An operational view of Schrödinger's equation

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Introduction

We give an operational view of observables and the time independent Schrödinger equation as given the standard treatment of quantum mechanics of say Prugovecki.

One can view this from a pedagogical perspective — as a means to motivate the standard treatment from more basic assumptions.

The standards

Pure states: are unit vectors v in a separable Hilbert space $\mathcal{H}$

Observables: are densely defined $A : \mathcal{H} \to \mathcal{H}$ such that $A = A^\dagger$.

The time independent Schrödinger's equation:

If H is the observable for energy, then the one-parameter family of evolution operators U_t $t \in \mathbb{R}$ on states is given by

$$U_t = e^{-iHt}$$

Here U_t is a unitary operator with $U_t(v)$ the state of the system at time t given that the state at time 0 is v.

Observable

What is an observable such as "position"

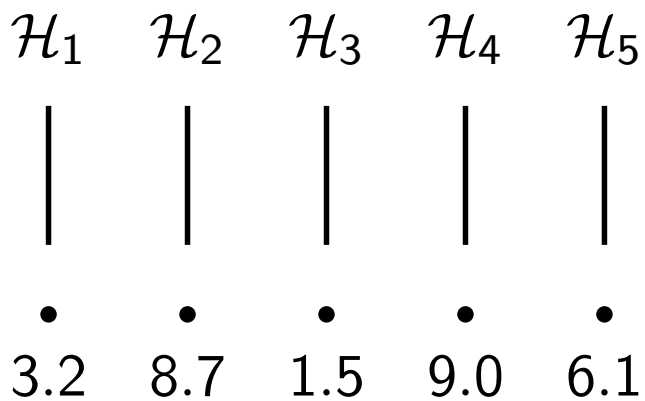
Not so easy to answer because of things like the 2-slit experiment. We take a very cautious path, even separating the thing we measure from our assignment of numerical value to it.

Observable quantity: A thing that can be measured by a finitary experiment.

Scaling: An assignment of numerical values to the outcomes of a finitary experiment.

First assumption

An n-ary experiment corresponds to an n-ary direct product decomposition $\mathcal{H} \simeq \mathcal{H}_1 \times \cdots \times \mathcal{H}_n$.



This $\mathcal{H} \simeq \mathcal{H}_1 \times \cdots \times \mathcal{H}_n$ is the usual orthogonal decomposition via projectors $P_1, \ldots, P_n$.

Interpretation

It is easy enough to translate this into self-adjoint operators. Let $\lambda_1, \ldots, \lambda_n$ be the numerical values attached to the n outcomes of the experiment. Set

$$A = \lambda_1 P_1 + \cdots + \lambda_n P_n$$

Then A is a self-adjoint operator, and for each unit vector v, the probability of the i^{th} outcome when in state v is $\| P_i v \|^2$.

Note, there are no oddities here. The domain of our operator is all of $\mathcal{H}$, no infinities occur, everything is fine.

Second assumption

We assume that our system has a base "natural frequency".

$$E_t v = e^{-it}$$

A system initially in state v is in state $E_t v$ at time t.

This "natural frequency" assumes no outside influences.

Schrödinger equation

Suppose the observable quantity H for energy has n outcomes, corresponds to the decomposition $\mathcal{H} \simeq \mathcal{H}_1 \times \cdots \times \mathcal{H}_n$, and has scaling $\lambda_1, \ldots, \lambda_n$.

A vector v gives an n-tuple $(v_1, \ldots, v_n)$ where $v_i = P_i v$. Set

$$U_t v = (e^{-i\lambda_1} v_1, \ldots, e^{-i\lambda_n} v_n)$$

Components at higher energy vibrate more rapidly.

Comments on the finite outcome setting

We call observable quantities with finitely many outcomes, together with their scalings, physical observables.

They are easy to motivate physically.

Key point to me

Physical observables and their role dynamics are easily extended past the Hilbert space setting to other structures X equipped with a one-parameter family of evolution operators $(X, (E_t)_{\mathbb{R}})$.

Moving to general observables and dynamics

We will view general observables as idealizations, or limits, of the physically realizable physical observables.

We call these "idealized observables" and consider dynamics using these rather than a physical observable for energy.

Purpose this talk

To make precise the relationship between physical and idealized observables and their dynamics in an operationally motivated way.

