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The Truth Value Algebra of Type-2 Fuzzy Sets

Order Convolutions of Functions on the Unit Interval

Contents

Preface

Type-2 fuzzy sets, which have come to play an increasingly important role in applications, were introduced by Lotfi Zadeh in 1975, extending the notion of both ordinary fuzzy sets, and interval-valued fuzzy sets. The truth value algebras of each of these types of fuzzy sets are of particular interest. For example, they determine the operations on the other entities involved. For ordinary fuzzy sets and for interval-valued fuzzy sets, these algebras are well known and familiar. For type-2 fuzzy sets, the truth value algebra is rather complicated, with many features. This book is a detailed exposition of some of these features. Our interest in this algebra is twofold: as the truth value algebra of type-2 fuzzy sets, and as an algebra of interest in its own right.

The material in this book is based on results of many authors over the past 40 years, especially those of Dubois & Prade, Goguen, Karnik & Mendel, Kawaguchi & Miyakoshi, Mizumoto & Tanaka, Winter, as well as those of the authors. Our focus is on fundamental properties of the type-2 truth value algebra rather than applications of type-2 fuzzy sets. However, care is taken to illustrate how these basic properties are related to usage of the type-2 truth value algebra in applications. Recent books with a greater emphasis on applications include those of Castillo, Melin, and Mendel.

The study of the type-2 truth value algebra involves a range of mathematics that is of increasing use in fuzzy set theory in general. This includes the areas of lattice theory, universal algebra, and category theory. This book has been written to be self-contained and accessible to those with a standard undergraduate mathematics background. It is intended that this book could be used as a one-semester reading course for those wishing to learn not only about the type-2 truth value algebra, but about the surrounding mathematics. In essence, it is a treatment of various related disciplines from the perspective of one unifying theme. Various possibilities for such a course are allowed by the dependencies of the chapters, as shown below.

$$\begin{tikzcd}[arrows=dash]
 & & 1 \arrow[d] & & \\
 & & 2 \arrow[d] & & \\
 & & 3 \arrow[dll] \arrow[dl] \arrow[d] \arrow[dr] \arrow[drr] & & \\
4 & 5 & 6 \arrow[d] & 7 \arrow[d] & 10 \\
 & & 9 & 8 &
\end{tikzcd}$$

Chapter 1 begins with a brief overview of classical fuzzy sets and interval-valued fuzzy sets. It continues with a description of the algebra that is the subject of this book. Its elements consist of all mappings of the unit interval into itself, and its basic operations are convolutions of operations on the unit interval. Chapter 2 is devoted to developing the properties of those operations. There are two partial orders on the truth value algebra of type-2 fuzzy sets, given by the basic operations corresponding to union and intersection. These two operations are different, and neither is a lattice order. These partial orders are investigated in Chapter 2.

This algebra contains isomorphic copies of the truth value algebras of fuzzy sets and interval-valued fuzzy sets. These and other subalgebras, including the convex normal functions, are investigated in Chapter 3. A fundamental object associated with any algebra is its group of automorphisms, and this group is determined in Chapter 4. Some characteristic subalgebras are detailed there, including the ones mentioned above.

In ordinary fuzzy sets, t-norms and t-conorms are an important topic of interest. Chapter 5 investigates the convolutions of these operations to give analogous operations for type-2 fuzzy sets. These new operations satisfy many equalities, and these equalities are given in some detail.

The subalgebra of convex normal functions, and related subalgebras, is the focus of Chapter 6. These subalgebras are motivated by applications and give a tractable setting between the interval-valued fuzzy sets and the full type-2 setting. They have many desirable mathematical properties. The most appealing of them is a complete, completely distributive lattice with a compact Hausdorff metric space topology on which continuous t-norms from the unit interval lift to operations that preserve arbitrary joins and meets.

Chapters 7 and 8 deal with matters related to the axiomatics. Chapter 7 investigates the variety generated by the truth value algebra. It is shown that this variety is generated by a single finite 12-element algebra. The meet and join reduct of this algebra is generated by a 4-element algebra known as a bichain. This provides a decision procedure akin to the method of truth tables to determine when an equation holds in the truth value algebra. A syntactic decision procedure similar to finding a disjunctive normal form is also given for this task. In Chapter 8 we develop the theory of bichains in an effort to find an axiomatization of the algebra of truth values.

Chapter 9 gives a brief outline of, and motivation for, the use of Goguen categories of fuzzy relations in the study of ordinary fuzzy sets and fuzzy controllers. These are essentially categories of matrices with entries in the truth value algebra and underscore the connection between fuzzy controllers and linear algebra. Results of Chapter 6 show that a subalgebra of convex normal functions has properties needed to form a category of matrices with entries in this subalgebra, allowing a path to treat type-2 fuzzy controllers.

The final chapter, Chapter 10, treats an analog of the type-2 truth value algebra in the finite setting. Here the elements are all functions from one finite chain to another with operations again given by convolutions. Such algebras would be employed in practical situations implementing type-2 fuzzy sets.

There are several possibilities for reading this book or using it as a reading course. Chapters 1, 2, 3, 5, 6, 9 (5 and 6.8 could be omitted) provide the basics of the type-2 truth value algebra, its subalgebra of convex normal functions, and their application. It gives good exposure to lattice theory, analysis, category theory, and the basics of universal algebra. Chapters 1, 2, 3, 7, 8 (with either 4 or 10) also provide the basics of the truth value algebra, but from a more algebraic and axiomatic view. It gives good exposure to lattice theory and more advanced portions of universal algebra.

We thank Hung Nguyen and Mai Gehrke. Hung first suggested Elbert investigate the mathematics of fuzzy sets, and Mai introduced Elbert and Carol to connections between fuzzy set theory and universal algebra. We thank Sandor Jenei and Ram Prasad. Sandor suggested the use of equivalence almost everywhere for investigations of type-2 notions, and the key congruence c.a.e. of Chapter 6 grew from this. Results in Chapter 9 are the start of an ongoing investigation with Ram. We also thank Bob Stern who signed us for this book, and wish him joy in his retirement. We thank Mathematics Editor Sarfraz Kahn, Editorial Assistant Sherry Thomas, Project Coordinator Ashley Weinstein, Project Editor Karen Simon and the rest of the staff at Taylor & Francis for making the process of preparing this book for publication a very pleasant one.

Finally, portions of this book have been published earlier by the authors in journals and book chapters. We are grateful to several publishing companies for their permission to reuse this material. This includes Elsevier for [6, 44, 45, 46, 50, 108, 112, 115]; Springer for [47, 49]; Wiley for [113]; and World Scientific for [51, 109, 110, 111, 114].

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Author Biographies

John Harding earned his Ph.D. in mathematics from McMaster University in Hamilton in 1991. He spent two years as a postdoctoral fellow at Vanderbilt University supported by a National Sciences and Engineering Research Council fellowship, then three years as an assistant professor at Brandon University. John has been a permanent faculty member at New Mexico State University since 1996. He was promoted to full professor in 2005. John's area of specialty is order theory and its applications. His particular interests include applications to topology and logic, the foundations of quantum mechanics, completions, and fuzzy sets. He has authored or co-authored about 70 papers in this area. John currently serves on the editorial board of *Order*, the advisory board of *Mathematica Slovaca*, and as the president of the International Quantum Structures Association.

Carol Walker earned her Ph.D. in mathematics from New Mexico State University in 1963. She spent one year as a member of the Institute for Advanced Study in Princeton, supported by a National Science Foundation postdoctoral fellowship. She was a permanent faculty member at New Mexico State University from 1964 until her retirement in 1996. She served 14 years as department head and three years as associate dean of Arts and Sciences and director of the Arts and Sciences Research Center. Carol's area of specialty is algebra. Her particular interests include abelian group theory, applications of category theory to abelian groups and modules, and algebraic aspects of the mathematics of fuzzy sets. She has authored or co-authored more than 35 papers in these areas, as well as several textbooks and technical manuals.

Elbert Walker earned his Ph.D. in mathematics from the University of Kansas in 1955. He spent one year at the National Security Agency in Washington D.C., and one on the mathematics faculty at the University of Kansas. He joined the mathematics faculty at New Mexico State University in 1957, and retired as professor in 1987, after which he spent two years at the National Science Foundation in Washington D.C. His research interests include abelian group theory, statistics, and the mathematics of fuzzy sets and fuzzy logic. He has authored or coauthored about 95 research papers and several books in these areas.

Symbol List

Symbol	**Description**
$\wedge$ and $\vee$	min and max, or meet and join (p. 2).
$'$	negation, involution (p. 3).
$\leq$	partial order (p. 2).
$\mathbf{2}$	two element Boolean algebra (p. 3).
$\in$	membership symbol.
Y^X	all maps from X to Y, also written $\mathrm{Map}(X,Y)$ (p. 4).
$\{0,1\}$	two element set (p. 3).
$[0,1]$	unit interval (p. 4).
$\mathbb{I}$	unit interval, or algebra on unit interval (p. 4).
$\mathbb{I}^{[2]}$	intervals in $[0,1]$ (p. 6).
$\bigvee S$, $\bigwedge S$	least upper bound of S and greatest lower bound of S (p. 8).
$\sqcap$, $\sqcup$	convolution meet and join (p. 9).
*, 1_0, 1_1	convolution of $'$, 0, and 1, respectively (p. 9).
$\mathbb{M}$	$\mathrm{Map}([0,1],[0,1])$ with appropriate operations (p. 10).
f^L	smallest increasing function above f (p. 10).
f^R	smallest decreasing function above f (p. 10).
$x \leq_\wedge y$	partial order in a semilattice (p. 21).
$x \leq_\vee y$	partial order in a semilattice (p. 21).
$\sqsubseteq_\sqcap$, $\sqsubseteq_\sqcup$	partial orders on $\mathbb{M}$ or $\mathrm{m^n}$ (pp. 28, 197).
1_a	characteristic function of a (p. 37).
$\mathbb{S}$	singletons (pp. 37, 72).
$1_{[a,b]}$	characteristic function of interval $[a,b]$ (p. 39).
$\mathbb{S}^{[2]}$	characteristic functions of closed intervals (p. 40).
$\mathbb{N}$	normal functions (pp. 41, 70, 195).
$\mathbb{C}$	convex functions (pp. 42, 70, 196).
$\mathbb{L}$	convex normal functions (pp. 45, 70, 197).
$\mathbb{K}$	convex normal and upper or lower functions (p. 47).
$\mathbb{LU}$	lower functions and upper functions (p. 46).
1_A	characteristic function of A (p. 50).
$\mathbb{E}$	characteristic functions of subsets of $[0,1]$ (p. 50).
$\mathrm{Supp}(f)$	support of f (p. 51).
$\mathrm{Aut}(A)$	automorphism group of an algebra A (p. 56).
p_A	product $p \cdot 1_A$ for $p \in [0,1]$ (p. 61).

p_a	product $p \cdot 1_a$ for $p \in [0,1]$ (p. 61).
$\mathrm{Perm}(S)$	permutation group of a set S (p. 61).
$\overline{1}$	constant function with value 1 (p. 67).
P	point functions in M (p. 72).
$\vartriangle$, $\triangledown$	t-norm and t-conorm (p. 83).
$\blacktriangle$, $\blacktriangledown$	convolution of t-norm $\vartriangle$ and t-conorm $\triangledown$ (p. 87).
$a \ll b$	a is way below b (p. 101).
LSC, USC	lower semicontinuous, upper semicontinuous (p. 101).
$\mathrm{I}^\dagger$	the closed interval [0,2] (p. 104).
$f^\dagger$	straightened version of f (pp. 104, 210).
D	certain decreasing functions (p. 106).
L_1	convex strictly normal functions (p. 106).
D_1	strictly normal functions in D (p. 107).
L_u	upper semicontinuous convex normal functions (p. 109).
D_u	band semicontinuous functions (p. 111).
X	decreasing functions from $[0,1]$ to $[0,2]$ (p. 111).
Θ	a relation on X (p. 112).
Φ	a relation on L (p. 114).
$\hat{f}$	least USC pointwise above f (p. 119).
$\bigsqcup$, $\bigsqcap$	join and meet in L_u (pp. 120, 123).
$\mathcal{V}$	variety of algebras (p. 128).
$\mathcal{V}(A)$	variety generated by A (p. 128).
φ_a	a homomorphism from M onto E (p. 134).
2^C	complex algebra of a bounded chain C (p. 135).
A^L, A^R	upset and downset of A (p. 135).
3	the three-element bounded chain $0 < u < 1$ (p. 137).
5	the five-element bounded chain with involution (p. 137).
$\mathrm{Con}(A)$	lattice of congruences on A (p. 145).
$\mathrm{Sub}(\mathcal{V})$	lattice of subvarieties of $\mathcal{V}$ (p. 147).
$\prod_I A_i$	product of algebras A_i $(i \in I)$ (p. 128).
Δ, ∇	smallest and largest congruence relations on an algebra (p. 146).
$\cdot$ and $+$	semilattice operations (p. 148).
$\leq_\cdot$	meet order on a Birkhoff system (p. 148).
$\leq_+$	join order on a Birkhoff system (p. 148).
B	a four element bichain (p. 148).
2_l, 2_s	two-element bichains (p. 149).
$3_l, \ldots, 3_s$	three-element bichains (p. 149).
BS	variety of Birkhoff systems (p. 150).
DL	variety of distributive lattices (p. 150).
SL	subvariety of BS with equal semilattice operations (p. 150).
mDB	subvariety of BS satisfying meet distributive law (p. 150).
jDB	subvariety of BS satisfying join distributive law (p. 150).
DB	subvariety of BS satisfying both distributive laws (p. 150).
BCh	variety generated by all bichains (p. 151).
δ_{xy}	Kronecker delta function (p. 166).

