# Modal Compact Hausdorff Spaces I

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# Overview

Our aim is to lift aspects of modal logic from the setting of zero-dimensional compact Hausdorff spaces, to that of general compact Hausdorff spaces.

The talk will be in two parts.

# Preliminaries — Modal Logic

Definition A modal space $(X, R)$ is a pair where $X$ is a Stone space and $R$ is a relation on $X$ that satisfies

1. $R[x]$ is closed for each $x \in X$
2. $R^{-1}U$ is clopen for each clopen $U \subseteq X$.

Definition A modal algebra $(B, \Diamond)$ is a pair where $B$ is a Boolean algebra and $\Diamond$ is an operation that satisfies

1. $\Diamond 0 = 0$
2. $\Diamond(x \vee y) = \Diamond x \vee \Diamond y$.

# Preliminaries — Modal Logic

Proposition For $(X, R)$ a modal space $(\text{Clop } X, \Diamond)$ is a modal algebra where $\Diamond U = R^{-1}U$.

Proposition For $(B, f)$ a modal algebra $(B^*, R)$ is a modal space where $B^*$ is the space of prime filters and $p\,R\,q \Leftrightarrow \Diamond q \subseteq p$.

These extend to a dual equivalence between the category MS of modal spaces and continuous p-morphisms $(f \circ R = R \circ f)$, and the category MA of modal algebras and their homomorphisms.

## Preliminaries — Coalgebras and Modal Logic

Definition For $X$ a Stone space, let $\mathcal{F}X$ be its closed sets and for each open $U \subseteq X$ define

$$\begin{aligned} \Diamond U &= \{F \in \mathcal{F}X : F \cap U \neq \varnothing\} \\ \Box U &= \{F \in \mathcal{F}X : F \subseteq U\} \end{aligned}$$

The Vietoris space $\mathcal{V}X$ is the space on the set $\mathcal{F}X$ having the sets $\Diamond U$ and $\Box U$ where $U \subseteq X$ is open as a sub-basis.

Theorem $\mathcal{V}$ extends to a functor from the category Stone of Stone spaces to itself.

# Preliminaries — Coalgebras and Modal Logic

Relations $R$ on a set $X$ correspond to functions from $X$ to its power set $\mathcal{P}X$ by considering $x \rightsquigarrow R[x]$.

Proposition For a relation $R$ on a Stone space $X$, TFAE

1. $(X, R)$ is a modal space.
2. The map $x \rightsquigarrow R[x]$ is a continuous map from $X$ to $\mathcal{V}X$.

Continuous maps from $X$ to $\mathcal{V}X$ are coalgebras for the Vietoris functor on Stone. They form a category with morphisms certain commuting squares, and this category is isomorphic to MS.

# Modal Compact Hausdorff Spaces

The Vietoris construction was introduced as a generalization of the Hausdorff metric to general Hausdorff spaces. It is well known that $\mathcal{V}$ yields a functor from the category KHaus of compact Hausdorff spaces to itself. This leads naturally to the following ...

Definition A modal compact Hausdorff space is pair $(X, R)$ where

1. $X$ is a compact Hausdorff space.
2. The map $x \rightsquigarrow R[x]$ is a continuous map from $X$ to $\mathcal{V}X$.

In effect, MKH-spaces are defined to be concrete realizations of coalgebras for the Vietoris functor on KHaus. Indeed ...

# Modal Compact Hausdorff Spaces

Proposition With morphisms being continuous p-morphisms, the category MKHaus of modal compact Hausdorff spaces is equivalent to the category of coalgebras for the Vietoris functor $\mathcal{V}$ on KHaus.

For a more direct description ...

Proposition For $R$ a relation on a compact Hausdorff space $X$, then $(X, R)$ is an MKH-space iff

1. $R[x]$ is closed for each $x \in X$.
2. $R^{-1}U$ is open for each open $U \subseteq X$.
3. $R^{-1}F$ is closed for each closed $F \subseteq X$.

# Modal Compact Hausdorff Spaces

Any modal space is a modal compact Hausdorff space, and one does not have to look hard to find new examples.

Example The real unit interval $[0, 1]$ with the usual topology and relation $\leq$ is an MKH-space that is not a modal space.

We seek algebraic counterparts to MKH-spaces to play the role modal algebras play for modal spaces. This is Part II.

First we pave the way ...

