[N]| = 1$, then $|\varphi[N']| = 1$, and again L would be Boolean. If $\varphi(a) = \varphi(b)$ for $a \in N$, $b \in N'$, then $\varphi(a) = \varphi(a \wedge b) = 0$, and L would be

Boolean. If L is not Boolean, we have $\varphi[N]$, $\varphi[N']$ disjoint non-trivial sublattices of L such that $\varphi[N] \cup \varphi[N'] = L\backslash\{0,1\}$, giving $L \in H$ with $\mathcal{A}(L) = \{\varphi[N], \varphi[N']\}$.

4. For $M = \Pi_{\mathfrak{A}} M_i$, where $\{M_i\}_I$ is a family in A^*, consider $L' = \Pi_{\mathfrak{A}} L_i$, where $\{L_i\}_I$ is a family in A such that $M_i \in \mathcal{A}(L_i)$ for all $i \in I$. If $\mathcal{A}(L') = \{N,N'\}$, the proof of 1 in this lemma gives M isomorphic to N or N'. Then if $L \in H$ with $M \in \mathcal{A}(L)$, L is isomorphic to L'.

5. Let M be a non-trivial subalgebra of N, and $G \in H$ such that $N \in \mathcal{A}(G)$. Define M' as $\{x' \in G | x \in M\}$, and L' as $M \cup M' \cup \{0,1\}$. Then L' is a subalgebra of G, $L' \in H$ and $\mathcal{A}(L') = \{M,M'\}$. Then if $L \in H$ with $M \in \mathcal{A}(L)$, L will be isomorphic to L'.

6. Let M be a non-trivial image of N under the homomorphism φ. Assume $G \in H$ with $\mathcal{A}(G) = \{N,N'\}$ for some N'. Extend φ to a homomorphism $\bar{\varphi}$: $G \to L'$ for some L'. Then $\bar{\varphi}[N] = M$ and $\bar{\varphi}[N']$ are disjoint sublattices of L' as $\bar{\varphi}(a) = \bar{\varphi}(b)$ for $a \in N$, $b \in N'$ would imply $\bar{\varphi}(b) = \bar{\varphi}(a \wedge b) = 0$, so $\bar{\varphi}(x \vee b) = \bar{\varphi}(x) = 1$ for all $x \in N$, a contradiction of M being non-trivial. Clearly $\varphi[N] \cup \varphi[N'] = L'\backslash\{0,1\}$, so $\mathcal{A}(L') = \{M, \bar{\varphi}[N']\}$, and if $L \in H$ with $M \in \mathcal{A}(L)$, then L is isomorphic to L'.

<u>Proposition</u> For $L \in H$, $\mathcal{A}(L) = \{M, M'\}$, L is subdirectly irreducible iff M is subdirectly irreducible.

Proof. Let $\mathcal{L}(L)$ and $\mathcal{L}(M)$ be the congruence lattices of L and M respectively. If M is not subdirectly irreducible, there exists a family $\{X_i\}_I$ in $\mathcal{L}(M)$ such that $\bigwedge_I X_i = \Delta_M$, and for all $i \in I$, $X_i \neq \Delta_M$. For each $i \in I$ define $Y_i = X_i \cup \{(c',d') \mid (c,d) \in X_i\} \cup \Delta_L$. Then for all $i \in I$, $Y_i \in \mathcal{L}(L)$, $Y_i \neq \Delta_L$, and $\bigwedge_I Y_i = \Delta_L$. Therefore L is not subdirectly irreducible. Conversely, assume L is not subdirectly irreducible. There exists a family $\{X_i\}_I$ in $\mathcal{L}(L)$ such that $\bigwedge_I X_i = \Delta_L$ and for all $i \in I$, $X_i \neq \Delta_L$. Assume $X_i \cap M^2 = \Delta_M$ for some $i \in I$. Then there exists $p \in M$, $q \notin M$ such that $(p,q) \in X_i$. If $q = 0$, then for all $r \in M'$ $(1,r) \in X_i$, and $X_i \cap M^2 = M^2$, similarily if $q = 1$. If $q \in M'$ then $(1,q) \in X_i$, and $X_i \cap M^2 = M^2$. Therefore, for all $i \in I$, $X_i \neq \Delta_M$, $\bigwedge_I X_i = \Delta_M$ and as $X_i \in \mathcal{L}(M)$, M is not subdirectly irreducible.