$R \circ S$	relational product (p. 166).
Id_X	identity relation on X (p. 167).
Set	category of sets (p. 167).
Mat	category of matrices (p. 168).
Mat_D	category of D-matrices (p. 171).
Rel_D	category of D-relations (p. 171).
FRel	category of fuzzy relations (p. 172).
2 FRel	category of type-2 fuzzy relations (p. 172).
$\ast$-product	matrix product (p. 172).
$\mathrm{FRel}_\vartriangle$	category using t-norm for composition (p. 172).
$2\,\mathrm{FRel}_\blacktriangle$	category using type-2 t-norm for composition (p. 172).
$\overline{\psi}$	center of mass (p. 177).
$A \otimes A'$	Kronecker product of matrices (p. 178).
$A \otimes_\vartriangle A'$	$\vartriangle$-Kronecker product of matrices (p. 179).
$\dagger$	operation on dagger category (p. 184).
$J(L)$	join irreducible elements of a lattice L (p. 190).
$D(P)$	all downsets of P (p. 191).
$p\!\downarrow$	principal downset generated by p (p. 191).
FDist	a category of finite distributive lattices (p. 192).
FPos	a category of finite posets (p. 192).
n	an algebra on the set $\{1, 2, \ldots, n\}$ (p. 193).
m^n	an algebra on the set $\mathrm{Map}(\mathrm{n}, \mathrm{m})$ (p. 193).
N_k	the set of functions of height k (p. 195).
$\mathrm{L}(\mathrm{m}^\mathrm{n})$	the subalgebra L of convex normal functions of m^n (p. 197).
$f \odot g$	the meet of f and g in the join order of m^n (p. 199).
$f \oplus g$	the join of f and g in the meet order of m^n (p. 199).
$\sqsubseteq$	double order on finite algebra m^n (p. 203).
$\mathrm{D}_1(\mathrm{m}^\mathrm{n})$	an algebra of decreasing n-tuples (p. 209).
$\mathrm{D}_2(\mathrm{m}^\mathrm{n})$	decreasing functions from $\mathrm{n}-1$ to $2\,\mathrm{m}-1$ (p. 210).
$\mathrm{D}_3(\mathrm{m}^\mathrm{n})$	strictly decreasing functions from $\mathrm{n}-1$ to $2\,\mathrm{m}+\mathrm{n}-3$ (p. 210).
$\mathrm{K\,L}(\mathrm{m}^\mathrm{n})$	a certain Kleene subalgebra of $\mathrm{L}(\mathrm{m}^\mathrm{n})$ (p. 212).
$\mathrm{H}(\mathrm{m}^\mathrm{n})$	an algebra of decreasing n-tuples from $\{1, 2, \ldots, m\}$ (p. 213).

Chapter 1

The Algebra of Truth Values

Type-2 fuzzy sets were introduced by Lotfi Zadeh [118, 119], extending the notion of both ordinary fuzzy sets [117], and interval-valued fuzzy sets. They have come to play an increasingly important role in applications [11, 12, 13, 78, 102]. The truth value algebras of these types of fuzzy sets are of particular interest because they determine the operations on the other entities involved. For ordinary fuzzy sets and for interval-valued fuzzy sets, these algebras are well known and quite familiar [20, 86, 108, 117]. For type-2 fuzzy sets, the truth value algebra is rather complicated, with many features. This chapter introduces this algebra, defines and simplifies its operations, and gives a number of examples illustrating these operations. Our interest in this algebra is twofold: as the truth value algebra of type-2 fuzzy sets, and as an algebra of interest in its own right.

1.1 Preliminaries

We begin with a brief discussion of several kinds of algebras that arise in connection with operations on truth values. For basic references on this material see [3, 10].

Definition 1.1.1 *An* ***n-ary operation*** *on a set A is a mapping $f : A^n \to A$. An* ***algebra*** *is a set with a family of n_i-ary operations. A specification of the number of these operations and their arities is the* ***type*** *of the algebra.*

There are many different kinds of algebras frequently encountered in various branches of mathematics and their applications. For example, a group

$(G, *, ^{-1}, e)$ is an algebra of type 2, 1, 0, meaning that it has a binary operation $*$, a unary operation $^{-1}$, and a constant, or nullary operation, e, that satisfy certain conditions. The algebras of primary interest here will be ones that have ties to logic, and truth values of fuzzy sets. We next describe several of the most important kinds of such algebras and related concepts.

Definition 1.1.2 *A* ***lattice*** $(L, \wedge, \vee)$ *is an algebra with two binary operations* $\wedge$ *and* $\vee$ *called* ***meet*** *and* ***join*** *that satisfy the following conditions:*

1. $x \wedge x = x$; $x \vee x = x$. *(idempotent)*
2. $x \wedge y = y \wedge x$; $x \vee y = y \vee x$. *(commutative)*
3. $x \wedge (y \wedge z) = (x \wedge y) \wedge z$; $x \vee (y \vee z) = (x \vee y) \vee z$. *(associative)*
4. $x \wedge (x \vee y) = x$; $x \vee (x \wedge y) = x$. *(absorption)*

We recall that a **binary relation** R on a set X is a set of ordered pairs (x_1, x_2) of elements of X; that is, a relation is a subset of $X \times X$. It is usual to write $x_1 R x_2$ to indicate that (x_1, x_2) is one of the ordered pairs in R.

Definition 1.1.3 *A* ***partial order*** *is a binary relation* $\leq$ *on a set with the properties*

1. $a \leq a$. *(reflexive)*
2. *If* $a \leq b$ *and* $b \leq a$ *then* $a = b$. *(antisymmetric)*
3. *If* $a \leq b$ *and* $b \leq c$ *then* $a \leq c$. *(transitive)*

A partial order is ***linear****, or a* ***chain****, if for each* a, b *either* $a \leq b$ *or* $b \leq a$. *A* ***poset*** *is a partially-ordered set* $(S, \leq)$.

Associated with any lattice is a partial ordering $\leq$ given by $x \leq y$ if and only if $x \wedge y = x$, or equivalently, if and only if $x \vee y = y$. In fact, lattices may be alternatively defined as partially-ordered sets where any two elements have a greatest lower bound $x \wedge y$ and a least upper bound $x \vee y$ (Exercise 1).

Definition 1.1.4 *A* ***distributive lattice*** *is a lattice* $(L, \wedge, \vee)$ *that satisfies the following two equivalent conditions known as the* ***distributive laws****.*

1. $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.
2. $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.

Definition 1.1.5 *A* ***bounded lattice*** *is a lattice* $(L, \wedge, \vee, 0, 1)$ *with constants* 0 *and* 1 *that satisfy the following:*

1. $x \wedge 0 = 0$; $x \vee 0 = x$.

2. $x \wedge 1 = x$; $x \vee 1 = 1$.

Definition 1.1.6 *An* ***involution****, or* ***negation****, on a poset is a unary operation $'$ that is of period two, so $x'' = x$, and is order inverting, so $x \leq y$ implies that $y' \leq x'$.*

Definition 1.1.7 *A* ***De Morgan algebra*** *is a bounded distributive lattice with an involution $'$. Any De Morgan algebra satisfies the following conditions known as De Morgan's laws:*

$$(x \wedge y)' = x' \vee y'; (x \vee y)' = x' \wedge y' \qquad \text{(De Morgan's laws)}$$

Definition 1.1.8 *A* ***Kleene algebra*** *is a De Morgan algebra that satisfies Kleene's inequality:*

$$x \wedge x' \leq y \vee y' \qquad \text{(Kleene's inequality)}$$

Definition 1.1.9 *A* ***Boolean algebra*** *is a De Morgan algebra that also satisfies complementation:*

$$x \wedge x' = 0; \; x \vee x' = 1 \qquad \text{(complementation)}$$

We note that every Boolean algebra is a Kleene algebra and that every Kleene algebra is a De Morgan algebra. We also note that De Morgan's laws can be derived from complementation and distributivity, so they do not need to be assumed in defining a Boolean algebra (Exercise 4).

As a final comment, we shall often refer to an algebra such as $(L, \wedge, \vee)$ simply by its underlying set L. If there is the possibility of confusion as to the operations, we will specify them.

1.2 Classical and fuzzy subsets

We begin with a discussion of the classical notion of the subsets of a set, and its algebraic structure. This is centered around the following Boolean algebra.

Definition 1.2.1 *Let* $\mathbf{2}$ *be the 2-element Boolean algebra with underlying set* $\{0, 1\}$*. This is the* ***truth value algebra for classical sets****.*

We use a well-known technique that adapts itself nicely to generalization, and view a **subset** A **of a set** S to be a mapping $A : S \to \{0, 1\}$. The idea is that for $s \in S$, s is an element of the subset A if $A(s) = 1$, and s is not an element of the subset A if $A(s) = 0$.

Definition 1.2.2 $\mathrm{Map}(S, \{0,1\})$ *is the set of all subsets of the set S; that is, all maps from S into $\{0,1\}$. The notation $\{0,1\}^S$ is also used for this.*

The set S has no operations on it, so to define operations on $\mathrm{Map}(S, \{0,1\})$ we use the operations on $\{0,1\}$ given by the Boolean algebra structure 2. In particular, we define on $\mathrm{Map}(S, \{0,1\})$ binary operations $\wedge$ and $\vee$, a unary operation $'$, and two constants 0 and 1 as **pointwise operations** as follows.

Definition 1.2.3 *Operations $\wedge$, $\vee$, $'$, 0 and 1 on $\mathrm{Map}(S, \{0,1\})$ are as follows for each $A, B : S \to \{0,1\}$ and each $s \in S$:*

1. $(A \wedge B)(s) = A(s) \wedge B(s)$.
2. $(A \vee B)(s) = A(s) \vee B(s)$.
3. $A'(s) = (A(s))'$.
4. $0(s) = 0$.
5. $1(s) = 1$.

The set $(\mathrm{Map}(S, \{0,1\}), \wedge, \vee, ', 0, 1)$ *is the* ***algebra of subsets of*** S.

It is easy to see that the operation $\wedge$ on $\mathrm{Map}(S, \{0,1\})$ is associative since it is defined as a pointwise operation from that of the algebra 2, and the corresponding operation of 2 is associative. This simple idea can be extended in the following proposition whose proof is left as an exercise (Exercise 5).

Proposition 1.2.4 *An equation holds in the algebra of subsets of a set S if and only if it holds in the truth value algebra* 2.

It follows immediately that the algebra of subsets of a set S is a Boolean algebra. The operations $\wedge$, $\vee$ and $'$ of this Boolean algebra correspond to the usual intersection, union, and complementation for subsets of S.

We next turn our attention to fuzzy subsets. The starting point is the following definition of one of our basic algebras of interest.

Definition 1.2.5 *Let* $\mathrm{I} = [0,1]$ *be the unit interval, and consider* 0 *and* 1 *as constants. The operations $\wedge$, $\vee$ and $'$ on I are as follows:*

1. $x \wedge y = \min\{x, y\}$.
2. $x \vee y = \max\{x, y\}$.
3. $x' = 1 - x$.

The algebra $(\mathrm{I}, \wedge, \vee, ', 0, 1)$ *is the* ***truth value algebra for fuzzy sets***.

The following result describes a basic property of this algebra I. Its proof is not difficult, and is left as an exercise (Exercise 6).

Proposition 1.2.6 *The algebra* I *is a Kleene algebra, but not a Boolean algebra.*

The following extends the definition of a classical subset of a set.

Definition 1.2.7 *A* ***fuzzy subset*** *A of a set S is a mapping $A : S \to [0, 1]$.*

The idea here is the basic one that for $s \in S$, the value $A(s)$ is the "degree" to which s belongs to the set A. The mapping A is commonly referred to as a **fuzzy set**. Fuzzy sets were introduced by Lotfi Zadeh in 1965 [117].

Definition 1.2.8 $\mathrm{Map}(S, [0, 1])$ *is the set of all fuzzy subsets of the set S; that is, all maps from S into $[0, 1]$.*

We again define operations on the set of fuzzy subsets of S as the pointwise operations obtained from the algebra I. Here we remark on our choice of the operations $\wedge$, $\vee$, $'$, 0 and 1 for this algebra. There are, of course, operations on the interval $[0, 1]$ other than these that are of interest in fuzzy matters, for example t-norms, and t-conorms, but a discussion of these will be postponed until Chapter 5.

Definition 1.2.9 *The operations $\wedge$, $\vee$, $'$, 0 and 1 on $\mathrm{Map}(S, [0, 1])$ are as follows, for each $A, B : S \to \{0, 1\}$ and each $s \in S$:*

1. $(A \wedge B)(s) = A(s) \wedge B(s)$.
2. $(A \vee B)(s) = A(s) \vee B(s)$.
3. $A'(s) = (A(s))'$.
4. $0(s) = 0$.
5. $1(s) = 1$.

The algebra $(\mathrm{Map}(S, [0, 1]), \wedge, \vee, ', 0, 1)$ *is the* ***algebra of fuzzy subsets of*** *S.*

Since the operations of the algebra of fuzzy subsets are defined pointwise from those of the Kleene algebra I, the algebra of fuzzy subsets of S is also a Kleene algebra (Exercise 8).

We next turn attention to interval-valued fuzzy sets. These were introduced independently by Zadeh [118], Grattan-Guiness [37], Jahn [56], and Sambuc [95], all in 1975. Recall that for a fuzzy subset $A : S \to [0, 1]$, one interprets $A(s)$ as the "degree" to which the element s belongs to A. One view is that representing this degree with a single real number is too restrictive, and that assigning an interval of real numbers to it is more realistic.

This leads to the following. Here we describe a closed interval $[a, b]$ of the reals unit interval by giving the ordered pair (a, b) of its endpoints, as is standard [3].

Definition 1.2.10 *The set of intervals in* $[0,1]$ *is written as* $\mathrm{I}^{[2]} = \{(a,b) : a,b \in \mathrm{I}$ *and* $a \le b\}$. *The algebra on* $\mathrm{I}^{[2]}$ *has operations* $\wedge$, $\vee$, $'$, 0 *and* 1 *on* $\mathrm{I}^{[2]}$ *using those of* I *as follows:*

1. $(a,b) \wedge (c,d) = (a \wedge c, b \wedge d)$.
2. $(a,b) \vee (c,d) = (a \vee c, b \vee d)$.
3. $(a,b)' = (b', a')$.
4. $0 = (0,0)$.
5. $1 = (1,1)$.

The algebra $(\mathrm{I}^{[2]}, \wedge, \vee, ', 0, 1)$ *is the* ***truth value algebra for interval-valued fuzzy sets****.*

The notation $\mathrm{I}^{[2]}$ is standard for a general method of creating a distributive lattice from the closed intervals of a given one. (See [3].) The algebra $\mathrm{I}^{[2]}$ has many interesting properties. (See, for example, [29].) The job of establishing the following basic property is left as an exercise (Exercise 7).

Proposition 1.2.11 *The algebra* $\mathrm{I}^{[2]}$ *is a De Morgan algebra, but it is not a Kleene algebra.*

The established pattern continues; that is, we can look at fuzzy subsets using $\mathrm{I}^{[2]}$ as a truth value algebra.

Definition 1.2.12 *An* ***interval-valued fuzzy subset*** *of* S *is a mapping* $A : S \to \mathrm{I}^{[2]}$.

As noted, for $s \in S$, the "degree" of membership of s in A is now an interval of real numbers.

Definition 1.2.13 $\mathrm{Map}(S, \mathrm{I}^{[2]})$ *is the set of interval-valued fuzzy subsets of* S*; that is, all maps from* S *to* $\mathrm{I}^{[2]}$.

Operations are then defined on $\mathrm{Map}(S, \mathrm{I}^{[2]})$ as pointwise operations from $\mathrm{I}^{[2]}$ exactly as in Definitions 1.2.3 and 1.2.9. This gives us $(\mathrm{Map}(S, \mathrm{I}^{[2]}), \wedge, \vee, ', 0, 1)$, the **algebra of interval-valued fuzzy subsets** of S. Again, as a basic consequence of defining operations pointwise, we have the following.