# Preliminaries — Isbell duality

Definition A compact regular frame $L$ is a frame whose top is compact and satisfies for each $a \in L$

$$a = \bigvee \{ b : b \prec a \}$$

Here $b \prec a$ means $\neg b \vee a = 1$ where $\neg b$ is pseudocomplement.

Example If $X$ is a KHaus space the frame $L = \Omega X$ of open sets is compact regular. Here $\neg B = \mathbf{I} - B$ and $B \prec A$ means $\mathbf{C}B \subseteq A$.

## Preliminaries — Isbell duality

Proposition In any compact regular frame

1. $0 \prec 0$
2. $1 \prec 1$
3. $a \prec b \Rightarrow a \leq b$
4. $a \leq b \prec c \leq d \Rightarrow a \prec d$
5. $a, b \prec c, d \Rightarrow a \vee b \prec c \wedge d$
6. $a \prec b \Rightarrow$ exists $c$ with $a \prec c \prec b$ (interpolation)
7. $a = \bigvee\{b : b \prec a\}$.

These are easy to see if we think of $\Omega X$.

Don't stare at them too long, the point is we'll see them again.

# Preliminaries — Isbell duality

Definition A point of $L$ is a frame homomorphism $p : L \to 2$. Let $\mathfrak{p}L$ be the points of $L$ topologized by the sets $\varphi(a) = \{p : p(a) = 1\}$.

Theorem (Isbell) The category KRFrm is dually equivalent to the category KHaus via the functors $\Omega$ and $\mathfrak{pt}$.

As a break from preliminaries, we provide an alternate approach ...

# Preliminaries — Alternate to Isbell duality

Definition For $L$ a KRFrm and $A \subseteq L$ define

1. $\twoheaduparrow A = \{b : a \prec b \text{ for some } a \in A\}$
2. $\twoheaddownarrow A = \{b : b \prec a \text{ for some } a \in A\}$

Call $A$ a round filter if $A = \twoheaduparrow A$ and a round ideal if $A = \twoheaddownarrow A$.

Proposition The set of prime round filters of $L$ topologized by the sets $\varphi(a) = \{F : a \in F\}$ is homeomorphic to $\mathfrak{p}L$.

This gives an alternate path to Isbell duality quite like Stone duality. There is also a version with prime round ideals, and an analog to the prime ideal theorem.

# Preliminaries — de Vries duality

Definition A de Vries algebra $(A, \prec)$ is a complete Boolean algebra $A$ with relation $\prec$ that satisfies

1. $0 \prec 0$
2. $1 \prec 1$
3. $a \prec b \Rightarrow a \leq b$
4. $a \leq b \prec c \leq d \Rightarrow a \prec d$
5. $a, b \prec c, d \Rightarrow a \vee b \prec c \wedge d$
6. $a \prec b \Rightarrow$ exists $c$ with $a \prec c \prec b$ (interpolation)
7. $a = \bigvee\{b : b \prec a\}$.
8. $a \prec b \Rightarrow \neg b \prec \neg a$.

# Preliminaries — de Vries duality

For $X$ a compact Hausdorff space, the regular open sets ($U = \mathbf{IC}U$) form a complete Boolean algebra $\mathcal{RO}\,X$ with finite meets given by intersection and joins by **IC** applied to union.

Example $(\mathcal{RO}\,X, \prec)$ is a de Vries algebra where $U \prec V$ iff $\mathbf{C}U \subseteq V$.

This is the canonical example, as all occur this way. Roughly, Isbell says we can recover a compact Hausdorff $X$ from its open sets, de Vries that we can recover it from its regular opens and $\prec$.

Still, a few more examples help ...

# Preliminaries — de Vries duality

Example For B a complete Boolean algebra, $(B, \leq)$ is de Vries.

Example Let $B = \mathcal{P}\,\mathbb{N}$ be the power set of the natural numbers and define $S \prec T$ iff $S \subseteq T$ and at least one of $S, T$ is finite or cofinite.

Example For $B$ any Boolean algebra define $\prec$ on its MacNeille completion by $x \prec y$ iff there is $a \in B$ with $x \leq a \leq y$.

Note The third example includes the second.
Note We cannot recover $\prec$ from $B$ as with an MKR frame.

## Preliminaries — de Vries duality

Definition A filter $F$ of a de Vries algebra $A$ is round if $F = \Uparrow F$. The maximal round filters are called ends. The set $\mathcal{E}A$ of ends of $A$ is topologized by the basis of sets $\varphi(a) = \{E : a \in E\}$.

