Convolution algebras

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Overview

This is a story that begins in one place, and winds through other topics. The aim is an overview, rather than full detail.

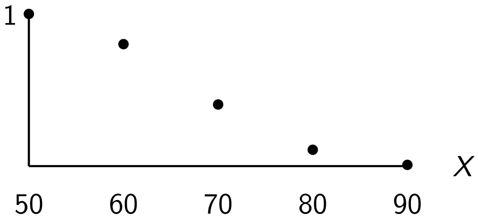
This work was done with Carol and Elbert Walker.

Type-1 fuzzy subsets

$X = \{50, 60, 70, 80, 90\}$

A type-1 fuzzy subset of X is a map $\textsc{Cold} : X \to [0, 1]$

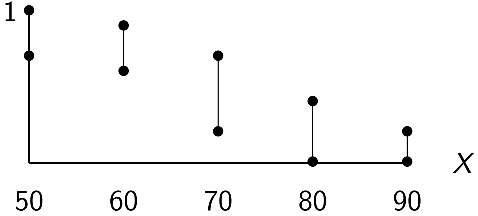
The expert's opinion that 60 degrees F is cold is 0.8.



Interval valued fuzzy subsets

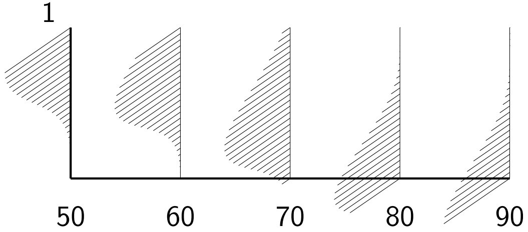
This is a map $\text{COLD} : X \to \{(a,b) \in [0,1]^2 : a \le b\}$.

The expert's opinion that 60 F is cold is between [0.6,0.9].



Type-2 fuzzy subsets

A type-2 fuzzy subset is $\textsc{Cold} : X \to \{f \mid f : [0,1] \to [0,1]\}$



Truth value algebras

The truth value algebras for fuzzy sets, interval valued fuzzy sets, and type-2 fuzzy sets are

$$\mathbb{I} = [0,1]$$

$$\mathbb{I}^{[2]} = \{(a,b) : a \leq b \in \mathbb{I}\}$$

$$\mathbf{M} = \{f \mid f : \mathbb{I} \to \mathbb{I}\}$$

$\mathbb{I}$ and $\mathbb{I}^{[2]}$ sit in $\mathbf{M}$ as characteristic functions of points and intervals

Operations

I and $\mathsf{I}^{[2]}$ have naturally defined operations of $\wedge, \vee, *, 0, 1$ making them De Morgan algebras.

Definition (Zadeh) Define the following operations on M

1. $(f \sqcap g)(x) = \bigvee\{(f(y) \wedge g(z)) : y \wedge z = x\}$
2. $(f \sqcup g)(x) = \bigvee\{(f(y) \wedge g(z)) : y \vee z = x\}$
3. $f^*(x) = f(1 - x) = \bigvee\{f(y) : y^* = x\}$
4. $0(x) = 1$ if $x = 0$ and 0 otherwise
5. $1(x) = 1$ if $x = 1$ and 0 otherwise

These are convolutions of the corresponding operations on I.

(For polynomials $(p \cdot q)(n) = \sum\{p(i) \cdot q(j) : i + j = n\}$)

Equations

Theorem M satisfies the equations for De Morgan algebras except that absorption and distributivity are weakened to the following.

1. $x \sqcap (x \sqcup y) = x \sqcup (x \sqcap y)$
2. $(x \sqcap y) \sqcup (x \sqcap z) \sqcup (y \sqcap z) = (x \sqcup y) \sqcap (x \sqcup z) \sqcap (y \sqcup z)$

M is not a lattice.

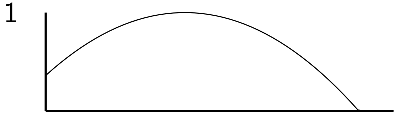
The unbalanced distributive laws do not hold.

M is a type of thing known as a De Morgan Birkhoff system.

We return later to see why these things are true.

A related algebra

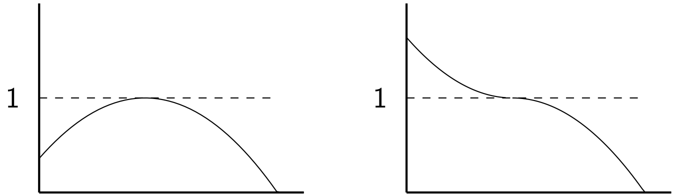
Definition A function $f : \mathbf{I} \to \mathbf{I}$ is convex normal if it goes up to 1, then down.



Convex normal functions are a not too restrictive setting for our desired use as belief functions.

A related algebra

We can "straighten" each convex normal function by reflecting its increasing part in the line $y = 1$.



Straightening takes the convolution operations to the pointwise operations on the lattice of decreasing functions from $\mathbb{I}$ to $[0, 2]$.

A related algebra

Theorem The convex normal functions are a subalgebra of M. For the quotient L of this subalgebra modulo agreement c.a.e.