Proposition 1.2.14 *The algebra of interval-valued fuzzy subsets of a set* S *is a De Morgan algebra.*

We note that the classical subsets of S are included in the fuzzy subsets of S, and that the fuzzy subsets of S are included in the interval-valued fuzzy subsets of S by considering a real number to be a 1-element interval.

We next begin the study of type-2 fuzzy sets, a notion that encompasses all the previous ones, and many intermediate possibilities. A complete discussion of these remarks will be undertaken in Chapter 3.

1.3 The truth value algebra of type-2 fuzzy sets

Type-2 fuzzy sets were introduced by Zadeh in 1975 in [118, 119]. The basic idea is to generalize the notion of ordinary fuzzy sets and interval-valued fuzzy sets, affording wider applicability. The "degree" of membership of an element s in a set A will no longer be simply 0 or 1 as with classical subsets, or a single real number as in fuzzy subsets, or even an interval of real numbers as in interval-valued fuzzy sets. Now the "degree" of membership will be a function $f : [0,1] \to [0,1]$.

Definition 1.3.1 *For sets X and Y,*

$$\mathrm{Map}(X,Y) = \{f : f \text{ is a function from } X \text{ to } Y\}$$

The notation Y^X is also used for this.

The set $\mathrm{Map}(\mathrm{I},\mathrm{I})$ will play a primary role throughout this book. We will sometimes find it convenient to give it a shorter name, and will also use M to denote this set. We note that the elements of $\mathrm{Map}(\mathrm{I},\mathrm{I})$ are precisely the fuzzy subsets of I as discussed in the previous section. In the context of type-2 fuzzy sets, the members of $\mathrm{Map}(\mathrm{I},\mathrm{I})$ are sometimes called **fuzzy truth values** or **membership grades** of type-2 fuzzy sets.

Definition 1.3.2 *A* ***type-2 fuzzy subset*** *of a set S is a map $A : S \to \mathrm{Map}(\mathrm{I},\mathrm{I})$.*

Of course, the matter arises of putting the correct algebraic structure on $\mathrm{Map}(\mathrm{I},\mathrm{I})$. This will be done using operations on both the domain I and range I of mappings in $\mathrm{Map}(\mathrm{I},\mathrm{I})$ through the general technique of **convolution**. The concept of convolution arises in many areas of mathematics. The specific form in which we shall employ it is described below.

Definition 1.3.3 *Let U and V be sets, and suppose that U has a binary operation $\circ$ on it, V has a binary operation $\triangle$ on it, and $\square$ is another appropriate operation with which one may define a binary operation $*$ on the set $\mathrm{Map}(U,V)$ by the formula*

$$(f * g)(u) = \underset{x \circ y = u}{\square} (f(x) \bigtriangleup g(y)) \tag{1.1}$$

The operation $*$ is the ***convolution*** *of $\triangle$ with $\circ$ via $\square$.*

We have written this definition for $\circ$ and $\triangle$ binary operations. It applies when both are n-ary operations of the same n, including the cases that they are both unary or both nullary. We have been deliberately vague regarding the nature of $\square$. In the applications to type-2 fuzzy sets, $\square$ will be an operation on the subsets of V. In general, it is an operation on certain indexed families of elements of V. We next consider two familiar examples of convolutions.

Example 1.3.4 Polynomials $p(x)$ in a variable x and with real coefficients can be considered as functions $p : \mathbb{N} \to \mathbb{R}$ from the natural numbers (including 0) to the reals that have finite support; that is, are nonzero at only finitely many places. For example, $p(x) = 5 - \pi x + x^2$ would be the function that has the values $5, -\pi, 1$ at $0, 1, 2$, respectively, and has value 0 otherwise. For two such polynomials p and q, the k^{th} coefficient of the product is given by

$$(p \cdot q)(k) = \sum_{i+j=k} p(i)q(j)$$

This operation is a convolution. Here the set U is $\mathbb{N}$, the set V is $\mathbb{R}$, the operation $\circ$ on U is addition $+$ on $\mathbb{N}$, and the operation $\triangle$ on V is multiplication on $\mathbb{R}$. The operation $\square$ is the operation that associates to any finite sequence of real numbers its sum.

An extension of this example is of basic importance in many branches of analysis and physics.

Example 1.3.5 Given two integrable functions $f, g : \mathbb{R} \to \mathbb{R}$, their convolution is defined by

$$(f * g)(t) = \int_{-\infty}^{\infty} f(x)g(t-x)dx$$

This is seen in our context with U and V both being the reals $\mathbb{R}$, and with the binary operation $\circ$ being addition, and $\triangle$ being multiplication. The operation $\square$ takes the family of real numbers $f(x)g(y)$ indexed over all $x+y = t$, reindexes this in an obvious way as the family $f(x)g(t-x)$ where $x \in \mathbb{R}$, and associates to this the indicated integral.

We turn next to the matter of employing convolutions to define operations on Map(I, I). In this context, these convolutions are sometimes referred to as **Zadeh's extension principle** [118]. In each case the operation we will produce will be a convolution with $\circ$ being one of the basic operations of I. To describe the operations $\triangle$ and $\square$ used, we require the following.

Definition 1.3.6 *A lattice $(L, \wedge, \vee)$ is a **complete lattice** if every subset S of L has a greatest lower bound (meet), and a least upper bound (join), under the partial ordering $\leq$ of the lattice. We denote these as follows:*

1. *$\bigvee S$ = the least upper bound of S.*
2. *$\bigwedge S$ = the greatest lower bound of S.*

We note that the greatest lower bound of the empty set is the largest element 1 of the lattice, and the least upper bound of the empty set is the least element 0.

It is a basic property of the real numbers that the unit interval I is a complete lattice. In this setting, the least upper bound of a subset S is sometimes written $\sup S$, and the greatest lower bound of S is sometimes written $\inf S$.

We return to the topic of complete lattices arising in type-2 fuzzy sets in Chapter 6, and employ these results in Chapter 9. But now we are prepared for the primary definition of this book.

Definition 1.3.7 *The operations* $\sqcap$, $\sqcup$, *, 1_0, *and* 1_1 *on* $\mathrm{Map}(\mathrm{I},\mathrm{I})$ *are as follows:*

1. $(f \sqcap g)(x) = \bigvee_{y \wedge z = x}(f(y) \wedge g(z))$.
2. $(f \sqcup g)(x) = \bigvee_{y \vee z = x}(f(y) \wedge g(z))$.
3. $f^*(x) = f(1-x)$.
4. $1_0(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0. \end{cases}$
5. $1_1(x) = \begin{cases} 0 & \text{if } x = 1 \\ 1 & \text{if } x \neq 1. \end{cases}$

The algebra $(\mathrm{Map}(\mathrm{I},\mathrm{I}), \sqcap, \sqcup, {}^*, 1_0, 1_1)$ *is the* ***truth value algebra for type-2 fuzzy sets****.*

It is of interest to see that the choice of operations on $\mathrm{Map}(\mathrm{I},\mathrm{I})$ follows a basic pattern. To see this, we note that the meet operation $\bigwedge$ of I can be applied to sets of any size. When applied to 2-element sets it is the binary meet operation $\wedge$, when applied to 1-element sets it is the identity operation, and when applied to the empty set it returns the largest element 1.

Proposition 1.3.8 *The operations* $\sqcap$, $\sqcup$, *, 1_0, *and* 1_1 *on* $\mathrm{Map}(\mathrm{I},\mathrm{I})$ *are the convolutions of the operations* $\wedge$, $\vee$, $'$, 0, *and* 1 *of* I *with the meet operation of* I*, using for* $\Box$ *the infinite join operation* $\bigvee$*.*

Proof. From the definition, it is clear that $\sqcap$ is the convolution of the operation $\wedge$ on I with $\wedge$ on I using for $\Box$ the infinite join operation $\bigvee$, and that $\sqcup$ is the convolution of $\vee$ with $\wedge$ using $\bigvee$. Since $y' = x$ if and only if $1 - y = x$, which occurs if and only if $x' = y$, we have

$$f^*(x) = f(1-x) = f(x') = \bigvee_{y'=x} f(y)$$

Thus * is the convolution of the negation $x' = 1 - x$ of I with the unary meet operation using $\bigvee$. The constant 1_0 is the convolution of the constant 0 of I with the empty meet operation using $\bigvee$, and the constant 1_1 is the convolution of the constant 1 of I with the empty meet operation using $\bigvee$. Establishing these facts is a bit of arcane reasoning with the empty set, and is left to the reader. ∎

The fact that the operations on $\mathrm{Map}(\mathrm{I},\mathrm{I})$ arise from convolutions is useful primarily to see that they have a natural source, and are not *ad hoc* creations.

In working with these operations, their origins in convolutions will not be applied. In fact, we shall try very hard to simplify their descriptions to make working with these operations far more tractable. The degree to which this can be done is striking. We begin this task in the following section.

1.4 Simplifying the operations

It is convenient to use M for the set $\mathrm{Map}(\mathrm{I},\mathrm{I})$. We continue our convention to use M also to refer to the algebra $(\mathrm{Map}(\mathrm{I},\mathrm{I}), \sqcap, \sqcup, ^*, 1_0, 1_1)$. The elements of M are functions $f : \mathrm{I} \to \mathrm{I}$. In addition to the convolution operations defined on M, we may consider the pointwise operations $\wedge$, $\vee$, $'$, 0, 1 on it coming from operations on the range I. In this section, we use these pointwise operations in conjunction with the following auxiliary operations L and R, to greatly simplify the descriptions of $\sqcap$ and $\sqcup$.

Definition 1.4.1 *For $f \in \mathrm{M}$, let f^L and f^R be the elements of* M *defined by*

$$f^L(x) = \bigvee_{y \le x} f(y) \quad \text{and} \quad f^R(x) = \bigvee_{y \ge x} f(y) \tag{1.2}$$

For $f, g \in \mathrm{M}$, having $f \le g$ in the **pointwise order** means that $f(x) \le g(x)$ for all $x \in \mathrm{I}$. In this case, we often say that f is **below** g, or that g is **above** f. We adhere to the convention of increasing versus strictly increasing when speaking of monotone functions. For clarity, this is defined below.

Definition 1.4.2 *Let $f \in \mathrm{M}$. We use the following terminology:*

1. *f is **increasing** if $x \le y$ implies $f(x) \le f(y)$.*
2. *f is **decreasing** if $x \le y$ implies $f(y) \le f(x)$.*

*The definitions of **strictly increasing** and **strictly decreasing** are obtained if $\le$ is replaced by strictly less than $<$. A function is **monotone** if it is either increasing or decreasing.*

The following observation, whose proof is left as an exercise (Exercise 10), provides the meaning behind the definitions of f^L and f^R.

Proposition 1.4.3 *Let $f \in M$.*

1. *f^L is the least increasing function above f.*
2. *f^R is the least decreasing function above f.*

This situation is illustrated in Figure 1.1.

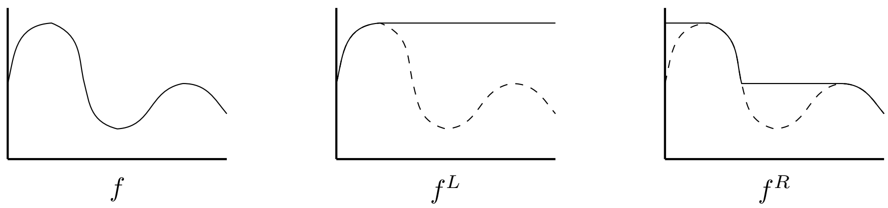


FIGURE 1.1: Computations of f^L and f^R

Before proceeding, we will need some basic properties of the real unit interval I relating to infinite versions of distributivity. We will return to this topic in a more general setting in Chapter 6.

Proposition 1.4.4 *Let a, a_j $(j \in J)$ and b, b_k $(k \in K)$ be elements of* I. *Then*

1. $\bigvee_{j \in J} (a \wedge a_i) = a \wedge \bigvee_{j \in J} a_j$.

2. $\bigvee_{j \in J, k \in K} (a_j \wedge b_k) = \bigvee_{j \in J} a_j \wedge \bigvee_{k \in K} b_k$.

The first statement is known as ***meet continuity****.*

With respect to the pointwise operations $\vee$ and $\wedge$, M is a lattice, and these operations on M are far easier to compute with than the convolution operations $\sqcup$ and $\sqcap$ defined in Definition 1.3.7. The following theorem expresses each of the operations $\sqcup$ and $\sqcap$ in terms of pointwise operations and the operations L and R in two alternate forms, and is basic in deriving properties of these operations. Similar expressions also appear in [14], [19], [21], and [24]. The fact that the unit interval I is meet continuous is a basic tool in deriving properties of the operations.

Theorem 1.4.5 *The following hold for all $f, g \in$ M:*

$$f \sqcup g = (f \wedge g^L) \vee (f^L \wedge g) = (f \vee g) \wedge (f^L \wedge g^L) \tag{1.3}$$

$$f \sqcap g = (f \wedge g^R) \vee (f^R \wedge g) = (f \vee g) \wedge (f^R \wedge g^R) \tag{1.4}$$

Proof. Let $f, g \in$ M. Beginning with the definition of $\sqcup$ from Definition 1.3.7,

$$(f \sqcup g)(x) = \bigvee_{y \vee z = x} (f(y) \wedge g(z))$$

Note that $y \vee z = x$ if and only if either $y = x$ and $z \leq x$, or $y < x$ and $z = x$. So this expression becomes

$$\bigvee_{z \leq x} (f(x) \wedge g(z)) \;\vee\; \bigvee_{y \leq x} (f(y) \wedge g(x))$$

Using the meet continuity of I and the definition of the operation L, this becomes

$$(f(x) \wedge g^L(x)) \vee (f^L(x) \wedge g(x))$$

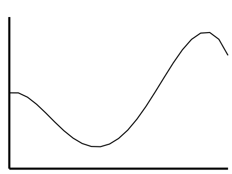
This yields $f \sqcup g = (f \wedge g^L) \vee (f^L \wedge g)$. Then using the fact that $(M, \wedge, \vee)$ is a distributive lattice, and applying several applications of the distributive law,

$$f \sqcup g = (f \vee f^L) \wedge (f \vee g) \wedge (g^L \vee f^L) \wedge (g^L \vee g)$$

This then simplifies to provide $f \sqcup g = (f \vee g) \wedge (f^L \wedge g^L)$. This establishes (1.3). In a totally analogous manner, we get the formulas stated for $f \sqcap g$. ■

1.5 Examples

In this section, we provide a number of examples to illustrate the various operations in M. To begin, consider the functions f and g shown below.