Theorem $\mathcal{E}A$ is a compact Hausdorff space whose de Vries algebra of regular open sets is isomorphic to $A$.

Lets turn this into a categorical equivalence ...

# Preliminaries — de Vries duality

Definition A morphism between de Vries algebras $A$ and $B$ is a map $f : A \to B$ that satisfies

1. $f(0) = 0$
2. $f(a \wedge b) = f(a) \wedge f(b)$
3. $a \prec b \Rightarrow \neg f(\neg a) \prec f(b)$.
4. $f(a) = \bigvee\{f(b) : b \prec a\}$

Definition The composite $f * g$ of de Vries morphisms is given by

$$(f * g)(a) = \bigvee\{(f \circ g)(b) : b \prec a\}$$

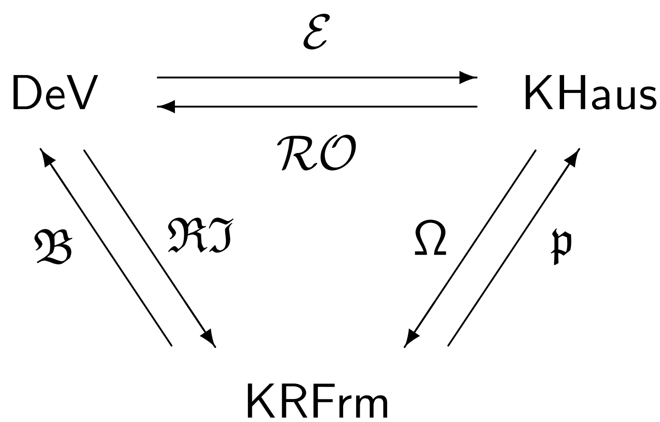
# Preliminaries — de Vries duality

Warning! Composition $*$ of de Vries morphisms is not the usual function composition, and de Vries morphisms are not necessarily even Boolean algebra homomorphisms. Odd things can happen.

Definition DeV is the category of de Vries algebras and their morphisms under the composition $*$.

Theorem DeV is dually equivalent to KHaus via the end functor $\mathcal{E}$ and regular open functor $\mathcal{RO}$.

# Summary of Dualities




$\mathfrak{B}$ = the functor taking all regular elements ($a = \neg\neg a$) of $L$
$\mathfrak{RI}$ = the frame of round ideals of a de Vries algebra.

# Dualities in the Stone Space Setting

For a Boolean algebra $B$ ...

- Its space $X$ is its Stone space and has a basis of clopens.
- Its frame $L$ is the ideal lattice of $B$. Its complemented elements are join dense.
- Its de Vries algebra $A$ is its MacNeille completion with $\prec$ as before. Its reflexive elements ($a \prec a$) are join dense.

# Extended Preliminaries — Vietoris for Frames

As KHaus and KRFrm are dually equivalent, the Vietoris functor $\mathcal{V}$ on KHaus transfers to a "Vietoris functor" $\mathcal{W}$ on KRFrm.

MKHaus is equivalent to the category of coalgebras for $\mathcal{V}$, hence is dually equivalent to the category of algebras for $\mathcal{W}$ ($\mathcal{W}L \to L$).

Johnstone has given a direct construction of $\mathcal{W}$ ...

## Extended Preliminaries — Vietoris for Frames

Theorem (Johnstone) For a frame $L$ let $\mathfrak{X} = \{\Box_a, \Diamond_a : a \in L\}$, $L^*$ be the free frame over $\mathfrak{X}$, and $\Theta$ the congruence generated by

1. $\Box_{a \wedge b} \;=\; \Box_a \wedge \Box_b$
2. $\Diamond_{a \vee b} \;=\; \Diamond_a \vee \Diamond_b$
3. $\Box_{a \vee b} \;\leq\; \Box_a \vee \Diamond_b$
4. $\Box_a \wedge \Diamond_b \;\leq\; \Diamond_{a \wedge b}$
5. $\Box_{\bigvee S} \;=\; \bigvee\{\Box_s : s \in S\}$ ($S$ directed in $L$)
6. $\Diamond_{\bigvee S} \;=\; \bigvee\{\Diamond_s : s \in S\}$ ($S$ directed in $L$)

Then $\mathcal{W} L = L^*/\Theta$ is the Vietoris frame of $L$.