<u>Proposition.</u> For $A \subseteq H$, $L \in H$ and $M \in \mathcal{A}(L)$, L is subdirectly irreducible and contained in the equational class generated by A (denoted $\langle A \rangle$) iff M is subdirectly irreducible and is contained in $\langle A^* \rangle$.

Proof. The above lemma shows that for $L \in H$ with $M \in \mathcal{A}(L)$, $L \in HSP_u(A)$ iff $M \in HSP_u(A^*)$. Jónsson showed (page 147, [1]) that the subdirectly irreducibles in $\langle B \rangle$ are those in $HSP_u(B)$. Our result follows directly.

<u>Proposition.</u> If A is a class of ortholattices, and K a variety of ortholattices, the subdirectly irreducibles in $\langle A \cup K \rangle$ are those in $HSP_u(A) \cup K$.

Proof. For $M = \Pi_{\mathfrak{A}} N_i$ for some family $\{N_i\}_I$ in $A \cup K$, let $I_1 = \{i | N_i \in A\}$ and $I_2 = \{i | N_i \in K\}$. As $\mathfrak{A}$ is an ultrafilter over I, exactly one of I_1, I_2 is an element of $\mathfrak{A}$. If $I_1 \in \mathfrak{A}$, then M is the ultraproduct of the family $\{N_i\}_{I_1}$ in A by the utrafilter $\mathfrak{A}_1 = \mathcal{P}(I_1) \cap \mathfrak{A}$. If $I_2 \in \mathfrak{A}$ then M is the ultraproduct of a family $\{N_i\}_{I_2}$ by the ultrafilter $\mathfrak{A}_2 = \mathcal{P}(I_2) \cap \mathfrak{A}$. Then $P_u(A \cup K) = P_u(A) \cup K$. By Jónsson's theorem (page 147, [1]), the subdirectly irreducibles in $\langle A \cup K \rangle$ are those in $HSP_u(A \cup K)$. But we have $HSP_u(A \cup K) = HS(P_u(A) \cup K) = HSP_u(A) \cup K$.

<u>Theorem.</u> There is an isomorphic copy of the lattice of self-dual lattice varieties below $K_{3,6} \cap K_{3,7}$, having OML as a zero.

Proof. Define a mapping γ from the lattice of ortholattice varieties of the form $\langle A \cup OML \rangle$ for $A \subseteq H$ to the lattice of self-dual lattice varieties by $\gamma(\langle A \cup OML \rangle) = \langle A^* \rangle$. For $A \subseteq H$, $\langle A \cup OML \rangle \subseteq K_{3,6} \cap K_{3,7}$, as if $L \in \langle A \cup OML \rangle$, and L is subdirectly irreducible, then $L \in HSP_u(A) \cup OML \subseteq H \cup OML$. But for $L \in H$, $\gamma(a,b) = 0$, and $b \wedge (b' \vee a) \neq b \wedge a$ implies $a \vee (a' \wedge b) \neq a \vee b$ for all $a,b \in L$, so $L \in K_{3,6} \cap K_{3,7}$. It is only left to show that γ is a lattice isomorphism onto the self-dual varieties of lattices.

(i) γ is well defined. For $A,B \subseteq H$, if $\langle A^* \rangle \neq \langle B^* \rangle$ then there exists a non-trivial subdirectly irreducible lattice $M \in \langle A^* \rangle \setminus \langle B^* \rangle$. There must exist a subdirectly irreducible $L \in H$ with $M \in \mathcal{A}(L)$ and $L \in \langle A \rangle \setminus \langle B \rangle$. Then $L \in \langle A \cup OML \rangle \setminus \langle B \cup OML \rangle$.

(ii) γ is one to one. For $A,B \subseteq H$ assume $\langle A \cup OML \rangle \neq \langle B \cup OML \rangle$. Then there exists a subdirectly irreducible $L \in H$ with $\mathcal{A}(L) = \{M,M'\}$ such that $L \in \langle A \cup OML \rangle \setminus \langle B \cup OML \rangle$. Then $M \in \langle A^* \rangle \setminus \langle B^* \rangle$.