1. L is a complete, completely distributive DeMorgan algebra
2. L is a compact Hausdorff topological algebra
3. $\int_0^1 |f(x) - g(x)|\, dx$ is a metric on it

Further, the convolution $\triangle$ of any continuous t-norm on I gives a commutative quantale structure $(\mathsf{L}, \triangle, \bigvee)$.

Interlude

We began in a practical place, with the truth value algebra M for type-2 fuzzy sets. This was vaguely tied to convolutions.

Let's return to convolutions in a more deliberate way ...

Definition A relational structure $\mathfrak{X} = (X, (R_i)_I)$ is a set X with a family R_i of n_i-ary relations on it.

Note: A binary operation $+ : X \times X \to X$ is a ternary relation.

The convolution algebra

Definition For a relational structure $\mathfrak{X}$ and complete lattice L, the convolution algebra $L^{\mathfrak{X}}$ is the lattice L^X with additional operations f_i is given by

$$f_i(\alpha_1,\ldots,\alpha_n)(x) = \bigvee\{\alpha_1(x_1)\wedge\cdots\wedge\alpha_{n_i}(x_{n_i}) : (x_1,\ldots,x_{n_i},x)\in R_i\}$$

The dual convolution algebra $L^{\mathfrak{X}-}$ has additional operations

$$g_i(\alpha_1,\ldots,\alpha_n)(x) = \bigwedge\{\alpha_1(x_1)\vee\cdots\vee\alpha_{n_i}(x_{n_i}) : (x_1,\ldots,x_{n_i},x)\in R_i\}$$

The double convolution algebra has both $L^{\mathfrak{X}*} = (L^X,(f_i)_I,(g_i)_I)$.

Our motivating example

Out type-2 fuzzy truth value algebra M is the convolution algebra

$$\mathsf{M} = \mathsf{I}^{\mathsf{I}}$$

where I on the bottom is the unit interval as a lattice and I in the exponent is the unit interval $(\mathsf{I}, \vee, \wedge, *, 0, 1)$ with operations viewed as ternary, binary, and unary relations. Here $x^* = 1 - x$.

We give another well-known motivating example ...

Complex algebras

If you consider a group $\mathcal{G} = (G, \cdot, ^{-1}, e)$ as a relational structure with a ternary, binary, and unary relation, its complex algebra is the usual group complex $\mathcal{G}^+$ where

$$\begin{aligned} A ; B &= \{ab : a \in A, b \in B\} \\ A^{\smile} &= \{a^{-1} : a \in A\} \\ 1' &= \{e\} \end{aligned}$$

These have been studied apparently since Frobenius. They are basic examples of relation algebras.

Complex algebras

For a relational structure $\mathfrak{X}$, its complex algebra $\mathfrak{X}^+$ is the power set $\mathcal{P}(X)$ with operations f_i where

$$f_i(A_i, \ldots, A_{n_i}) = \{x : \exists(x_1, \ldots, x_{n_i}, x) \in R_i \text{ with } x_j \in A_j \text{ each } j\}$$

Its dual complex algebra $\mathfrak{X}^-$ is $(\mathcal{P}(X), (g_i)_I)$ where

$$g_i(A_1, \ldots, A_{n_i}) = \{x : \forall(x_1, \ldots, x_{n_i}, x) \in R_i \Rightarrow \exists j \text{ with } x_j \in A_j\}$$

Its double complex algebra $\mathfrak{X}^*$ is $(\mathcal{P}(X), (f_i)_I, (g_i)_I)$.

Fundamental relationship

Proposition For $\mathfrak{X}$ a relational structure

1. $\mathfrak{X}^+ \simeq 2^{\mathfrak{X}}$
2. $\mathfrak{X}^- \simeq 2^{\mathfrak{X}-}$
3. $\mathfrak{X}^* \simeq 2^{\mathfrak{X}*}$

Theorem If L is a non-trivial complete lattice

1. L a frame $\Rightarrow$ $\mathfrak{X}^+$ and $L^{\mathfrak{X}}$ satisfy the same equations.
2. L a dual frame $\Rightarrow$ $\mathfrak{X}^-$ and $L^{\mathfrak{X}-}$ "
3. L completely distributive $\Rightarrow$ $\mathfrak{X}^*$ and $L^{\mathfrak{X}*}$ "

Note This provides the properties of M described earlier by considering the complex algebra I^+ of the unit interval.

Categorical aspects

Definition Lat is the category whose objects are complete lattices and whose morphisms are maps that preserve $\wedge, \bigvee$. Let Lat^- have the same objects with maps that preserve $\vee, \bigwedge$ and Lat^* have the same objects with maps that preserve $\bigwedge, \bigvee$.

Definition Rel_τ is the category of relational structures of type τ with morphisms being p-morphisms.

Definition Alg_τ is the category whose objects are complete lattices with additional operations of type τ and homomorphisms preserving $\wedge, \vee$ and the additional operations.