Note Several identities are familiar from positive modal logic.

# Extended Preliminaries — Vietoris for de Vries?

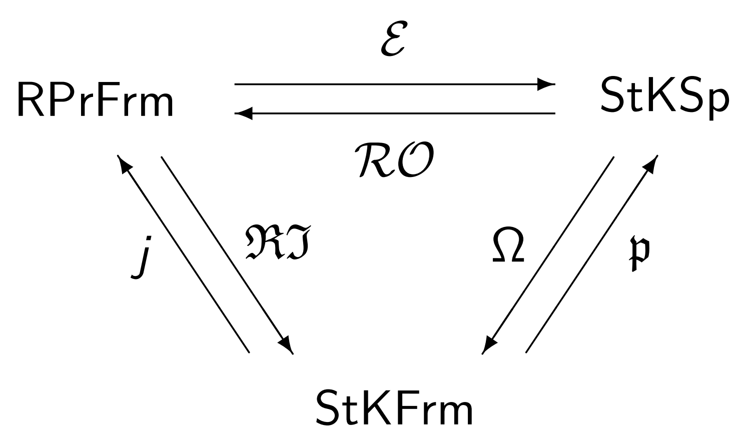
As DeV and KHaus are dually equivalent, there is a Vietoris functor $\mathcal{Z}$ for de Vries algebras corresponding to $\mathcal{V}$ for spaces.

MKHaus will be dually equivalent to the category of algebras $(\mathcal{Z}\,A \to A)$ for the Vietoris de Vries functor.

Problem Give a direct construction of $\mathcal{Z}$.

# Extended Preliminaries — Stably Compact Spaces

Before ending, we mention recent work (BH) extending these dualities to stably compact spaces and "regular proximity frames".



## Extended Preliminaries — Stably Compact Spaces

Recall, $X$ is stably compact if it is compact, locally compact, sober, and the intersection of two compact saturated sets is compact.

A proximity frame $\mathcal{L} = (L, \prec)$ is roughly a frame $L$ with proximity $\prec$ that satisfies the conditions for a de Vries algebra not involving $\neg$. Regularity is subtle and hard to describe quickly.

The functors are clear except $j$ and $\mathcal{RO}$. Here $j$ is "regularization" and $\mathcal{RO}$ is regular open meaning $A = \mathbf{I}_\tau \mathbf{C}_\pi A$ where $\pi =$ patch.

# Conclusions

This conclues the preliminaries. The plan is to lift this situation to the modal setting.

On to Part II ...

Thank you for listening.

Papers at www.math.nmsu.edu/~jharding

# Modal Compact Hausdorff Spaces II

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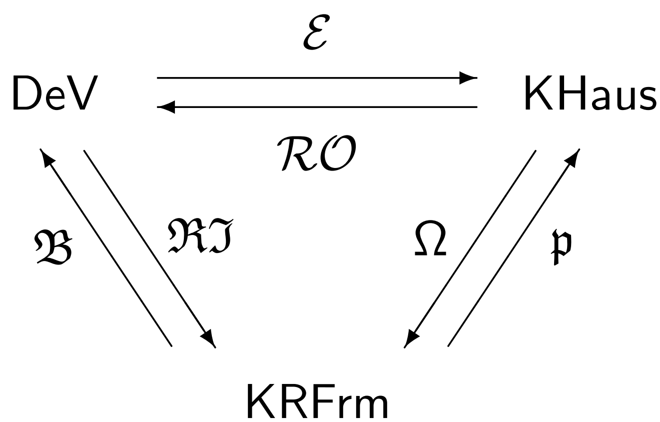
# Recall

We have the following equivalences and dual equivalences ...




We aim to lift these to the modal setting.

# Modal Compact Hausdorff Spaces

One part is already settled.

Definition Call $(X, R)$ a modal compact Hausdorff space if $X$ is a compact Hausdorff space and $R$ is a relation on $X$ that satisfies

1. $R[x]$ is closed for each $x \in X$.
2. $R^{-1}U$ is open for each open $U \subseteq X$.
3. $R^{-1}F$ is closed for each closed $F \subseteq X$.

Definition MKHaus is the category of modal compact Hausdorff spaces and continuous p-morphisms ($f \circ R = R \circ f$).