(iii) γ is order preserving. For $A,B \subseteq H$, assume $\langle A \cup OML \rangle \subseteq \langle B \cup OML \rangle$. Then, $\langle B \cup OML \rangle = \langle A \cup B \cup OML \rangle$, and $\gamma(\langle A \cup OML \rangle) = \langle A^* \rangle \subseteq \langle (A \cup B)^* \rangle = \gamma(\langle B \cup OML \rangle)$. Assume $\gamma(\langle A \cup OML \rangle) \subseteq \gamma(\langle B \cup OML \rangle)$ and $\langle A \cup OML \rangle \not\subseteq \langle B \cup OML \rangle$. Then there exists a subdirectly irreducible $L \in H$ with $\mathcal{A}(L) = \{M,M'\}$ such that $L \in \langle A \cup OML \rangle \setminus \langle B \cup OML \rangle$. Then we have $L \in \langle A \rangle \setminus \langle B \rangle$ which implies $M \in \langle A^* \rangle \setminus \langle B^* \rangle$, contrary to the assumption that $\langle A^* \rangle \subseteq \langle B^* \rangle$.

(iv) γ is onto the self-dual varieties of lattices. Take K any self-dual variety of lattices. Define $A = \{L \in H \mid \mathcal{A}(L) \subseteq K\}$. It is clear that $A^* \subseteq K$, but for any non-trivial $M \in K$ there exists an $L \in H$ with $\mathcal{A}(L) = \{M,M'\}$ for some M'. As there exists a dual isomorphism from M to M', $M' \in K$, and so $L \in A$ and $A^* = K$. Then $\gamma(\langle A \cup OML \rangle) = \langle A^* \rangle = K$.

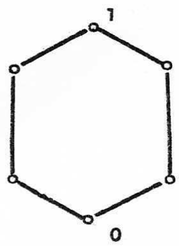

FIGURE 1

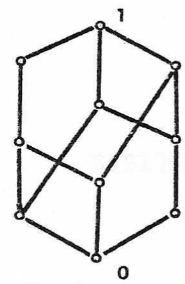

FIGURE 2

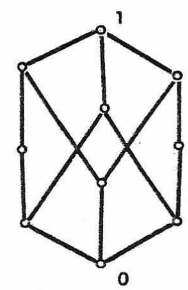

FIGURE 3

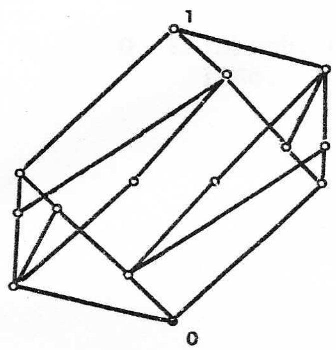

FIGURE 4

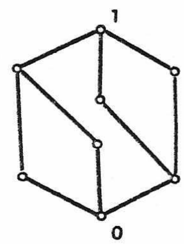

FIGURE 5

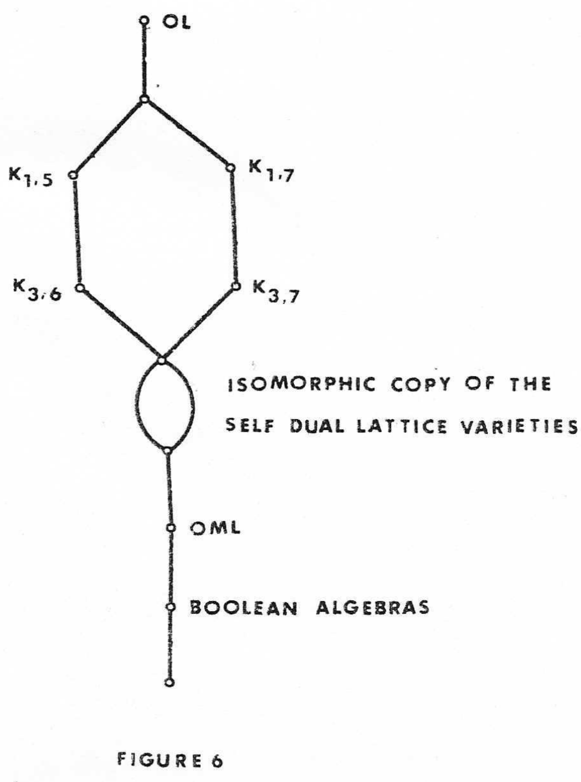

FIGURE 6

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