Idealized observable quantities

A physical observable quantity corresponds to an n-ary experiment, hence to a decomposition $\mathcal{H} \simeq \mathcal{H}_1 \times \cdots \times H_n$.

This corresponds to a finite Boolean subalgebra B_F of the projection lattice, with the outcomes its atoms.

Ever finer experiments correspond to a directed family of Boolean algebras, their limit is a Boolean subalgebra of $Proj(\mathcal{H})$

An idealized observable quantity:

This is a Boolean subalgebra $B \leq Proj(\mathcal{H})$.

Scaling an idealized observable

A physical scaling associates numerical values to the n outcomes of an experiment.

What are "outcomes" of an idealized observable quantity B?

They are maximally consistent sets of outcomes of the physical observables quantities produce B, hence ultrafilters of B. So the set of outcomes of B is the Stone space Z of B.

Scaling an idealized observable

Theorem Let $\mathcal{A}$ be an abelian von Neumann algebra of operators on $\mathcal{H}$ with B its Boolean algebra of projections. The self-adjoint operators A affiliated with $\mathcal{A}$ are in bijective correspondence to the continuous functions $f : Z \to \mathbb{R} \cup \{\pm\infty\}$ that are real valued on a dense open set.

Further, a unit vector v induces a measure μ_v on Z and for the function f corresponding to A, the expected value of A when the system is in state v is given by $\int f \, d\mu_v$

This is the spectral theorem, as found in Kadisson + Ringrose.

Comments on Idealized observables

Self-adjoint operators correspond to certain limits of physical observables.

The partial domains and unboundedness of self-adjoint operators arise from this limiting process.

Self-adjoint operators are very special limits, B is complete, the function $f : Z \to \mathbb{R} \cup \{\pm\infty\}$ is even continuous rather than simply measurable, Do other limits play a role?

Dynamics

Dynamics using a physical observable for energy has a particularly simple form — writing $v = (v_1, \ldots, v_n)$, the components of v at higher energy vibrate more rapidly.

Can we precisely formulate the idea that the time-independent Schrödinger equation for an idealized operator H is a limiting version of this?

$$U_t = e^{-iHt}$$

Schrödinger's equation as a limit

Theorem

Let H be a self-adjoint operator. Then there is a sequence of physical observables H_n so that for any $v \in \mathcal{H}$ and $t \in \mathbb{R}$ we have

$$U_t^{H_n} v \longrightarrow e^{-iHt} v$$

Here $U_t^{H_n}$ is the evolution operator for the physical observable H_n.

Proof We sketch the key points.