# Modal Compact Regular Frames

Definition Call $(L, \Box, \Diamond)$ a modal compact regular frame if $L$ is a compact regular frame with unary operations $\Box$ and $\Diamond$ satisfying

1. $\Box$ preserves finite meets and $\Diamond$ preserves finite joins
2. $\Box$ and $\Diamond$ preserve directed joins.
3. $\Box(a \vee b) \leq \Box a \vee \Diamond b$
4. $\Box a \wedge \Diamond b \leq \Diamond(a \vee b)$

Definition A morphism of MKR-frames is a frame homomorphism that satisfies $f(\Box a) = \Box fa$ and $f(\Diamond a) = \Diamond fa$.

Definition MKRFrm is the resulting category.

# Modal Compact Regular Frames

Note The operations $\Box$ and $\Diamond$ are definable from one another.

$$\Box b = \bigvee\{\neg \Diamond \neg a : a \prec b\}$$

$$\Diamond b = \bigvee\{\neg \Box \neg a : a \prec b\}$$

Note Removing the axioms about the underlying frame and those with directed joins, these axioms define positive modal algebras.

Note These axioms are closely related to the construction of the Vietoris frame functor $\mathcal{W}$. Not a surprise!

# Modal Compact Regular Frames

Recall $\mathcal{W}L$ is a quotient of the free frame over the set of fomral symbols $\{\Box_a, \Diamond_a : a \in L\}$ by a certain frame congruence $\Theta$.

Proposition For $L$ a compact regular frame and $f : \mathcal{W}L \to L$ a frame homomorphism define operations $\Box_f$ and $\Diamond_f$ on $L$ by

1. $\Box_f(a) = f(\Box_a)$
2. $\Diamond_f(a) = f(\Diamond_a)$

Then $(L, \Box_f, \Diamond_f)$ is an MKRFrm and all such arise this way.

Corollary MKRFrm is equivalent to the category of algebras for the Vietoris frame functor, hence dually equivalent to MKHaus.

# Modal Compact Regular Frames

We realize this duality directly by lifting the functors $\mathfrak{p}$ and $\Omega$.

Proposition For $(X, R)$ a modal compact Hausdorff space, define operations $\Box$ and $\Diamond$ on its frame $\Omega X$ of opens by

1. $\Box U = -R^{-1} - U$
2. $\Diamond U = R^{-1} U$

Then $(\Omega X, \Box, \Diamond)$ is a modal compact regular frame.

Proposition This extends to a functor $\Omega : \mathsf{MKHaus} \to \mathsf{MKRFrm}$.

# Modal Compact Regular Frames

The other direction is more nuanced ...

Proposition For $(L, \Box, \Diamond)$ a modal compact regular frame, define $R$ on its space of points by $p\,R\,q$ iff $q(a) = 1$ implies $p(\Diamond a) = 1$. Then $(\mathfrak{p}\,L, R)$ is a modal compact Hausdorff space.

Proposition This extends to a functor $\mathfrak{p} : \mathsf{MKRFrm} \to \mathsf{MKHaus}$.

# Modal Compact Regular Frames

Using our alternate to Isbell duality via prime round filters ...

Proposition For $(L, \Box, \Diamond)$ a modal compact regular frame, define $R$ on its space $X$ of prime round filters by $P\, R\, Q$ iff $Q \subseteq \Diamond^{-1} P$. Then $(X, R)$ is a modal compact Hausdorff space.

Note This approach may be more familiar to modal logicians.

Note Defining $P\, R\, Q$ iff $\Box^{-1} P \subseteq Q$ is equivalent to the above just as in modal logic (but not in positive modal logic). This may work here because $\Box$ and $\Diamond$ determine each other.

# Modal Compact Regular Frames

We have rather beaten it to death, but ...

Theorem MKRFrm and MKHaus are dually equivalent via $\mathfrak{p}, \Omega$.

# Modal de Vries Algebras

Definition A pair $(A, \Diamond)$ where $A$ is a de Vries algebra and $\Diamond$ is an operation on $A$ is a modal de Vries algebra if $\Diamond 0 = 0$ and

1. $a_1 \prec a_2$ and $b_1 \prec b_2 \Rightarrow \Diamond(a_1 \vee b_1) \prec \Diamond a_2 \vee \Diamond b_2$
2. $a_1 \prec a_2$ and $b_1 \prec b_2 \Rightarrow \Diamond a_1 \vee \Diamond b_1 \prec \Diamond(a_2 \vee b_2)$

We call such $\Diamond$ a de Vries additive operator.

