The Decompositions Approach to Quantum Mechanics

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The role of projection operators

In the standard Hilbert space formulation of QM, projections play a central role. Our key ingredients.

$\mathcal{Q}$ = the orthomodular lattice of projections of $\mathcal{H}$

$\mathcal{S}$ = the convex set of states

$\mathcal{O}$ = the observables

$\mathcal{B}$ = the Borel algebra of $\mathbb{R}$

$\mathcal{G}$ = a Lie group

The Spectral Theorem

Observables correspond to σ-homomorphisms $E : \mathcal{B} \to \mathcal{Q}$

Gleason's Theorem

States correspond to σ-additive $s : \mathcal{Q} \to [0, 1]$

Wigner's Theorem

Unitary and anti-unitary maps of $\mathcal{H}$ correspond to $\mathrm{Aut}(\mathcal{Q})$

The dynamical group of the system

Is a continuous group homomorphism $\mathrm{U} : \mathbb{R} \to \mathrm{Aut}(\mathcal{Q})$.

Stone's Theorem

Dynamical groups are given by $\mathrm{U}_t = e^{\mathrm{iHt}}$ for some $\mathrm{H} \in \mathcal{O}$ called the Hamiltonian. This is an abstract form of Schrödinger's equation.

Group Representations

A continuous homomorphism $\prod : \mathcal{G} \to \mathrm{Aut}(\mathcal{Q})$.

Note: Do not forget the topology — on $\mathcal{G}$ and $\mathcal{H}$ and $\mathcal{Q}$.

Program

- Replace $\mathcal{H}$ with another structure S.
- To build an omp $\mathcal{Q}$ from S.
- To use this as a basis of developing aspects of QM.

Aims

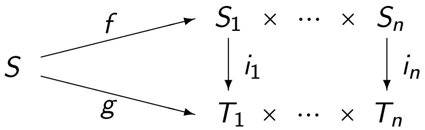
- Find structures that allow states, rich automorphism groups, topologies, and tensor products, etc.
- Give operational motivation for the components of QM
- Provide a setting to analyze why/if Hilbert space is the beating heart of QM.

Key idea

- View projections of $\mathcal{H}$ as direct product $\mathcal{H} \simeq \mathcal{H}_1 \times \mathcal{H}_2$.
- View superposition not as $u + v$ but as the ordered pair (u, v).

Definition An n-ary product map is an iso $f : S \longrightarrow S_1 \times \cdots \times S_n$.

Definition Two such maps are equivalent if there are iso's $i_1, \ldots, i_n$ making the following diagram commute.



Definition An n-ary decomposition of S is an equivalence class

$$[S \cong_f S_1 \times \cdots \times S_n]$$

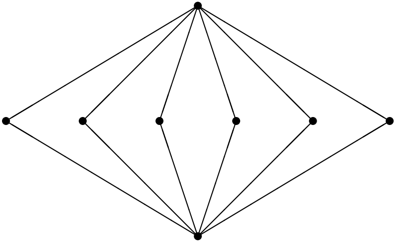
Definition $\mathcal{Q}(S)$ is all binary decompositions $[S \simeq S_1 \times S_2]$. Set

1. $0 = [S \cong \{*\} \times S]$
2. $1 = [S \cong S \times \{*\}]$
3. $\perp$ be the operation $[S \cong S_1 \times S_2]^{\perp} = [S \cong S_2 \times S_1]$
4. $\leq$ be the relation $[S \cong S_1 \times (S_2 \times S_3)] \leq [S \cong (S_1 \times S_2) \times S_3]$

Theorem $\mathcal{Q}(S)$ is an OMP in any of the following settings:

- sets
- sets with valuation $v : S \to [0, \infty)$
- G-sets
- groups, rings
- normed groups
- graphs
- topological spaces
- uniform spaces
- topological groups
- vector bundles (with or without inner product)
- An abstract object in a suitable type of category

Example — $S = \{a, b, c, d\}$ a 4-element set



$\mathcal{Q}(S)$

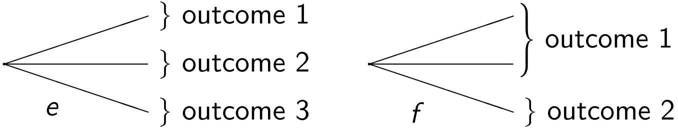
Physical interpretation – experiments

S represents a quantum system.

n-ary decompositions of S correspond to experiments with n outcomes, $\text{Outcome}_1, \ldots, \text{Outcome}_n$.

For an n-ary experiment e other experiments can be built from e.

Ex Ternary $e : S \to S_1 \times S_2 \times S_3$ gives binary $f : S \to (S_1 \times S_2) \times S_3$



Physical interpretation – observables

Cavemen know position means is it here, or is it here, or is it here.

- Position is a word for a family of compatible questions.
- Position in an interval can be measured. Position at a point is an ideal concept for a maximally consistent set of questions.
- Assigning numbers to "ideal questions" is called a scaling.