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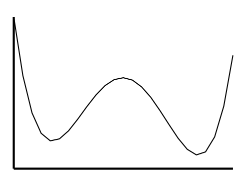
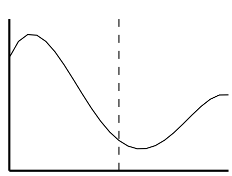
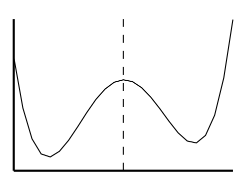


g

Since $f^*(x) = f(1 - x)$, computation of f^* and g^* is done by taking their mirror images in the line $x = 1/2$. These are shown below.

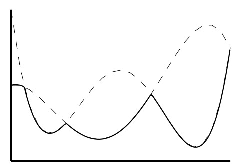


f^*

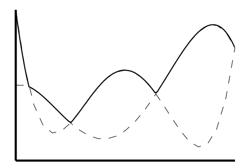


g^*

Since the pointwise meet $(f \wedge g)(x) = \min\{f(x), g(x)\}$, computation of pointwise meet $f \wedge g$ is done by taking the smaller of the values of f and g at each x. Similarly, since $(f \vee g)(x) = \max\{f(x), g(x)\}$, it is obtained by taking the larger of the values of f and g at each x.

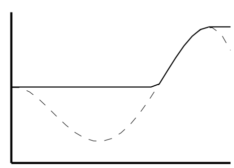


$f \wedge g$

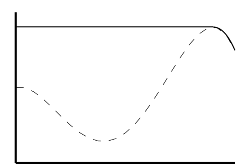


$f \vee g$

Recall that $f \sqcup g = (f \vee g) \wedge f^L \wedge g^L$ and $f \sqcap g = (f \vee g) \wedge f^R \wedge g^R$. Note that both of these expression involve the pointwise join $f \vee g$. Neither directly involves the pointwise meet $f \wedge g$. However, both use the concept of the pointwise meet, but applied to other functions. To break things into smaller steps when illustrating $\sqcup$ and $\sqcap$, we next describe the functions f^L and f^R.

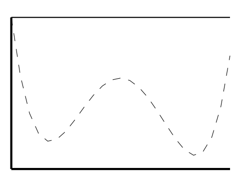


f^L

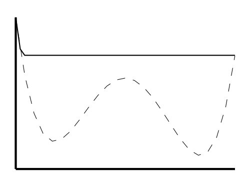


f^R

Similarly, g^L and g^R are as follows.



g^L



g^R

To compute $f \sqcup g = (f \vee g) \wedge f^L \wedge g^L$, we take the pointwise meet of the functions $f \vee g$, f^L and g^L. This amounts to cutting off the height of $f \vee g$ at the smaller of the levels of f^L and g^L. To compute $f \sqcap g = (f \vee g) \wedge f^R \wedge g^R$ is similar, using f^R and g^R. The results are shown below.

$f \sqcup g$ $\quad$ $f \sqcap g$

With some experience, it is not difficult to compute $\sqcup$ and $\sqcap$ directly, provided the functions are not so complex. Consider the functions h and k shown below. They are of a sort often encountered in fuzzy applications.

h $\quad$ k

When computing $h \sqcup k$ and $h \sqcap k$, we compute the pointwise join $h \vee k$ and cut the values off to remain below h^L and k^L for $h \sqcup k$, and to remain below h^R and k^R for $h \sqcap k$. The results are easily seen to be as follows. We note that the computation of $h \sqcap k$ results in h because the function h^R is zero toward the right half of its domain.

$h \sqcup k$ $\quad$ $h \sqcap k$

So far, the examples we have considered have all been continuous functions. This is not necessary, and it fact, our basic constants 1_0 and 1_1 are not continuous. We recall the standard device for depicting functions that are not continuous. When a circle is filled in, it indicates the value that a function takes at a point, and when a circle is open, it indicates values that a function takes close to the point, but not at the point. The function 1_0 that takes value 1 at $x = 0$ and value 0 everywhere else is as shown at left in the following, and the function 1_1 takes value 1 at $x = 1$ and 0 everywhere else is shown at right in the following.

1_0

1_1

We note that in these final two diagrams, the values of the functions lie along the x-axis at all but one point, so the function coincides with the x-axis for most of its domain.

1.6 Summary

In this chapter, we have outlined the role of the truth value algebras of sets, fuzzy sets, interval-valued fuzzy sets, and type-2 fuzzy sets. The operations on the truth value algebra M of type-2 fuzzy sets were given as convolutions of the operations $\wedge$, $\vee$, $'$, 0, and 1 of the unit interval I. Auxiliary operations L and R were introduced that made computation of the operations of M tractable. The chapter concluded with a number of examples illustrating the computation of the various operations in several situations.

1.7 Exercises

1. Prove that lattices may be alternatively defined as partially-ordered sets where any two elements have a greatest lower bound $x \wedge y$ and a least upper bound $x \vee y$.

2. Prove that a vector space over the reals can be considered as an algebra in the sense of Definition 1.1.1. (Hint: Vector addition is obvious. For scalar multiplication λv use one unary operation for each scalar λ).

3. Prove that the two equalities in Definition 1.1.4 are equivalent.

4. Prove that the De Morgan laws follow from complementation and distributivity, so that they do not need to be assumed in defining a Boolean algebra.

5. Prove Proposition 1.2.4, regarding equations in the algebra of subsets of a set.

6. Show that the algebra I is a Kleene algebra but not a Boolean algebra (Proposition 1.2.6).

7. Show that the algebra $I^{[2]}$ is a De Morgan algebra, but it is not a Kleene algebra (Proposition 1.2.11).

8. Prove that the algebra of fuzzy subsets of a set S is a Kleene algebra.

9. Show that the algebra of interval-valued fuzzy subsets of a set S is a De Morgan algebra. (Proposition 1.2.14).

10. Prove Proposition 1.4.3, which describes f^L and f^R.

11. Prove that f^* is the mirror image of f in the vertical line $x = 1/2$.

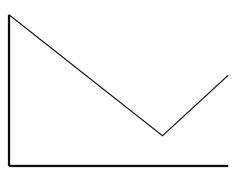
12. Verify that the complete lattice $([0,1], \leq)$ is meet continuous. (See Proposition 1.4.4.) Use this result to prove item 2 of Proposition 1.4.4.

13. Verify that the second equation in Theorem 1.4.5 holds.

14. Prove that the set of rational numbers in the interval $[0,1]$ forms a bounded lattice, but that this lattice is not complete.

15. The elements 1_0 and 1_1 of Map(I, I) can be considered nullary operations, and can be obtained by convolution of the nullary operations 1 and 0 on I. Show how this is done.

16. Consider the following functions f and g.



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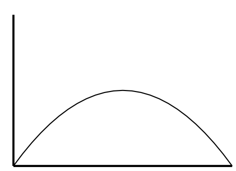
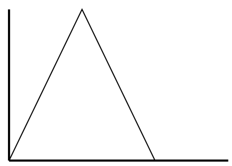


g

(a) Draw f^* and g^*.
(b) Draw $f \wedge g$ and $f \vee g$.
(c) Draw f^L, f^R, g^L and g^R.
(d) Draw $f \sqcap g$ and $f \sqcup g$.

17. Consider the following functions f and g.



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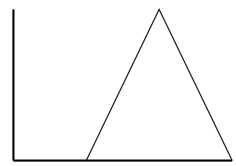
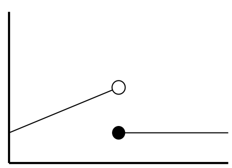


g

(a) Draw f^* and g^*.
(b) Draw $f \wedge g$ and $f \vee g$.
(c) Draw f^L, f^R, g^L and g^R.
(d) Draw $f \sqcap g$ and $f \sqcup g$.

18. Consider the following functions f and g.



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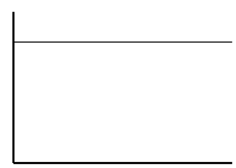


g

(a) Draw f^* and g^*.
(b) Draw $f \wedge g$ and $f \vee g$.
(c) Draw f^L, f^R, g^L and g^R.
(d) Draw $f \sqcap g$ and $f \sqcup g$.

Hint: $f \sqcup g$ is continuous, $f \sqcap g$ is not.

Chapter 2

Properties of the Type-2 Truth Value Algebra

The basic operations of the algebra M were introduced in Chapter 1. In this chapter, the fundamental properties of these operations are presented. They were developed by a number of authors [18, 81, 83, 118, 119]. This includes simple observations about these operations presented in Section 2.2. These will be used throughout this book. The basic equations satisfied by these operations are developed in Section 2.3. There are further equations satisfied by these operations, but this is a more delicate matter left to Chapter 8. Section 2.4 develops properties of two partial orderings on M that are defined through these operations. Section 2.5 develops fragments of distributivity that hold in M. These fragments of distributivity will be used essentially in Chapter 6. Development of the properties of these operations is greatly clarified by some general considerations that are presented in the preliminaries that follow.

2.1 Preliminaries

A study of the algebra of truth values of type-2 fuzzy sets will involve algebras of a more general nature than those encountered in Chapter 1. The algebras we will encounter are built from simpler structures, some of which we consider here.

Definition 2.1.1 *A* ***semilattice*** *is an algebra* $(S, *)$ *with a binary operation*

$*$ that is commutative, associative, and idempotent; that is, it satisfies the following equations:

1. $x * y = y * x$.
2. $(x * y) * z = x * (y * z)$.
3. $x * x = x$.

*An operation $*$ on S that is commutative, associative, and idempotent is called a **semilattice operation** on S.*

A lattice (Definition 1.1.2) $(L, \wedge, \vee)$ is an algebra with two binary operations $\wedge$ and $\vee$ that are both commutative, associative, and idempotent, hence are semilattice operations. If one omits the operation $\vee$ from a lattice, one is left with a semilattice $(L, \wedge)$; and if one omits the operation $\wedge$, one is left with a semilattice $(L, \vee)$.

Definition 2.1.2 *A **bisemilattice** is an algebra $(B, \wedge, \vee)$ with two binary operations, both of which are semilattice operations.*

The following result is immediate from the definition of a lattice (Definition 1.1.2).

Proposition 2.1.3 *A bisemilattice is a lattice if, and only if, it satisfies the absorption laws:*

1. $x \wedge (x \vee y) = x$.
2. $x \vee (x \wedge y) = x$.

We recall that lattices could be equivalently defined as partially-ordered sets where any two elements have a greatest lower bound $x \wedge y$ and a least upper bound $x \vee y$ (Exercise 1 of Chapter 1). There is a similar situation for semilattices. We first introduce some terminology to aid the discussion.

Definition 2.1.4 *A **meet semilattice** is a poset $(S, \leq)$ where any two elements have a greatest lower bound $x \wedge y$. A **join semilattice** is a poset $(S, \leq)$ where any two elements have a least upper bound $x \vee y$.*

Semilattices are algebras. Meet semilattices and join semilattices are certain types of partially-ordered sets. However, there are close connections between these notions. We begin with the following result whose proof is left as an exercise (Exercise 1).

Proposition 2.1.5 *If $(S, \leq)$ is a meet semilattice whose greatest lower bound operation is given by $x \wedge y$, then $(S, \wedge)$ is a semilattice. If $(S, \leq)$ is a join semilattice whose least upper bound operation is given by $x \vee y$, then $(S, \vee)$ is a semilattice.*

So we have two separate order-theoretic notions for semilattices, meet semilattices and join semilattices, but only one algebraic notion for a semilattice. The point is that each semilattice gives rise to two partial orderings, one that yields a meet semilattice and the other a join semilattice. This is the content of the following result.

Proposition 2.1.6 *Suppose that $(S, *)$ is a semilattice. Then with the relation $x \leq y$ if and only if $x * y = x$ we have $(S, \leq)$ is a meet semilattice with meet given by $*$. With the relation $x \leq y$ if and only if $x * y = y$ we have $(S, \leq)$ is a join semilattice with join given by $*$.*

Proof. We prove the first statement, the second is similar. Since $*$ is idempotent, $x * x = x$, giving $x \leq x$. So $\leq$ is reflexive. If $x \leq y$ and $y \leq x$, then $x * y = x$ and $y * x = y$. So since $*$ is commutative, $x = y$. If $x \leq y$ and $y \leq z$, then $x * y = x$ and $y * z = y$. So since $*$ is associative, $x * z = (x * y) * z = x * (y * z) = x * y = x$. Thus $x \leq z$, and thus $\leq$ is transitive.

Since $x * (x * y) = (x * x) * y = x * y$, and $y * (x * y) = x * (y * y) = x * y$, we have that $x * y \leq x$ and $x * y \leq y$. Suppose $z \leq x, y$. Then $z * x = z$ and $z * y = y$, so $z * (x * y) = (z * x) * y = z * y = z$. This shows that $z \leq x * y$. Thus $x * y$ is the greatest lower bound of x and y under $\leq$. Thus $(S, \leq)$ is a meet semilattice with meet given by $*$. ■

We note that the two methods of producing a partial ordering from a semilattice $(S, *)$ are clearly related to one another. Setting $x \leq_1 y$ if $x * y = x$, and $x \leq_2 y$ if $x * y = y$, it is easily seen that $x \leq_1 y$ if and only if $y \leq_2 x$. So the two partial orderings built from $(S, *)$ determine one another, and they are called **converses** of each other.

Definition 2.1.7 *Let $(B, \wedge, \vee)$ be a bisemilattice. Define two partial orderings $\leq_\wedge$ and $\leq_\vee$ on B as follows:*

1. *$x \leq_\wedge y$ if and only if $x \wedge y = x$.*
2. *$x \leq_\vee y$ if and only if $x \vee y = y$.*

*We call $\leq_\wedge$ the **meet order** and $\leq_\vee$ the **join order**.*

Since each semilattice gives rise to two partial orderings, each bisemilattice gives rise to four partial orderings. However, the two remaining partial orderings on B are the converses of the two given. To explain our choice of these two particular partial orderings, we consider matters in relation to lattices.

Proposition 2.1.8 *A bisemilattice $(B, \wedge, \vee)$ is a lattice if and only if the partial orderings $\leq_\wedge$ and $\leq_\vee$ agree.*

Proof. Suppose that $(B, \wedge, \vee)$ is a lattice. If $x \leq_\wedge y$, then by the absorption law, $x \vee y = y \vee (x \wedge y) = y$, giving $x \leq_\vee y$. Conversely, if $x \leq_\vee y$, then $x \vee y = y$, and by absorption, $x \wedge y = x \wedge (x \vee y) = x$. So $x \leq_\wedge y$. So in a lattice, the two partial orders agree.