Categorical aspects

Theorem There is a bifunctor, covariant in the first argument and contravariant in the second

$$\mathsf{Conv}(\,\cdot\,,\,\cdot\,) : \mathsf{Lat} \times \mathsf{Rel}_\tau \to \mathsf{Alg}_\tau$$

For objects it takes (L,$\mathfrak{X}$) to $\mathsf{L}^{\mathfrak{X}}$
For morphisms it takes f : L→ M, $p : \mathcal{Y} \to \mathfrak{X}$ to $\phi : \mathsf{L}^{\mathfrak{X}} \to \mathsf{M}^{\mathcal{Y}}$

$$\begin{array}{ccc} \mathfrak{X} & \xrightarrow{\;\;p\;\;} & \mathcal{Y} \\ \alpha \Big\downarrow & & \Big\downarrow \phi(\alpha) \\ \mathsf{L} & \xrightarrow{\;\;f\;\;} & \mathsf{M} \end{array}$$

Topoi

We go in one further direction, again motivated by our original algebra M of type-2 fuzzy truth values.

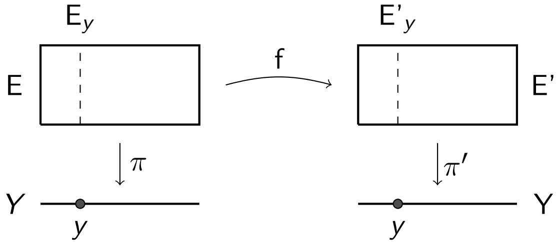
A topos is a type of category that has features similar to the category of sets.

Each topos has an “internal logic” describing how things appear within the topos.

Here we are interested in a particular type of topos, that of étalé spaces.

Étalé spaces

Definition An étalé space $\mathcal{E}$ over a topological space Y is a topological space E and a local homeomorphism π : E →Y.



A morphism of étalé spaces over Y is a continuous map f : E → E' mapping a "stalk" to the corresponding stalk.

Étalé spaces

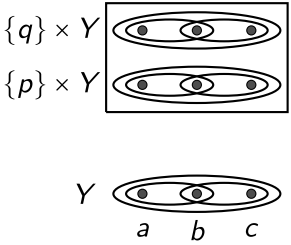
The category of étalé spaces is a topos. Its objects play the role of sets. In a way, they are "continuously varying sets".

Just as ordinary sets have subsets, an étalé has subobjects. There is an étalé that plays the role of the power set of a set, the power étalé $\mathcal{P}(\mathcal{E})$ whose "global elements" are subobjects of $\mathcal{E}$.

Theorem The subobjects of an étalé $\mathcal{E}$ are the open sets of E, naturally considered as étalés.

Étalé spaces

Definition For a set X, let the "constant étalé $\hat{X}$ is the space $X \times Y$ with obvious projection.

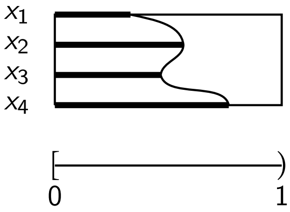


The constant étalé of $X = \{p, q\}$.

Definition Let Y be the topological space consisting of the set [0,1) with the half-intervals $[0,\lambda)$ as opens.

Note The frame $\mathcal{O}(Y)$ of open sets of Y is the unit interval [0,1].

Key observation (Höhle) Fuzzy subsets of X correspond to subobjects of the constant étalé $\hat{X}$ over Y.



So subobjects of the constant étalé $\hat{I}$ correspond to elements of M.

Relational étalés

An n-ary relation on a set X is a subset $R \subseteq X \times \cdots \times X$.

Definition An n-ary relation on an étalé $\mathcal{E}$ is a subobject of the product $\mathcal{E} \times \cdots \times \mathcal{E}$.

Definition A relational étalé is an étalé with a family of relations.

Definition For a relational étalé $\mathcal{E}$, the complex algebra $\mathcal{E}^+$ is the set of subobjects of $\mathcal{E}$ with operations given by relational image.

Theorem M is isomorphic to the complex algebra $\hat{I}^+$ of the constant relational étalé over I.

Further results

Theorem For Y a topological space, $\mathsf{L} = \mathcal{O}(\mathsf{Y})$ its lattice of open sets, and $\mathfrak{X}$ a relational structure

$$\mathsf{L}^{\mathfrak{X}} \simeq \hat{\mathfrak{X}}^{+}.$$

These are both external notions. A topos is like a set theory, it has a "internal" complex algebras as well created via power objects.

Theorem $\mathsf{L}^{\mathfrak{X}}$ is isomorphic to the algebra of global elements of the internal complex algebra of the constant étalé $\hat{\mathfrak{X}}$.

Corollary M is isomorphic to the global elements of the internal complex algebra of $\hat{\mathsf{I}}$.

Further results

It is tempting to conclude that the type-2 fuzzy truth value algebra is the internal complex algebra of the real unit interval $\mathbf{I}$ in the topos of étalés over Y.

Almost ... $\mathbf{I}$ is not the internal real unit interval in the topos!

These issues disappear if we modify our approach and replace the relational structure $\mathbf{I}$ with its rational counterpart. From a type-2 fuzzy standpoint, this seems to not be a large issue.

Thank You!