Definition A modal de Vries morphism $f : A \to B$ is a de Vries morphism that satisfies

1. $a_1 \prec a_2 \Rightarrow f(\Diamond a_1) \prec \Diamond f(a_2)$
2. $a_1 \prec a_2 \Rightarrow \Diamond f(a_1) \prec f(\Diamond a_2)$

# Modal de Vries Algebras

Paradigm When adapting to the de Vries setting, we often replace an equation with a pair of "inequalities" involving proximity.

Example We replace $\Diamond(a \vee b) = \Diamond a \vee \Diamond b$ with

1. $a_1 \prec a_2$ and $b_1 \prec b_2 \Rightarrow \Diamond(a_1 \vee b_1) \prec \Diamond a_2 \vee \Diamond b_2$
2. $a_1 \prec a_2$ and $b_1 \prec b_2 \Rightarrow \Diamond a_1 \vee \Diamond b_1 \prec \Diamond(a_2 \vee b_2)$

Example We replace $f(\Diamond a) = \Diamond f(a)$ with

1. $a_1 \prec a_2 \Rightarrow f(\Diamond a_1) \prec \Diamond f(a_2)$
2. $a_1 \prec a_2 \Rightarrow \Diamond f(a_1) \prec f(\Diamond a_2)$

# Modal de Vries Algebras

Modal de Vries algebras can behave in unexpected ways.

Note An additive operation $\Diamond$ on a de Vries algebra need not be de Vries additive, and a de Vries additive operation $\Diamond$ need not be order preserving, let alone additive.

Example Let $A = \mathcal{P}(\mathbb{N})$ with $a \prec b$ iff $a \subseteq b$ and at least one is finite or cofinite. Set $\Diamond a = 0$ if $a$ is finite, $\Diamond a = 1$ if $a$ is cofinite, and let $\Diamond$ be some random bijection on the rest. Then $\Diamond$ is de Vries additive but not even order-preserving.

# Modal de Vries Algebras

Definition MDV is the category of modal de Vries algebras and their morphisms with composition $*$ de Vries morphisms.

The unusual nature of composition makes itself felt.

Definition A map $f$ between modal de Vries algebras is called a structural isomorphism if it is a Boolean algebra isomorphism where $a \prec b$ iff $f(a) \prec f(b)$ and $f(\Diamond a) = \Diamond f(a)$.

Proposition A structural isomorphism between modal de Vries algebras is an MDV isomorphism, but not conversely!

# Modal de Vries Algebras

Definition We say a modal de Vries algebra $A$ is

1. lower continuous if $\Diamond a = \bigvee\{\Diamond b : b \prec a\}$
2. upper continuous if $\Diamond a = \bigwedge\{\Diamond b : a \prec b\}$

Definition LMDV and UMDV be the full subcategories of lower and upper continuous modal de Vries algebras.

Proposition Each de Vries algebra is isomorphic to a lower continuous one and to an upper continuous one.

Proposition LMDV and UMDV are equivalent to MDV.

# Modal de Vries Algebras

Behavior is much better in LMDV and UMDV.

Proposition In LMDV and UMDV

1. $\Diamond$ is order preserving.
2. Isomorphisms are structural isomorphisms.

Proposition Each member of UMDV is a modal algebra in the ordinary sense if we forget its proximity.

Back to extending our dualities ...

# From MDV to MKHaus

Recall for a de Vries algebra $A$, its ends $\mathcal{E}A$ (maximal round filters) form a compact Hausdorff space. Much as with modal algebras ...

Proposition For a modal de Vries algebra, define $R$ on its ends by $xRy \Leftrightarrow \Diamond[y] \subseteq x$. Then $(\mathcal{E}A, R)$ is a modal compact Hausdorff space.

Proposition $\mathcal{E}$ extends to a functor from MDV to MKHaus.