Suppose that $(B, \wedge, \vee)$ is a bisemilattice in which these two partial orderings agree. Note that $x \wedge (x \wedge y) = x \wedge y$, so $x \wedge y \leq_\wedge x$. Since these orders agree, then $x \wedge y \leq_\vee x$, giving $x \vee (x \wedge y) = x$. This is one of the absorption laws. The other is obtained in a similar manner. ■

The following result is immediate from Definition 2.1.7 and Proposition 2.1.6.

Proposition 2.1.9 *Let $(B, \wedge, \vee)$ be a bisemilattice. The following hold:*

1. *$(B, \leq_\wedge)$ is a meet semilattice in which meets are given by $\wedge$.*
2. *$(B, \leq_\vee)$ is a join semilattice in which joins are given by $\vee$.*

To conclude this section, we discuss several types of algebras related to bisemilattices that shall be of interest in the study of M.

Definition 2.1.10 *A* ***Birkhoff system*** *is a bisemilattice $(B, \wedge, \vee)$ that satisfies the following equation known as* ***Birkhoff's equation***.

$$x \wedge (x \vee y) = x \vee (x \wedge y) \tag{2.1}$$

Birkhoff systems have an extensive literature [7, 76, 88]. We will return to this matter in Chapter 8. We note that Birkhoff's equation is a weakening of the usual absorption laws of lattices, which can be written $x \wedge (x \vee y) = x = x \vee (x \wedge y)$. Indeed, the middle term is simply removed. Thus every lattice is a Birkhoff system, but not conversely.

Definition 2.1.11 *A* ***bounded bisemilattice*** $(B, \wedge, \vee, 0, 1)$ *is a bisemilattice with constants* 0 *and* 1 *that satisfy*

1. $x \vee 0 = x$.
2. $x \wedge 1 = x$.

Definition 2.1.12 *A* ***De Morgan Birkhoff system*** $(B, \wedge, \vee, ^*, 0, 1)$ *is a bounded Birkhoff system with a unary operation* * *that satisfies*

1. $x^{**} = x$.
2. $(x \wedge y)^* = x^* \vee y^*$.
3. $(x \vee y)^* = x^* \wedge y^*$.
4. $0^* = 1$.
5. $1^* = 1$.

The primary aim of this chapter is to establish that $(\mathrm{M}, \sqcap, \sqcup, ^*, 1_0, 1_1)$ is a De Morgan Birkhoff system.

2.2 Basic observations

In this section we collect basic observations that are used at numerous points throughout the remainder of the book. We begin with the following proposition that establishes properties of L and R. We note that f^{LR} will be used in place of $(f^L)^R$, and so forth.

Lemma 2.2.1 *The following hold for all $f, g \in \mathrm{M}$:*

1. $f \leq f^L$; $f \leq f^R$.
2. $f \leq g$ *implies* $f^L \leq g^L$ *and* $f^R \leq g^R$.
3. $f^{LL} = f^L$; $f^{RR} = f^R$.
4. $f^{LR} = f^{RL}$ *and this is a constant function with value* $\sup f$.

Proof. By Proposition 1.4.3, f^L is the least increasing function above f and f^R is the least decreasing function above f. From this, statements (1) − (3) follow immediately. For (4) we note that since f^L is the least increasing function above f, that f^L takes its largest value at $x = 1$, and this value is the supremum $\sup\ f$ of the values taken by f. Therefore f^{LR} is the constant function that takes value $\sup\ f$, and similar reasoning shows that f^{RL} is also a constant function taking this value. ∎

We next consider basic relationships among the pointwise operations $\vee$ and $\wedge$, the operations L and R, and the * operation of M. Here, we use f^{L*} for $(f^L)^*$, and so forth.

Lemma 2.2.2 *The following hold for $f, g \in \mathrm{M}$:*

1. $f^{**} = f$.
2. $f^{L*} = f^{*R}$; $f^{R*} = f^{*L}$.
3. $(f \wedge g)^* = f^* \wedge g^*$; $(f \vee g)^* = f^* \vee g^*$.
4. $(f \vee g)^L = f^L \vee g^L$; $(f \vee g)^R = f^R \vee g^R$.

Proof. The operation f^* takes the mirror image of f in the vertical line $x = 1/2$ (Exercise 11 of Chapter 1). So (1) is obvious. Note that applying * to an increasing function produces a decreasing function, and conversely. It follows that f^{L*} is the least decreasing function above f^*, hence equals f^{*R}. A similar argument shows $f^{R*} = f^{*L}$, establishing (2). Item (3) states that taking the pointwise meet of two functions and then reflecting them in $x = 1/2$

amounts to the same as reflecting them, then taking their pointwise meet, which is obvious. For (4), the pointwise join of two increasing functions is increasing, so $f^L \vee g^L$ is an increasing function above $f \vee g$. But any increasing function above $f \vee g$ lies above f^L and g^L, hence above $f^L \vee g^L$. So $f^L \vee g^L$ is the least increasing function above $f \vee g$. The second equation is similar. ■

We remark that the equations $(f \wedge g)^L = f^L \wedge g^L$ and $(f \wedge g)^R = f^R \wedge g^R$ need not hold (Exercise 4). However, since $(f \wedge g)^L \le f^L$ and $(f \wedge g)^L \le g^L$, we see that $(f \wedge g)^L \le f^L \wedge g^L$ and similarly $(f \wedge g)^R \le f^R \wedge g^R$.

Our final two lemmas provide relationships among the convolution operations $\sqcap$ and $\sqcup$, and L and R.

Lemma 2.2.3 *The following hold for* $f, g \in \mathrm{M}$.

1. $f^L \sqcup g^L = f^L \sqcup g = f \sqcup g^L = f^L \wedge g^L$.
2. $f^R \sqcap g^R = f^R \sqcap g = f \sqcap g^R = f^R \wedge g^R$.

Proof. Using Theorem 1.4.5,

$$\begin{aligned} f^L \sqcup g^L &= (f^L \wedge g^{LL}) \vee (f^{LL} \wedge g^L) \\ f^L \sqcup g &= (f^{LL} \wedge g) \vee (f^L \wedge g^L) \\ f \sqcup g^L &= (f \wedge g^{LL}) \vee (f^L \wedge g^L) \end{aligned}$$

Using Lemma 2.2.1, all three expressions simplify to $f^L \wedge g^L$. This proves item (1), and item (2) is similar. ■

Our final result of this section involves the most difficult computations. We must return to the original definition of the convolution operations, and make use of some of the more subtle order-theoretic properties of the interval I. However, the ultimate result is very simply stated and natural.

Lemma 2.2.4 *The following hold for* $f, g \in \mathrm{M}$.

1. $(f \sqcup g)^L = f^L \sqcup g^L$.
2. $(f \sqcup g)^R = f^R \sqcup g^R$.
3. $(f \sqcap g)^R = f^R \sqcap g^R$.
4. $(f \sqcap g)^L = f^L \sqcap g^L$.

Proof. For the first, note that the definitions of L and $\sqcup$ give

$$(f \sqcup g)^L(x) = \bigvee_{y \le x} (f \sqcup g)(y) = \bigvee_{y \le x} \; \bigvee_{u \vee v = y} (f(u) \wedge g(v))$$

Since $u \vee v = y$ for some $y \leq x$ if and only if $u, v \leq x$, combining this equation with the second item of Proposition 1.4.4 gives

$$(f \sqcup g)^L(x) = \bigvee_{u \leq x, v \leq x} (f(u) \wedge g(v)) = \bigvee_{u \leq x} f(u) \;\wedge \bigvee_{v \leq x} g(v)$$

This yields $(f \sqcup g)^L = f^L \wedge g^L$. Lemma 2.2.3 provides that $f^L \wedge g^L = f^L \sqcup g^L$, so this establishes item 1.

For item 2,

$$(f \sqcup g)^R(x) = \bigvee_{y \geq x} (f \sqcup g)(y) = \bigvee_{y \geq x} \; \bigvee_{u \vee v = y} (f(u) \wedge g(v))$$

Since $u \vee v = y$ for some $y \geq x$ if and only if $u \geq x$ or $v \geq x$, this gives

$$(f \sqcup g)^R(x) = \bigvee_{u \geq x, v \in \mathrm{I}} (f(u) \wedge g(v)) \;\vee \bigvee_{u \in \mathrm{I}, v \geq x} (f(u) \wedge g(v))$$

Using the second item of Proposition 1.4.4 twice, gives

$$(f \sqcup g)^R(x) = (\bigvee_{u \geq x} f(u) \;\wedge \bigvee_{v \in \mathrm{I}} g(v)) \;\vee (\bigvee_{u \in \mathrm{I}} f(u) \;\wedge \bigvee_{v \geq x} g(v))$$

Since $f^{RL}(x)$ is the supremum of the values taken by f, this expression then gives that $(f \sqcup g)^R = (f^R \wedge g^{RL}) \vee (f^{RL} \wedge g^R)$, and this is the formula for $f^R \sqcup g^R$. The remaining items follow by symmetry. ■

2.3 Properties of the operations

In this, the key section of this chapter, we give some equational properties of the algebra M. In particular, we show that it is a De Morgan Birkhoff system, as described in the preliminaries. This result will be built in stages.

Proposition 2.3.1 *The operations $\sqcap$ and $\sqcup$ of* M *are semilattice operations, meaning that they are commutative, associative, and idempotent. Thus* $(\mathrm{M}, \sqcap, \sqcup)$ *is a bisemilattice.*

Proof. We show the results for $\sqcup$, and those for $\sqcap$ are similar. For idempotence, Theorem 1.4.5 and Lemma 2.2.1 give

$$f \sqcup f = (f \wedge f^L) \vee (f^L \wedge f) = f$$

Commutativity is immediate since the expressions for $f \sqcup g$ in Theorem 1.4.5 are symmetric in these variables. For associativity, Theorem 1.4.5, Lemma 2.2.4 and Lemma 2.2.3 give

$$\begin{aligned}(f \sqcup g) \sqcup h &= [((f \vee g) \wedge (f^L \wedge g^L)) \vee h] \wedge [(f \sqcup g)^L \wedge h^L] \\ &= [((f \vee g) \wedge (f^L \wedge g^L)) \vee h] \wedge (f^L \wedge g^L \wedge h^L) \\ &= [(f \vee g \vee h) \wedge ((f^L \wedge g^L) \vee h)] \wedge (f^L \wedge g^L \wedge h^L) \\ &= (f \vee g \vee h) \wedge (f^L \wedge g^L \wedge h^L)\end{aligned}$$

A similar calculation for $f \sqcup (g \sqcup h) = (g \sqcup h) \sqcup f$ will also yield this expression. So $\sqcup$ is associative. ■

We remark that in the proof of this result, we have obtained an expression for $f \sqcup g \sqcup h$. This expression can be extended to more functions as follows.

$$\begin{aligned}f_1 \sqcup \cdots \sqcup f_n &= (f_1 \vee \cdots \vee f_n) \wedge f_1^L \wedge \cdots \wedge f_n^L \\ f_1 \sqcap \cdots \sqcap f_n &= (f_1 \vee \cdots \vee f_n) \wedge f_1^R \wedge \cdots \wedge f_n^R\end{aligned}$$

We will not use these, and leave the proof as an exercise (Exercise 9).

Proposition 2.3.2 *The bisemilattice* $(\mathrm{M}, \sqcap, \sqcup)$ *satisfies Birkhoff's equation*

$$f \sqcap (f \sqcup g) = f \sqcup (f \sqcap g)$$

Therefore $(\mathrm{M}, \sqcap, \sqcup)$ *is a Birkhoff system.*

Proof. Using Theorem 1.4.5, Lemma 2.2.4, and the distributive laws for $\wedge$ and $\vee$,

$$\begin{aligned}f \sqcup (f \sqcap g) &= [f \wedge (f \sqcap g)^L] \vee [f^L \wedge (f \sqcap g)] \\ &= [f \wedge (f^L \sqcap g^L)] \vee [f^L \wedge (f \sqcap g)] \\ &= [f \wedge (f^L \vee g^L) \wedge f^{LR} \wedge g^{LR}] \vee [f^L \wedge ((f \wedge g^R) \vee (f^R \wedge g))] \\ &= [f \wedge g^{RL}] \vee [(f^L \wedge f \wedge g^R) \vee (f^L \wedge f^R \wedge g)] \\ &= (f \wedge g^{RL}) \vee (f^L \wedge f^R \wedge g)\end{aligned}$$

and

$$\begin{aligned}f \sqcap (f \sqcup g) &= [f \wedge (f \sqcup g)^R] \vee [f^R \wedge (f \sqcup g)] \\ &= [f \wedge (f^R \sqcup g^R)] \vee [f^R \wedge (f \sqcup g)] \\ &= [f \wedge (f^R \vee g^R) \wedge f^{LR} \wedge g^{LR}] \vee [f^R \wedge ((f \wedge g^L) \vee (f^L \wedge g))] \\ &= [f \wedge g^{LR}] \vee [(f^R \wedge f \wedge g^L) \vee (f^R \wedge f^L \wedge g)] \\ &= (f \wedge g^{RL}) \vee (f^L \wedge f^R \wedge g)\end{aligned}$$

This establishes Birkhoff's equation. ■

We remark that in the course of the proof, we have obtained a useful description for the value of quantity obtained in Birkhoff's equation, namely

$$f \sqcap (f \sqcup g) = (f \wedge g^{LR}) \vee (f^R \wedge f^L \wedge g) = f \sqcup (f \sqcap g) \tag{2.2}$$

We now come to the goal of this section. The following equations for M are the ones most frequently encountered. However, there are other equations that are satisfied by M that are not consequences of these, as we will see in Chapter 8.

Theorem 2.3.3 *The algebra* $(\mathrm{M}, \sqcap, \sqcup, ^*, 1_0, 1_1)$ *is a De Morgan Birkhoff system. That is, it satisfies the following equations:*

1. $f \sqcup f = f$; $f \sqcap f = f$.
2. $f \sqcup g = g \sqcup f$; $f \sqcap g = g \sqcap f$.
3. $f \sqcup (g \sqcup h) = (f \sqcup g) \sqcup h$; $f \sqcap (g \sqcap h) = (f \sqcap g) \sqcap h$.
4. $f \sqcup (f \sqcap g) = f \sqcap (f \sqcup g)$.
5. $1_0 \sqcup f = f$; $1_1 \sqcap f = f$.
6. $f^{**} = f$.
7. $(f \sqcup g)^* = f^* \sqcap g^*$; $(f \sqcap g)^* = f^* \sqcup g^*$.