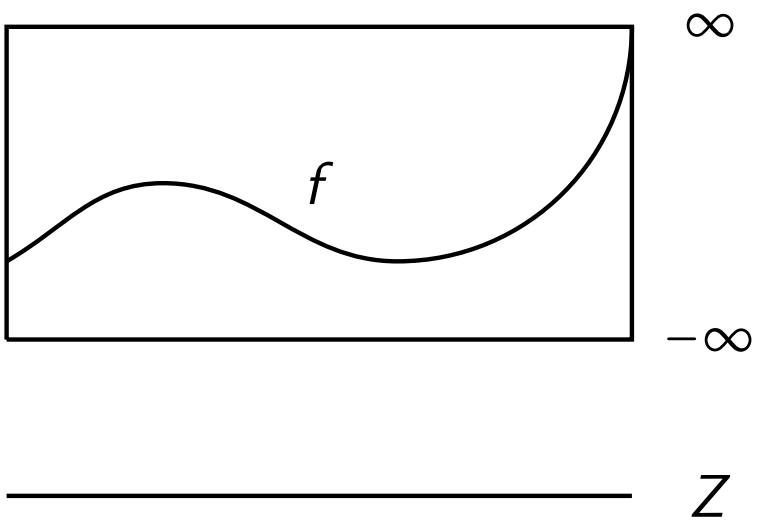
Let

$\mathcal{A}$ = an abelian v.n. algebra affiliated with H

B = the Boolean algebra of projections of $\mathcal{A}$

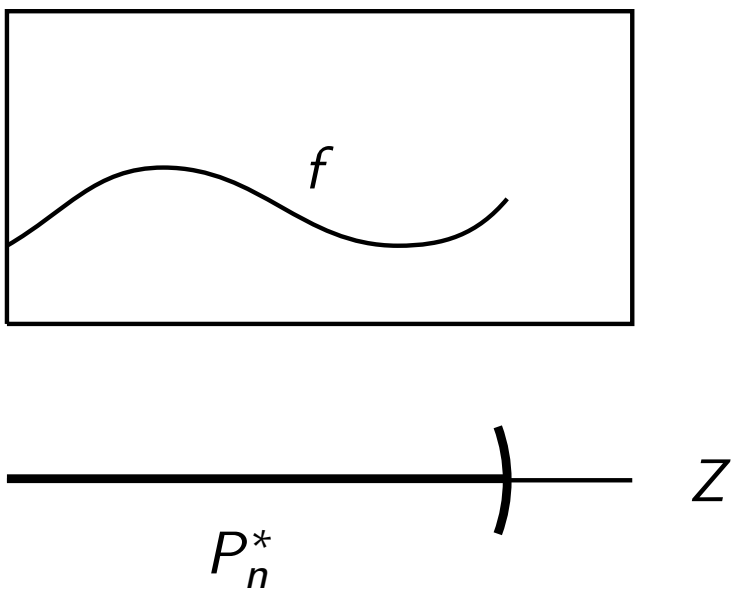
Z = the Stone space of B

f = the map from Z to $\mathbb{R} \cup \{\pm\infty\}$ corresponding to H



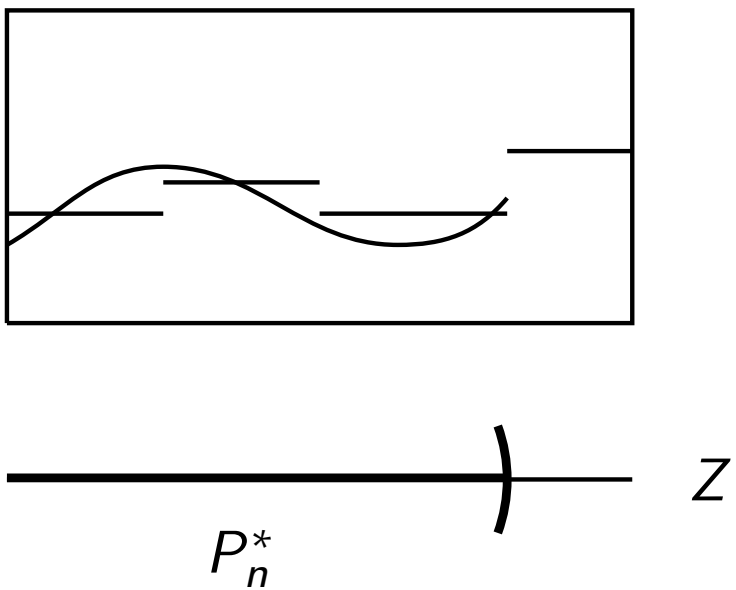
Since f is real valued on a dense open subset of Z we can find an increasing sequence P_n in B with

- $\bigvee P_n = 1$
- f real-valued on the clopen set P_n^* corresponding to P_n



For each n find a finitely valued f_n continuous on P_n^* so that

$$|\, f(z) - f_n(z) \,| < 1/n^2 \quad \text{for all } z \in P_n^*$$



Let H_n be the physical observable for f_n.

Then for any v with $P_n v = v$ and any $-n \leq t \leq n$

$$\| e^{-iHt} - e^{-iH_n t} \| < 1/n$$

It follows that for any v and any t that

$$\lim_{n \to \infty} e^{-iH_n t} v = e^{-iHt} v$$

Generalizations

Our interest is not pedagogy, but ways things could be different.

Alter the base "natural frequency"

The base "natural frequency" is $E_t v = e^{-it}$. Others could be used. Physical observables would be decompositions $\mathcal{H} \simeq \mathcal{H}_1 \times \cdots \times \mathcal{H}_n$ that are compatible with the "natural frequency"

Be looser with idealized observables

Now we require B complete and $f : Z \to \mathbb{R} \cup \{\pm\infty\}$ continuous. Incomplete B and measurable f would be the natural choice. Do these correspond to some known type of operators?

Generalizations

At the level of physical observables and their dynamics, very little of Hilbert spaces is required.

We need a structure X with a base "natural frequency" which is a one-parameter semigroup of endomorphisms. Our physical observables are decompositions $X \simeq X_1 \times \cdots \times X_n$.

Lets consider this when X is perhaps a normed group.

Thanks for listening.

Papers at www.math.nmsu.edu/~jharding