# From MKHaus to MDV

Proposition For $(X, R)$ an MKH space, define $\Diamond^L$ and $\Diamond^U$ on its de Vries algebra of regular open sets by setting

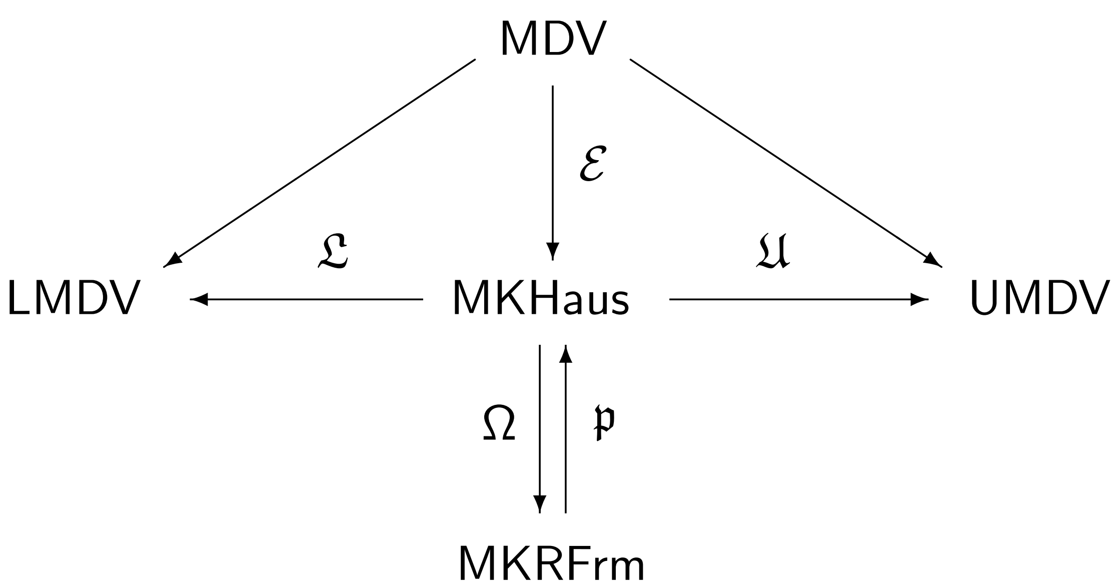
1. $\Diamond^L U = \mathbf{IC}R^{-1}U$
2. $\Diamond^U U = \mathbf{I}R^{-1}\mathbf{C}U$.

Then $(\mathcal{RO}X, \Box^L)$ and $(\mathcal{RO}X, \Box^U)$ are lower and upper continuous modal de Vries algebras.

Proposition These extend to functors $\mathfrak{L}$ and $\mathfrak{U}$ from MKHaus to LMDV and UMDV.

# Dualities in the Modal Setting

Theorem MKHaus is dually equivalent to each of the categories MKRFrm, LMDV, UMDV and MDV.



# Dualities in the Modal Setting

Corollary MKRFrm is equivalent to each of the categories LMDV, UMDV and MDV.

Remark There are direct, choice free proofs of these equivalences by lifting the round ideal functor $\mathfrak{R}$ and Booleanization functor $\mathfrak{B}$ to the modal setting.

We skip the details here.

# Links to completions

For a lattice $L$ with additional order preserving operations, one extends operations to the ideal completion from below

$$\overline{f}(x) = \bigvee\{f(a) : a \leq x\}$$

One extends operations to the MacNeille completion either by approximating from below or above (the lower and upper MC).

Note Ideal completions preserve all identities. Preservation of identities under lower or upper MacNeille completions is much more troublesome. We return to this.

# Links to completions

Proposition For a modal algebra $A$ with modal space $X$

1. The MKR frame of $X$ is the ideal completion of $A$.
2. The LMDV algebra of $X$ is the lower MC of $A$.
3. The UMDV algebra of $X$ is the upper MC of $A$.

For 2 and 3 the proximity is lifted to the MC as discussed before.

# Logical Aspects

Our results are a start to what one might hope. Take the positive fragment of propositional modal logic with $\wedge, \vee, \Box, \Diamond, 0, 1$ primitive.

Suppose $\varphi(\vec{x})$ is a positive modal formula.

Definition If $L$ is a MKR frame and $\vec{a}$ are elements in $L$, define $\varphi(\vec{a})$ to be the result of substituting $\vec{a}$ in the term $\varphi$.

Definition If $A$ is a MDV algebra and $\vec{a}$ are elements in $A$, define $\varphi(\vec{a})$ to be the result of substituting $\vec{a}$ in the term $\varphi$.

Definition If $X$ is a MKH space and $\vec{U}$ are open sets, so elements of $\Omega X$, let $\varphi(\vec{U})$ be the result of substituting $\vec{U}$ in the term $\varphi$.