Proof. We have already established that $(\mathrm{M}, \sqcap, \sqcup)$ is a Birkhoff system, so (1)–(4) hold. For (5) note that $f^L(0) = f(0)$, and that 1_0^L is the constant function 1. Therefore by Theorem 1.4.5

$$1_0 \sqcup f = (1_0 \wedge f^L) \vee (1_0^L \wedge f) = f$$

A similar calculation shows $1_1 \sqcap f = f$. Item (6) is given in Lemma 2.2.2. For (7) we use Theorem 1.4.5 and Lemma 2.2.2.

$$\begin{aligned}(f \sqcup g)^* &= ((f \vee g) \wedge f^L \wedge g^L)^* \\ &= (f^* \vee g^*) \wedge f^{L*} \wedge g^{L*} \\ &= (f^* \vee g^*) \wedge f^{*R} \wedge g^{*R} \\ &= f^* \sqcap g^*\end{aligned}$$

Similarly, $(f \sqcap g)^* = f^* \sqcup g^*$. ∎

To conclude this section, we note that M does not have the further property of being a lattice. For example, take f to have all its values larger than the supremum of the values of g. Then by (2.2), we obtain

$$f \sqcap (f \sqcup g) = (f \wedge g^{LR}) \vee (f^R \wedge f^L \wedge g) = g^{LR} \tag{2.3}$$

which is constant with value the supremum of the values of g, and has no value in common with f. The function f constant with value 1, and g constant with value anything less than 1, will work for this counterexample.

2.4 Two partial orderings

As mentioned in the preliminaries, to each bisemilattice, we associate two partial orderings, one for each of the semilattice operations. We specify the notation used when applying this to the algebra M.

Definition 2.4.1 *Define relations* $\sqsubseteq_{\sqcap}$ *and* $\sqsubseteq_{\sqcup}$ *on* M *as follows:*

$$\begin{array}{lll} f \sqsubseteq_{\sqcap} g & \text{if} & f \sqcap g = f \\ f \sqsubseteq_{\sqcup} g & \text{if} & f \sqcup g = g \end{array}$$

We call $\sqsubseteq_{\sqcap}$ *the* ***meet order*** *and* $\sqsubseteq_{\sqcup}$ *the* ***join order****.*

Applying results described in the preliminaries that apply to any bisemilattice, in particular in Proposition 2.1.9, we immediately obtain the following.

Proposition 2.4.2 *Both* $\sqsubseteq_{\sqcap}$ *and* $\sqsubseteq_{\sqcup}$ *are partial orders on* M*. Further, these relations have the following properties.*

1. *Under the partial order* $\sqsubseteq_{\sqcap}$*, any two elements* f *and* g *have a greatest lower bound. That greatest lower bound is* $f \sqcap g$*.*
2. *Under the partial order* $\sqsubseteq_{\sqcup}$*, any two elements* f *and* g *have a least upper bound. That least upper bound is* $f \sqcup g$*.*

The partial orderings $\sqsubseteq_{\sqcap}$ and $\sqsubseteq_{\sqcup}$ are defined through the convolution operations $\sqcap$ and $\sqcup$. In Theorem 1.4.5, we have simplified the descriptions of these convolution operations, and expressed them in terms of the pointwise operations $\wedge, \vee$ and the operations L, R. This allows the following.

Proposition 2.4.3 *The pointwise criteria for* $\sqsubseteq_{\sqcap}$ *and* $\sqsubseteq_{\sqcup}$ *are these:*

1. $f \sqsubseteq_{\sqcap} g$ *if and only if* $f^R \wedge g \leq f \leq g^R$*.*
2. $f \sqsubseteq_{\sqcup} g$ *if and only if* $f \wedge g^L \leq g \leq f^L$*.*

Proof. Theorem 1.4.5 states that

$$f \sqcap g = (f \wedge g^R) \vee (f^R \wedge g) = (f \vee g) \wedge f^R \wedge g^R$$

So if $f = f \sqcap g$, then $f^R \wedge g \leq f \leq g^R$. Conversely, if $f^R \wedge g \leq f \leq g^R$, then

$$f \sqcap g = (f \wedge g^R) \vee (f^R \wedge g) = f \vee (f^R \wedge g) = f$$

so $f \sqsubseteq_{\sqcap} g$. Item 2 follows similarly. ■

To conclude this section, we collect various properties of these orders.

Proposition 2.4.4 *The following hold for $f, g \in \mathrm{M}$:*

1. $f \sqsubseteq_{\sqcap} 1_1$ *and* $1_0 \sqsubseteq_{\sqcup} f$.
2. $f \sqsubseteq_{\sqcap} g$ *if and only if* $g^* \sqsubseteq_{\sqcup} f^*$.
3. *If* f *and* g *are decreasing, then* $f \sqsubseteq_{\sqcap} g$ *if and only if* $f \leq g$.
4. *If* f *is decreasing, then* $f \sqsubseteq_{\sqcap} g$ *if and only if* $f \leq g^R$.
5. *If* f *and* g *are increasing, then* $f \sqsubseteq_{\sqcup} g$ *if and only if* $g \leq f$.
6. *If* g *is increasing, then* $f \sqsubseteq_{\sqcup} g$ *if and only if* $g \leq f^L$.

These statements follow from Proposition 2.4.3 using observations from Section 2.2. For instance, in item (3), if both f and g are decreasing, then $f^R = f$ and $g^R = g$. So the condition for $f \sqsubseteq_{\sqcap} g$ becomes $f \wedge g \leq f \leq g$, which is equivalent simply to $f \leq g$. The proofs of the remaining items are left as an exercise (Exercise 14).

2.5 Fragments of distributivity

In this section, we consider the matter of when one type of binary operation, such as $\sqcap$, $\sqcup$, $\wedge$ or $\vee$, distributes over another. We begin with the following basic, but important, result that was discussed at length in Chapter 1.

Proposition 2.5.1 *The pointwise operations $\wedge$ and $\vee$ distribute over one another. That is,*

$$f \vee (g \wedge h) = (f \vee g) \wedge (f \vee h)$$
$$f \wedge (g \vee h) = (f \wedge g) \vee (f \wedge h)$$

We next consider another version of distributivity. It is of interest since it shows that the convolution operations $\sqcap$ and $\sqcup$ are **operators** on the lattice $(\mathrm{M}, \wedge, \vee)$ in the sense of [27, 59, 60], meaning that they distribute over $\vee$. We note that Lemma 2.2.2 shows that L, R and * are also operators on this lattice. We return to this matter in Chapter 6.

Proposition 2.5.2 *The operations $\sqcup$ and $\sqcap$ distribute over $\vee$. That is,*

$$f \sqcup (g \vee h) = (f \sqcup g) \vee (f \sqcup h) \tag{2.4}$$
$$f \sqcap (g \vee h) = (f \sqcap g) \vee (f \sqcap h)$$

Proof. Let $f, g, h \in \mathrm{M}$. Then by Theorem 1.4.5, Lemma 2.2.2, and the distributive law for the pointwise operations $\wedge$ and $\vee$, we have

$$\begin{aligned} f \sqcup (g \vee h) &= (f \wedge (g \vee h)^L) \vee (f^L \wedge (g \vee h)) \\ &= (f \wedge (g^L \vee h^L)) \vee ((f^L \wedge g) \vee (f^L \wedge h)) \\ &= (f \wedge g^L) \vee (f \wedge h^L) \vee (f^L \wedge g) \vee (f^L \wedge h) \\ &= (f \sqcup g) \vee (f \sqcup h) \end{aligned}$$

Thus $\sqcup$ distributes over $\vee$. The other argument follows by symmetry. ■

Various distributive laws do not hold in M: $\sqcup$ and $\sqcap$ do not distribute over $\wedge$; $\vee$ distributes over neither $\sqcup$ nor $\sqcap$, and similarly, $\wedge$ distributes over neither $\sqcup$ nor $\sqcap$; and $\sqcup$ and $\sqcap$ do not distribute over each other. There are easy examples to show this (Exercise 16).

While the distributive laws for the convolution operations do not generally hold in M, it will be essential in subsequent chapters to find circumstances in which they do hold. The following technical result is of primary importance in this task.

Lemma 2.5.3 *Let $f,\ g,\ h \in \mathrm{M}$.*

1. *$f \sqcup (g \sqcap h)$ is given by*

$$(f \wedge g^L \wedge h^{RL}) \vee (f \wedge g^{RL} \wedge h^L) \vee (f^L \wedge g \wedge h^R) \vee (f^L \wedge g^R \wedge h)$$

2. *$(f \sqcup g) \sqcap (f \sqcup h)$ is given by*

$$(f \sqcup (g \sqcap h)) \vee (f^L \wedge f^R \wedge g \wedge h^{RL}) \vee (f^R \wedge f^L \wedge g^{RL} \wedge h)$$

The term $f \sqcup (g \sqcap h)$ in item 2 may be expanded using item 1. Expressions for $f \sqcap (g \sqcup h)$ and $(f \sqcap g) \sqcup (f \sqcap h)$ are given by interchanging the operations L and R in these expressions.

Proof. Let $f,\ g,\ h \in \mathrm{M}$.

$$\begin{aligned} f \sqcup (g \sqcap h) &= [f \wedge (g \sqcap h)^L] \vee [f^L \wedge (g \sqcap h)] \\ &= [f \wedge (g^L \sqcap h^L)] \vee [f^L \wedge (g \sqcap h)] \\ &= [f \wedge ((g^L \wedge h^{RL}) \vee (g^{RL} \wedge h^L))] \vee [f^L \wedge ((g \wedge h^R) \vee (g^R \wedge h))] \\ &= (f \wedge g^L \wedge h^{RL}) \vee (f \wedge g^{RL} \wedge h^L) \vee (f^L \wedge g \wedge h^R) \vee (f^L \wedge g^R \wedge h) \end{aligned}$$

This establishes the first equation.

$$\begin{aligned}
(f \sqcup g) \sqcap (f \sqcup h) &= [(f \sqcup g) \wedge (f \sqcup h)^R] \vee [(f \sqcup g)^R \wedge (f \sqcup h)] \\
&= \{[(f \wedge g^L) \vee (f^L \wedge g)] \wedge (f^R \sqcup h^R)\} \vee [(f^R \sqcup g^R) \wedge (f \sqcup h)] \\
&= \{[(f \wedge g^L) \vee (f^L \wedge g)] \wedge [(f^R \wedge h^{RL}) \vee (f^{RL} \wedge h^R)]\} \vee \\
&\quad \{[(f^R \wedge g^{RL}) \vee (f^{RL} \wedge g^R)] \wedge [(f \wedge h^L) \vee (f^L \wedge h)]\} \\
&= [(f \wedge g^L) \wedge (f^R \wedge h^{RL})] \vee [(f \wedge g^L) \wedge (f^{RL} \wedge h^R)] \vee \\
&\quad [(f^L \wedge g) \wedge (f^R \wedge h^{RL})] \vee [(f^L \wedge g) \wedge (f^{RL} \wedge h^R)] \vee \\
&\quad [(f^R \wedge g^{RL}) \wedge (f \wedge h^L)] \vee [(f^R \wedge g^{RL}) \wedge (f^L \wedge h)] \vee \\
&\quad [(f^{RL} \wedge g^R) \wedge (f \wedge h^L)] \vee [(f^{RL} \wedge g^R) \wedge (f^L \wedge h)] \\
&= [f \wedge g^L \wedge h^{RL}] \vee [f \wedge g^L \wedge h^R] \vee [f^L \wedge f^R \wedge g \wedge h^{RL}] \vee \\
&\quad [f^L \wedge g \wedge h^R] \vee [f \wedge g^{RL} \wedge h^L] \vee [f^R \wedge f^L \wedge g^{RL} \wedge h] \vee \\
&\quad [f \wedge g^R \wedge h^L] \vee [f^L \wedge g^R \wedge h] \\
&= [f \wedge g^L \wedge h^{RL}] \vee [f \wedge g^{RL} \wedge h^L] \vee [f^L \wedge g \wedge h^R] \vee \\
&\quad [f^L \wedge g^R \wedge h] \vee [f^L \wedge f^R \wedge g \wedge h^{RL}] \vee [f^R \wedge f^L \wedge g^{RL} \wedge h]
\end{aligned}$$

This establishes the second. ■

Corollary 2.5.4 *For $f, g, h \in \mathrm{M}$, we have $f \sqcup (g \sqcap h) \le (f \sqcup g) \sqcap (f \sqcup h)$.*

2.6 Summary

In this chapter we gave basic properties of the various operations on M and their relations to one another. These were used to show that M is a De Morgan Birkhoff system. We described two partial orderings $\sqsubseteq_{\sqcap}$ and $\sqsubseteq_{\sqcup}$ on M and described basic properties of these partial orderings. Matters related to the distributivity of various operations over others were discussed.

For the convenience of the reader, the various properties of the operations obtained in this section are collected in the Appendix.

2.7 Exercises

1. Prove Proposition 2.1.5 demonstrating how meet semilattices and join semilattices determine semilattices.

2. Prove the second half of Proposition 2.1.6, showing how semilattices determine join semilattices.

3. Prove Proposition 2.1.9, which focuses on orders $\leq_\wedge$ and $\leq_\vee$.

4. Give an example of functions $f, g \in M$ with $(f \wedge g)^L \neq f^L \wedge g^L$. Provide pictures of the functions f, g, $f \wedge g$, f^L, g^L, $f^L \wedge g^L$ and $(f \wedge g)^L$. Show that $(f \wedge g)^L \leq f^L \wedge g^L$. (See discussion after Lemma 2.2.2.)

5. Prove that $f^R \sqcap g^R = f^R \sqcap g = f \sqcap g^R = f^R \wedge g^R$ (item 2 of Lemma 2.2.3).

6. Prove that $(f \sqcap g)^R = f^R \sqcap g^R$ and $(f \sqcap g)^L = f^L \sqcap g^L$ (Lemma 2.2.4, items 3 and 4).

7. Using the functions you chose for Exercise 4, provide pictures of $f \sqcap g$, $f^L \sqcap g^L$ and $(f \sqcap g)^L$, illustrating the fact that the latter two are the same.

8. Prove that $\sqcap$ is commutative, associative, and idempotent (Proposition 2.3.1).

9. Show that $f_1 \sqcup f_2 \sqcup f_3 \sqcup f_4 = (f_1 \vee f_2 \vee f_3 \vee f_4) \wedge f_1^L \wedge f_2^L \wedge f_3^L \wedge f_4^L$. (See comments following Proposition 2.3.1.)

10. Prove that $(f \sqcap g)^* = f^* \sqcup g^*$; that is, complete the proof of Theorem 2.3.3.

11. Prove that both $\sqsubseteq_\sqcap$ and $\sqsubseteq_\sqcup$ are partial orders on M (Proposition 2.4.2).

12. Prove that under the partial order $\sqsubseteq_\sqcap$, any two elements f and g have a greatest lower bound, and that greatest lower bound is $f \sqcup g$ (Proposition 2.4.2).

13. Show that for any $f, g \in M$, $f \sqsubseteq_\sqcup g$ if and only if $f \wedge g^L \leq g \leq f^L$ (Proposition 2.4.3).