Definitions

1. An observable quantity is a Boolean subalg B of $\mathcal{Q}(S)$.
2. Ideal questions are points of the Stone space Z of B.
3. A scaling is a measurable map $f : Z \to \mathbb{R} \cup \{\pm\infty\}$.
4. An observable is an observable quantity + scaling

Notes

- Finite observable quantities $\mathcal{B}$ correspond to n-ary experiments
- $C(Z)$ gives a calculus of compatible observables A^2, e^A, $A + B$

Physical interpretation – states

A state is a (σ) additive map $\sigma : \mathcal{Q}(S) \to [0, 1]$

Setup B is an observable quantity with Stone space Z, scaling f

Proposition Each state σ gives a probability measure μ_σ on Z.

Definition

$$\begin{aligned} \mu_\sigma(f^{-1}(U)) &= \text{probability of a result in } U \text{ when in state } \sigma \\ \int_Z f \, d\mu_\sigma &= \text{the expected value} \end{aligned}$$

Note

The spectral theorem says that in the setting of Hilbert spaces, a self-adjoint operator A gives B, Z, f and all behaves as described.

Physical interpretation – automorphisms

The automorphism group Aut $\mathcal{Q}(S)$ gives symmetries of questions
A Wigner theorem gives the relation between Aut S and Aut $\mathcal{Q}(S)$.

Proposition There is a group homomorphism $\Gamma\colon \text{Aut } S \to \text{Aut } \mathcal{Q}(S)$.

Theorem For S an infinite set, Aut $S \equiv$ Aut $\mathcal{Q}(S)$.

Notes

- In the Hilbert space setting Γ is neither one-one or onto.
- Both kinds of defects of Γ affect group representations.
- Wigner's theorems are known for vector spaces groups, etc.
- Related to the Fundamental Theorem of Projective Geometry.
- The result for sets is quite complex.

Physical interpretation – group representations

Definition A representation of G in S is a group homomorphism

$$\Pi : G \to \text{Aut } S$$

Notes

- This amounts to enriching S to a structure $S^{\Pi} = (S, (\pi_g)_G)$.
- $\mathcal{Q}(S^{\Pi})$ again forms an omp and we can apply all so far to it.
- If our objects S lie in some category $\mathcal{C}$, then a representation of G is a functor from the 1-element category G to $\mathcal{C}$.
- $\mathcal{C}^G$ is the category of our enriched structures S^{Π}.
- Such $\Pi : G \to \text{Aut } S$ gives $\Pi' : G \to \text{Aut } \mathcal{Q}(S)$
- Continuity is of interest when $G, S, \mathcal{Q}(S)$ have topologies.

Physical interpretation – dynamics

Definition An internal clock of S is a representation $\mathrm{E} : \mathbb{R} \to \mathrm{Aut}\ \ S$

Definition A Hamiltonian of S is an observable H of S^E associated with a finite scaling $\lambda_1, \ldots, \lambda_n$ and decomposition

$$S^E \simeq S_1^{E_1} \times \cdots \times S_n^{E_n}$$

Theorem A Hamiltonian H of a system S with internal clock E gives a dynamical group $\mathrm{U} : \mathbb{R} \to \mathrm{Aut}\ S^E$ where

$$\mathrm{U}(t) = \mathrm{E}_1(\lambda_1 t) \times \cdots \times \mathrm{E}_n(\lambda_n t)$$

Notes

- In the usual Hilbert space setting let $\mathrm{E}_t(v) = e^{it} v$.
- Topological structure on S allows infinite Hamiltonians.
- A clock gives a “natural frequency”.
- At higher energy things vibrate more rapidly.

Example

Now suppose that some observable we call the Hamiltonian has the following decomposition and scaling.

$$
\begin{array}{ccccc}
S_1 & S_2 & S_3 & S_4 & S_5 \\
| & | & | & | & | \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
3 & 8 & 1 & 9 & 6
\end{array}
$$

Then the dynamical operator U of the system takes has

$$U_t(a_1, \ldots, a_5) = (\, E_1(3t)(a_1), \ldots, E_5(6t)\,(a_5)\,)$$

Physical interpretation – compound systems

For systems with structures S_1, S_2 we want a structure S for the compound system so that

1. There is $f : \mathcal{Q}(S_1) \times \mathcal{Q}(S_2) \to \mathcal{Q}(S)$
2. This f preserves orthogonal joins in each argument
3. For states σ_i of $\mathcal{Q}(S_i)$, there is a state ω of $\mathcal{Q}(S)$ with

$$\omega(f(q_1, q_2)) = \sigma_1(q_1)\sigma_2(q_2)$$

Note

These requirements are realized with a suitable monoidal structure on the category of structures.

So where are we left ...

Many features come "for free"

- The structure of questions
- Automorphisms
- Observables
- Dynamics
- These come with operational motivation.

The primary issues are states and compound systems

- Analytic structure on S seems needed to get enough states.
- Monoidal structure on $\mathcal{C}$ seems needed for compound systems.

Two interesting settings to consider

Normed groups with operators and vector bundles both have the following features.

- A rich supply of states.
- An underlying monoidal structure.
- Close to Hilbert setting, yet significantly more general.

The underlying mathematics can become challenging. A Gleason theorem for trivial bundles was partly solved by T. Yang.

Thanks for listening.

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