## Satisfaction

Definition For formulas $\varphi, \psi$ define

1. $L \vDash \varphi \vdash \psi$ iff $\varphi(\vec{a}) \leq \psi(\vec{a})$ for each $\vec{a} \in L$.
2. $A \vDash \varphi \vdash \psi$ iff $\varphi(\vec{a}) \prec \psi(\vec{b})$ for each $\vec{a} \prec \vec{b} \in A$.
3. $X \vDash \varphi \vdash \psi$ iff $\varphi(\vec{U}) \leq \psi(\vec{U})$ for each $\vec{U} \in \Omega X$.

Proposition If $X$ is a MKH space, $L$ is its MKR frame, $A$ and $B$ are its LMDV and UMDV algebras, these are equivalent.

1. $X \vDash \varphi \vdash \psi$
2. $L \vDash \varphi \vdash \psi$
3. $A \vDash \varphi \vdash \psi$
4. $B \vDash \varphi \vdash \psi$

# Satisfaction

If $B$ is a modal algebra, the LMDV and UMDV algebras associated to its dual space are its lower and upper MC's.

The above says the lower and upper MC satisfy the same $\varphi \vdash \psi$. One is a modal algebra, the other not. How can this be?

Satisfaction is only up to proximity. So the lower and upper MC's satisfy the same sequents produced by "doubling" our identities according to our paradigm.

# A Sahlqvist Theorem

Note $X \vDash \varphi \vdash \psi$ uses assignments of variables to open sets of $X$. Having the underlying Kripke frame $K$ satisfy $\varphi \vdash \psi$ uses assignments to arbitrary subsets of $X$.

Definition $\varphi \vdash \psi$ is Sahlqvist if $\varphi$ is built from $\Box^n$ applied to variables or constants using $\wedge$ and $\Diamond$.

Theorem If $\varphi \vdash \psi$ is a Sahlqvist sequent, these are equivalent.

1. $X \vDash \varphi \vdash \psi$
2. $K \vDash \varphi \vdash \psi$ where $K$ is the Kripke frame underlying $X$.

Proof $2 \Rightarrow 1$ is trivial. We show $\neg\, 2 \Rightarrow \neg\, 1$.

1. Say $x \in \varphi(\vec{S})$ and $x \notin \psi(\vec{S})$ for some sets $\vec{S}$.

2. Show $x \in \varphi(\vec{S}) \Rightarrow x \in \varphi(\vec{F})$ for some closed $\vec{F} \subseteq \vec{S}$.

   This is the method of "closed assignments" for Sahlqvist formulas using each $R^n[x]$ closed and $x \in \Box^n S$ iff $R^n[x] \subseteq S$.

3. Then $x \in \varphi(\vec{F})$ and $x \notin \psi(\vec{F})$ for some closed $\vec{F}$.

4. $\psi(\vec{F}) = \bigcap\{\psi(\vec{U}) : \vec{F} \subseteq \vec{U} \text{ open}\}$

   This is the key step. It is proved by induction using infinite distributivity, continuity of $R$, and Esakia's Lemma that $R^{-1}$ commutes with directed intersections of closed sets.

5. Then $x \in \varphi(\vec{U})$ and $x \notin \psi(\vec{U})$ for some open $\vec{U}$.

# A Sahlqvist Theorem

We can combine this with standard results from modal logic.

Corollary For $\varphi \vdash \psi$ Sahlqvist, there is a first order formula $\Phi$ in the language with one binary relation so that these are equivalent

1. $X \vDash \varphi \vdash \psi$
2. $X$ satisfies $\Phi$

Proposition For a modal compact Hausdorff space $X$

1. $R$ is reflexive iff $X \vDash a \leq \Diamond a$
2. $R$ is symmetric iff $X \vDash a \leq \Box\Diamond\, a$
3. $R$ is transitive iff $X \vDash \Diamond\Diamond\, a \leq \Diamond a$

# Directions for Further Work

A few directions and problems to consider.

1. Consider modalities for other categories equivalent to KHaus.
2. Give a direct construction of the Vietoris functor on DeV.
3. Extend our dualities to stably compact spaces.
4. We showed $X$ and its unerlying Kripke frame satisfy the same Sahlqvist sequents. How about positive sequents?
5. What is the logic of $([0,1], \tau, \leq)$?
6. Consider languages with $\neg$ or infinite disjunctions $\bigvee$.

Thank you for listening.

Papers at www.math.nmsu.edu/~jharding