14. Prove items 1, 2, and 4 of Proposition 2.4.4.

15. Show that $f \sqcap (g \vee h) = (f \sqcap g) \vee (f \sqcap h)$ (Proposition 2.5.2).

16. Prove the following:

    (a) $\sqcup$ and $\sqcap$ do not distribute over $\wedge$.

(b) $\vee$ distributes over neither $\sqcup$ nor $\sqcap$.

(c) $\wedge$ distributes over neither $\sqcup$ nor $\sqcap$.

(d) $\sqcup$ and $\sqcap$ do not distribute over each other.

17. Sketch the functions f, g defined by

$$f(x) = \begin{cases} 1/4 & \text{if} \quad 0 \le x < 1/3 \\ 1/2 & \text{if} \quad 1/3 \le x \le 1/2 \\ 1/3 & \text{if} \quad 1/2 \le x \le 3/4 \\ 7/8 & \text{if} \quad 3/4 < x \le 1 \end{cases}$$

and

$$g(x) = \begin{cases} 3/4 & \text{if} \quad 0 \le x < 1/3 \\ 1/8 & \text{if} \quad 1/3 \le x \le 3/4 \\ 1 & \text{if} \quad 3/4 < x \le 1 \end{cases}$$

Then, sketch $f \sqcap g$, $f^L \sqcap g^L$, and $(f \sqcap g)^L$.

18. Using the functions of Exercise 17, sketch $f \sqcup g$, $f^R \sqcup g^R$, and $(f \sqcup g)^R$.

Chapter 3

Subalgebras of the Type-2 Truth Value Algebra

Type-2 fuzzy sets are mappings into the algebra M. This latter algebra is the one for which we have been deriving properties. We will see in Theorems 3.2.3 and 3.3.5 that M contains as subalgebras isomorphic copies of the algebras $([0,1], \vee, \wedge, ', 0, 1)$ and $([0,1]^{[2]}, \vee, \wedge, ', 0, 1)$. This legitimizes the claim that type-2 fuzzy sets are generalizations of type-1 and of interval-valued fuzzy sets. But M contains many other subalgebras of interest. This chapter examines several of these subalgebras.

3.1 Preliminaries

Here we follow the definitions and notation given in the Preliminaries of Chapter 1.

Definition 3.1.1 *Let A be a set and $f : A^n \to A$ be an n-ary operation on A. A subset $S \subseteq A$ is* ***closed under the operation*** *f if for each $s_1 \ldots, s_n \in S$, $f(s_1, \ldots, s_n) \in S$.*

So for example, a subspace S of a vector space V is closed under the operations of vector addition and scalar multiplication. (See Exercise 2 of

Chapter 1.) If you add two vectors in S, you get a vector in S, and if you multiply a vector in S by a scalar, you get a vector in S. A vector space also has a nullary operation, or constant, namely, the zero vector. Each subspace is also closed under this nullary operation, meaning simply that it contains the zero vector. We come now to the primary topic of this section.

Definition 3.1.2 *For an algebra A, a subset $S \subseteq A$ is a **subalgebra** of A if S is closed under the operations of A.*

If S is a subalgebra of A, then each n-ary operation $f : A^n \to A$ of the algebra A restricts to an n-ary operation on S. This **restriction** is properly denoted $f|S^n : S^n \to S$, but we shall commonly refer to it also simply as f. With these restricted operations, S forms an algebra of the same type as A. This is the familiar notion that a subspace of a vector space is itself a vector space, and that a subgroup of a group is itself a group.

Proposition 3.1.3 *If A is an algebra and S is a subalgebra of A, then any equation that is valid in A is also valid in S.*

This is simple to see. To illustrate, suppose an algebra A has a binary operation + and that A satisfies the commutative equation $x + y = y + x$. This means that for all $x, y \in A$ that $x + y = y + x$. Therefore, if S is a subalgebra of A, we clearly have that $x + y = y + x$ for all $x, y \in S$. However, it is possible for a subalgebra S of A to satisfy equations that are not valid in A. This situation arises, for instance, when S is a commutative subgroup of a non-commutative group A.

Proposition 3.1.4 *If A is an algebra and S_j $(j \in J)$ is a family of subalgebras of A, then the intersection $S = \bigcap_J S_j$ is a subalgebra of A.*

The proof of this result is left as an exercise (see Exercise 1). It has as a simple consequence that for any subset $X \subseteq A$, that there is a smallest subalgebra of A that contains X, namely the intersection of the set of all subalgebras of A that contain X. This allows the following definition.

Definition 3.1.5 *For A an algebra and $X \subseteq A$, the smallest subalgebra of A that contains X is called the **subalgebra generated by** X.*

A familiar instance is that the subalgebra of a vector space V generated by a set X is the span of X. In Chapter 7, we shall see that M is **locally finite**, meaning that each subalgebra generated by a finite set is finite. This will have a number of important consequences.

Definition 3.1.6 *Let A and B be algebras of the same type and $\alpha : A \to B$. We say α is a **homomorphism** if for each n-ary operation f of A, and for each $a_1, \ldots, a_n \in A$, we have*

$$\alpha(f(a_1, \ldots, a_n)) = f(\alpha(a_1), \ldots, \alpha(a_n))$$

For instance, a homomorphism $\alpha : L \to M$ from a lattice L to a lattice M satisfies $\alpha(x \wedge y) = \alpha(x) \wedge \alpha(y)$ and $\alpha(x \vee y) = \alpha(x) \vee \alpha(y)$ for all $x, y \in L$. If we consider homomorphisms between bounded lattices, where 0 and 1 are extra constant operations, then homomorphisms must additionally satisfy $\alpha(0) = 0$ and $\alpha(1) = 1$.

Definition 3.1.7 *A homomorphism $\alpha : A \to B$ is called an **isomorphism** if it is one-to-one and onto. Two algebras A and B are said to be **isomorphic** if there is an isomorphism from one to the other.*

The proof of the following result is left as an exercise (Exercise 6).

Proposition 3.1.8 *Suppose A and B are algebras and $\alpha : A \to B$ is a one-to-one homomorphism. Then the **image** $\alpha(A) = \{\alpha(a) : a \in A\}$ of α is a subalgebra of B and α is an isomorphism from A to its image.*

3.2 Type-1 fuzzy sets

Type-1 fuzzy sets take values in the unit interval $[0,1]$. To realize type-1 fuzzy sets as special type-2 fuzzy sets, we realize the algebra $([0,1], \wedge, \vee, ', 0, 1)$ as a subalgebra of $\mathbb{M}$.

Definition 3.2.1 *For each $a \in [0,1]$, the **characteristic function** of a is the function $1_a : [0,1] \to [0,1]$ that takes a to 1 and all other elements to 0. The functions 1_a are called **singletons**. The set of all singletons is denoted $\mathbb{S}$.*

These characteristic functions of points in $[0,1]$ are clearly in one-to-one correspondence with $[0,1]$, but much more is true.

Proposition 3.2.2 *The following hold for $a, b \in [0,1]$:*

1. *The characteristic function of $a \vee b$ is $1_a \sqcup 1_b$.*
2. *The characteristic function of $a \wedge b$ is $1_a \sqcap 1_b$.*
3. *The characteristic function of a' is 1_a^*.*
4. *The characteristic functions of 0 and 1 are 1_0 and 1_1.*

Proof. For item 1, by (1.3), $1_a \sqcup 1_b = (1_a \vee 1_b) \wedge (1_a^L \wedge 1_b^L)$. The function $1_a \vee 1_b$ takes value 1 at a and b and 0 otherwise. Since $a \vee b$ is the larger of a and b, the function $1_a^L \wedge 1_b^L$ takes value 1 for all $x \geq a \vee b$ and is 0 otherwise. It follows that $1_a \sqcup 1_b$ is the characteristic function of $a \vee b$. Item 2 is similar, and item 3 follows immediately from the definition $1_a^*(x) = 1_a(1 - x)$. Item 4 is by definition. ∎

This result may be rephrased by saying that the mapping $\alpha : [0,1] \to \mathrm{M}$ defined by $\alpha(a) = 1_a$ is a homomorphism. We sometimes indicate a mapping such as this with the notation $a \mapsto 1_a$. Since this mapping is obviously one-to-one, we have the following as a consequence of Proposition 3.1.8.

Theorem 3.2.3 *The mapping $a \mapsto 1_a$ is an isomorphism from the algebra $([0,1], \wedge, \vee, ', 0, 1)$ onto the subalgebra of M consisting of its characteristic functions of points; that is, the subalgebra S of singletons.*

Since S is isomorphic to $([0,1], \wedge, \vee, ', 0, 1)$, it is a Kleene algebra. In fact, its underlying lattice is a chain. While the characteristic functions of points are closed under all the basic operations of M, hence form a subalgebra with respect to this full type, we shall also be interested in subsets of M that are closed under perhaps only $\sqcap$ and $\sqcup$. Such subsets are subalgebras of $(\mathrm{M}, \sqcap, \sqcup)$.

Definition 3.2.4 *A subalgebra of $(\mathrm{M}, \sqcap, \sqcup)$ that is a lattice is a* ***sublattice*** *of M. If it is additionally a chain, it is called a* ***subchain*** *of M.*

Since $(\mathrm{M}, \sqcap, \sqcup)$ is a Birkhoff system, each of its subalgebras is also a Birkhoff system, and hence a bisemilattice. The following is an immediate consequence of Propositions 2.1.3 and 2.1.8.

Proposition 3.2.5 *For a subalgebra A of $(\mathrm{M}, \sqcap, \sqcup)$, the following statements are equivalent:*

1. *A is a lattice.*
2. *A satisfies the absorption laws.*
3. *The partial orderings $\sqsubseteq_\sqcap$ and $\sqsubseteq_\sqcup$ agree on A.*

We next turn our attention to how the subalgebra S is situated in M. Our aim is to show that it is a **maximal subchain** of M, meaning that it is a subchain of M, and that there is no other subchain of M that properly contains it. We begin with the following.

Lemma 3.2.6 *Suppose $a \in [0,1]$ and $f \in \mathrm{M}$. Then the following hold:*

1. *If $1_a \sqsubseteq_\sqcap f$, then $f(x) = 0$ for $0 \leq x < a$ and $f^R(a) = 1$.*
2. *If $f \sqsubseteq_\sqcup 1_a$, then $f(x) = 0$ for $a < x \leq 1$ and $f^L(a) = 1$.*

Proof. Suppose that $1_a \sqsubseteq_\sqcap f$. Then by Proposition 2.4.3, $1_a^R \wedge f \leq 1_a \leq f^R$. Now

$$1_a^R(x) = \begin{cases} 1 & \text{if} \quad 0 \leq x \leq a \\ 0 & \text{if} \quad a < x \leq 1 \end{cases}$$

is the characteristic function of the interval $[0,a]$. Thus for $0 \leq x < a$, we have $1_a^R(x) \wedge f(x) = f(x) \leq 1_a(x) = 0$. And $1_a(a) = 1 \leq f^R(a)$. On the other hand, if $f \sqsubseteq_\sqcup 1_a$, we have $1_a^L \wedge f \leq 1_a \leq f^L$. This says that for $a < x \leq 1$, $1_a^L(x) \wedge f(x) = f(x) \leq 1_a(x) = 0$, and $1_a(a) = 1 \leq f^L(a)$. ■

Proposition 3.2.7 *The only subchain of* $\mathbb{M}$ *that contains the subalgebra* $\mathbb{S}$ *is* $\mathbb{S}$ *itself.*

Proof. Suppose C is a subchain of $\mathbb{M}$ that contains $\mathbb{S}$. By Proposition 3.2.5, the partial orders $\sqsubseteq_\sqcap$ and $\sqsubseteq_\sqcup$ agree on C, and we denote these simply by $\sqsubseteq$. Suppose $f \in C$ and let $a = \sup\{x : 1_x \sqsubseteq f\}$. Suppose $y < a$. Then there is x with $y < x < a$, hence with $1_x \sqsubseteq f$. By Lemma 3.2.6, we have f is identically 0 on $[0, x)$, hence $f(y) = 0$. Thus f is identically 0 on $[0, a)$. Suppose $a < y$. Then there is z with $a < z < y$, hence with $f \sqsubseteq 1_z$. By Lemma 3.2.6, we have f is identically 0 on $(z, 1]$, hence $f(y) = 0$. Thus f is identically 0 on $(a, 1]$. Choose any number, say $b = 0.5$. Then since f is comparable to 1_b, Lemma 3.2.6 implies that either $f^L(b) = 1$ or $f^R(b) = 1$. Since $f(x) = 0$ for all $x \neq a$, it follows that $f(a) = 1$, and hence $f = 1_a$. ∎

3.3 Interval-valued fuzzy sets

Interval-valued type-1 fuzzy sets take values in $[0,1]^{[2]}$, so are realized as ordered pairs (a, b) of real numbers where $a \leq b$. Such ordered pairs correspond to non-empty closed intervals $[a, b]$ of the real unit interval, where we allow the possibility of singleton intervals $[a, a]$. To realize interval-valued fuzzy sets as special type-2 fuzzy sets, we realize the algebra $([0,1]^{[2]}, \wedge, \vee, 1_0, 1_1)$ as a subalgebra of $\mathbb{M}$.

Definition 3.3.1 *For* $a, b \in [0,1]$ *with* $a \leq b$*, the* ***characteristic function*** *of the interval* $[a, b]$ *is the function* $1_{[a,b]}$ *from* $[0,1]$ *to* $[0,1]$ *that takes value* 1 *at each* $x \in [a, b]$ *and takes value* 0 *otherwise.*

The following basic result is a simple consequence of the definitions.

Proposition 3.3.2 *For* $a, b \in [0,1]$ *with* $a \leq b$*, the characteristic function of the interval* $[a, b]$ *is equal to* $1_a^L \wedge 1_b^R$.

Clearly these characteristic functions of closed intervals are in one-to-one correspondence with the elements of $[0,1]^{[2]}$, but much more is true. First, a simple lemma.

Lemma 3.3.3 *Suppose* $a, b \in [0,1]$*. Then the following hold:*

$$1_a^L \wedge 1_b^L = (1_a \sqcup 1_b)^L \quad \text{and} \quad 1_a^R \vee 1_b^R = (1_a \sqcup 1_b)^R \tag{3.1}$$

Proof. By Proposition 3.2.2, $1_a \sqcup 1_b$ is the characteristic function of the point $a \vee b$. The result follows from a simple calculation using the definitions of L and R. ∎

Proposition 3.3.4 *For $f = 1_a^L \wedge 1_b^R$ and $g = 1_c^L \wedge 1_d^R$ with $a \leq b$ and $c \leq d$, the following hold:*

1. $f \sqcup g = (1_a \sqcup 1_c)^L \wedge (1_b \sqcup 1_d)^R$.
2. $f \sqcap g = (1_a \sqcap 1_c)^L \wedge (1_b \sqcap 1_d)^R$.
3. $f^* = (1_b^*)^L \wedge (1_a^*)^R$.

Proof. For item 1, note

$$\begin{aligned}
f \sqcup g &= (1_a^L \wedge 1_b^R) \sqcup (1_c^L \wedge 1_d^R) \\
&= [(1_a^L \wedge 1_b^R) \wedge (1_c^L \wedge 1_d^R)^L] \vee [(1_a^L \wedge 1_b^R)^L \wedge (1_c^L \wedge 1_d^R)] \\
&= [(1_a^L \wedge 1_b^R) \wedge 1_c^L] \vee [1_a^L \wedge (1_c^L \wedge 1_d^R)] \\
&= [(1_a^L \wedge 1_c^L) \wedge 1_b^R] \vee [(1_a^L \wedge 1_c^L) \wedge 1_d^R] \\
&= (1_a^L \wedge 1_c^L) \wedge (1_b^R \vee 1_d^R) \\
&= (1_a \sqcup 1_c)^L \wedge (1_b \sqcup 1_d)^R
\end{aligned}$$

The first equality in the proof is a substitution, and the second is by (1.3) on page 11. The third equality is by making the simple computations $(1_a^L \wedge 1_b^R)^L = 1_a^L$ and $(1_c^L \wedge 1_d^R)^L = 1_c^L$, making use of the fact that these are characteristic functions of intervals. The fourth equality is trivial, the fifth is the distributive laws for the pointwise operations, and the final step is by (3.1).

The proof of item 2 is similar. For item 3, f is the characteristic function of the interval $[a, b]$, so f^* is its mirror image in the line $x = 1/2$. So f^* is the characteristic function of the interval $[1 - b, 1 - a] = [b', a']$, and by Proposition 3.2.2 this is $(1_b^*)^L \wedge (1_a^*)^R$. ■

By Definition 1.2.10, the constants of $\mathrm{I}^{[2]}$ are $(0, 0)$ and $(1, 1)$. A simple computation shows that $1_0^L \wedge 1_0^R = 1_0$ and $1_1^L \wedge 1_1^R = 1_1$. Together with Proposition 3.3.4, this shows that the mapping $\alpha : \mathrm{I}^{[2]} \to \mathrm{M}$ defined by $\alpha(a, b) = 1_a^L \wedge 1_b^R$ is a homomorphism. Since it is clearly one-to-one, we have the following as a consequence of Proposition 3.1.8.

Theorem 3.3.5 *The mapping $(a, b) \mapsto 1_a^L \wedge 1_b^R$ is an isomorphism from $([0, 1]^{[2]}, \wedge, \vee, ', 0, 1)$ onto the subalgebra of M consisting of its characteristic functions of closed intervals.*

We will denote this subalgebra of M by $\mathrm{S}^{[2]}$. Since it is isomorphic to $\mathrm{I}^{[2]}$, it is a De Morgan algebra. Further, since each singleton is a closed interval, the characteristic function of a point is the characteristic function of a closed interval. Since both are subalgebras of M, we have the following.

Corollary 3.3.6 *The algebra S of characteristic functions of points is a subalgebra of the algebra $\mathrm{S}^{[2]}$ of characteristic functions of closed intervals, which in turn is a subalgebra of M.*

The usual interpretation of interval type-2 fuzzy sets is as members of Map(I, I) that are characteristic functions of closed intervals; that is, as a subalgebra of M. It seems that most applications of type-2 fuzzy sets are restricted to these interval type-2 ones, and in the discussion of type-2 fuzzy sets in [77], the emphasis is almost entirely on interval type-2 fuzzy sets. If attention is restricted to interval type-2 fuzzy sets, then one may work in the simpler algebra $\mathrm{I}^{[2]}$ of Definition 1.2.10. This is described in detail in [29]. However, we will see that there are many other natural and interesting classes of type-2 fuzzy sets that require the fuller description of M.

3.4 Normal functions

The usual definition of normality of an element f of M, is that $f(x) = 1$ for some x. We use a slightly weaker definition that coincides with the usual definition for continuous functions. This will allow greater flexibility.

Definition 3.4.1 *An element $f \in \mathrm{M}$ is* ***normal*** *if the least upper bound of the values it attains is* 1*. It is* ***strictly normal*** *if it attains the value* 1*. Let* N *be the set of all normal functions of* M*.*

There are several convenient ways to express this condition in terms of our operations on this algebra. The proof of the following proposition is immediate from definitions.

Proposition 3.4.2 *The following four conditions are equivalent:*

1. *f is normal.*
2. *$f^{RL} = 1$.*
3. *$f^L(1) = 1$.*
4. *$f^R(0) = 1$.*

Proposition 3.4.3 *The set* N *of all normal functions is a subalgebra of* M*.*

Proof. Here is the verification that it is closed under $\sqcup$. We use Lemma 2.2.3.

$$\begin{aligned}(f \sqcup g)^L(1) &= (f^L \sqcup g^L)(1)\\ &= (f^L \wedge g^L)(1)\\ &= f^L(1) \wedge g^L(1)\\ &= 1\end{aligned}$$

The other verifications are just as easy. ∎

The subalgebra N contains the subalgebra of characteristic functions of closed intervals that was discussed in the previous section. Normal functions will play an important role throughout our studies.

3.5 Convex functions

The common usage of a "convex" function from calculus, is a function that lies above its tangents. It is synonymous with "concave upward." There is a different usage common in fuzzy sets. An **α-cut** of f is $\{x : f(x) \geq \alpha\}$. In fuzzy theory, a convex function is one whose α-cuts are convex sets; that is, intervals. We reformulate this definition in a way that is more suited to our needs.

Definition 3.5.1 *An element $f \in \mathrm{M}$ is **convex** if whenever $x \leq y \leq z$, then $f(y) \geq f(x) \wedge f(z)$. Let C be the set of all convex functions in M.*

Proposition 3.5.2 *Every monotone function is convex, and the pointwise meet of convex functions is convex.*

Proof. Suppose f is monotone and that $x \leq y \leq z$. If f is increasing, then $f(x) \leq f(y) \leq f(z)$, and if f is decreasing, then $f(z) \leq f(y) \leq f(x)$. In either case, $f(y) \geq f(x) \wedge f(z)$, hence f is convex. For the second statement, suppose that f, g are convex, and set $h = f \wedge g$. Then if $x \leq y \leq z$, we have

$$h(y) = f(y) \wedge g(y) \geq f(x) \wedge f(z) \wedge g(x) \wedge g(z) = h(x) \wedge h(z).$$

Thus $h = f \wedge g$ is convex. ∎

There are a number of equivalent ways to express convexity.

Proposition 3.5.3 *For an element $f \in \mathrm{M}$, the following are equivalent:*

1. *f is convex.*
2. *$f = f^L \wedge f^R$.*
3. *f is the meet of an increasing function and a decreasing function.*

Proof. For 1 implies 2, suppose that f is convex. Then $f(y) \geq f(x) \wedge f(z)$ for all $x \leq y$ and for all $z \geq y$. Thus $f(y) \geq f^R(y) \wedge f^L(y) = (f^R \wedge f^L)(y)$, so $f \geq f^R \wedge f^L$. But it is always true that $f \leq f^R \wedge f^L$. Thus $f = f^R \wedge f^L$. That 2 implies 3 is trivial since f^L is increasing and f^R is decreasing. That 3 implies 1 follows immediately from Proposition 3.5.2. ∎

The above proposition essentially states that a convex function is one that first is increasing, then is decreasing. Of course, this colloquial view is not sufficiently precise, yet it contains the essential point.

Proposition 3.5.4 *The set* C *of convex functions is a subalgebra of* M.

Proof. The constants 1_0 and 1_1 are convex since they are monotone. Suppose that f is convex. Then by Proposition 3.5.3, $f = f^L \wedge f^R$, and so by Lemma 2.2.1, $f^* = (f^L \wedge f^R)^* = f^{L*} \wedge f^{R*} = f^{*L} \wedge f^{*R}$. Hence by Proposition 3.5.3, f^* is convex. Now suppose that f and g are convex. Then

$$\begin{aligned} f \sqcup g &= (f^L \wedge g) \vee (f \wedge g^L) \\ &= (f^L \wedge g^L \wedge g^R) \vee (f^L \wedge f^R \wedge g^L) \\ &= (f^L \wedge g^L) \wedge (f^R \vee g^R) \end{aligned}$$

Since $f^L \wedge g^L$ is increasing and $f^R \vee g^R$ is decreasing, $f \sqcup g$ is the pointwise meet of an increasing function and a decreasing function. So by Proposition 3.5.3, $f \sqcup g$ is convex. Now $f \sqcap g$ is convex since $(f \sqcap g)^* = f^* \sqcup g^*$ is convex. ■

Definition 3.5.5 *For* $f \in \mathrm{M}$, *let* $\Gamma(f) = f^L \wedge f^R$.

If A is an algebra and B is a subalgebra of A, a **retraction** of A onto B is a homomorphism $\alpha : A \to B$ such that $\alpha(b) = b$ for each $b \in B$. The following result, essentially from [110], gives a remarkable view of how the convex functions C sit inside M.

Theorem 3.5.6 *Consider the map* $\Gamma : \mathrm{M} \to \mathrm{C}$ *and let* $f \in \mathrm{M}$.

1. Γ *is a retraction from* M *onto* C.
2. $\Gamma(f)$ *is the least element of* C *that lies above* f *in the join order.*
3. $\Gamma(f)$ *is the largest element of* C *that lies below* f *in the meet order.*

Proof. Surely Γ is a map from M to C, and for each $f \in \mathrm{C}$ we have $\Gamma(f) = f$. To show that Γ is a retraction, it remains to show that it is a homomorphism. Suppose $f, g \in M$. Then using Lemmas 2.2.3 and 2.2.4 and Theorem 1.4.5,

$$\begin{aligned} \Gamma(f \sqcup g) &= (f \sqcup g)^L \wedge (f \sqcup g)^R \\ &= (f^L \sqcup g^L) \wedge (f^R \sqcup g^R) \\ &= (f^L \wedge g^L) \wedge (f^R \vee g^R) \wedge f^{LR} \wedge g^{LR} \\ &= (f^R \vee g^R) \wedge f^L \wedge g^L \end{aligned}$$

On the other hand, using Theorem 1.4.5, the observation that $(f^L \wedge f^R)^L = f^L$ since $f \leq f^L \wedge f^R$, and the distributive law,

$$\begin{aligned} \Gamma(f) \sqcup \Gamma(g) &= [(f^L \wedge f^R) \vee (g^L \wedge g^R)] \wedge (f^L \wedge f^R)^L \wedge (g^L \wedge g^R)^L \\ &= [(f^L \wedge f^R) \vee (g^L \wedge g^R)] \wedge f^L \wedge g^L \\ &= (f^L \wedge f^R \wedge g^L) \vee (f^L \wedge g^L \wedge g^R) \\ &= (f^R \vee g^R) \wedge f^L \wedge g^L \end{aligned}$$

So Γ preserves $\sqcup$. A similar calculation shows that it preserves $\sqcap$. By Lemma 2.2.2, $(f^L \wedge f^R)^* = (f^L)^* \wedge (f^R)^* = (f^*)^R \wedge (f^*)^L$. This shows that Γ preserves $*$. The constants 1_0 and 1_1 are convex, so they are preserved. Thus Γ is a homomorphism, hence a retraction.

For the second statement, since $\Gamma(f)$ is the least convex function pointwise above f, it follows that $f \sqcup \Gamma(f) = (f \vee \Gamma(f)) \wedge f^L \wedge \Gamma(f)^L = \Gamma(f) \wedge f^L = \Gamma(f)$. So $f \sqsubseteq_\sqcup \Gamma(f)$. Suppose g is another convex function with $f \sqsubseteq_\sqcup g$. Since Γ is a homomorphism, it preserves the join order. So $\Gamma(f) \sqsubseteq_\sqcup \Gamma(g) = g$. The third statement is similar. ∎

The following theorem gives a powerful and useful condition for determining if a given function is convex. (See [24, 81, 108].)

Theorem 3.5.7 *Given $f \in \mathrm{M}$, the distributive laws*

$$\begin{aligned} f \sqcup (g \sqcap h) &= (f \sqcup g) \sqcap (f \sqcup h) \\ f \sqcap (g \sqcup h) &= (f \sqcap g) \sqcup (f \sqcap h) \end{aligned} \tag{3.2}$$

hold for all $g, h \in \mathrm{M}$ if and only if f is convex. Moreover, one of these distributive laws holds for a given f and for all g, h if and only if the other holds.

Proof. Lemma 2.5.3 gives expressions for $f \sqcup (g \sqcap h)$ and $(f \sqcup g) \sqcap (f \sqcup h)$. From these, it follows that $(f \sqcup g) \sqcap (f \sqcup h)$ is equal to $f \sqcup (g \sqcap h) \vee K$, where

$$K = (f^L \wedge f^R \wedge g \wedge h^{RL}) \vee (f^R \wedge f^L \wedge g^{RL} \wedge h)$$

If f is convex, then $f^L \wedge f^R = f$, and using the description of $f \sqcup (g \sqcap h)$ in Lemma 2.5.3, then $K \leq f \sqcup (g \sqcap h)$. This gives $f \sqcup (g \sqcap h) = (f \sqcup g) \sqcap (f \sqcup h)$. The other distributive law follows similarly.

Suppose that $f \sqcup (g \sqcap h) = (f \sqcup g) \sqcap (f \sqcup h)$ holds for all g and h. Let h be the function that is 1 everywhere, and $g = 1_0$. Lemma 2.5.3 gives that $f \sqcup (g \sqcap h)$ is equal to f. However

$$K = (f^L \wedge f^R \wedge 1_0) \vee (f^R \wedge f^L \wedge 1_0^{RL})$$

Since $f \sqcup (g \sqcap h) = (f \sqcup g) \sqcap (f \sqcup h)$, we have $K \leq f$. Then since 1_0^{RL} is the constant function 1, we have $f^R \wedge f^L \leq f$, hence $f = f^L \wedge f^R$. By Proposition 3.5.3, this yields that f is convex. The other distributive law follows similarly. ∎

Here are two other cases where a distributive law holds.

Proposition 3.5.8 *Let $f, g, h \in \mathrm{M}$.*

1. *If g, h are decreasing, then $f \sqcap (g \sqcup h) = (f \sqcap g) \sqcup (f \sqcap h)$.*
2. *If g, h are increasing, then $f \sqcup (g \sqcap h) = (f \sqcup g) \sqcap (f \sqcup h)$.*

Proof. For item 1, note that Lemma 2.2.3 provides that $f \sqcap g^R = f^R \sqcap g$, and Lemma 2.2.4 provides that $(g \sqcup h)^R = g^R \sqcup h^R$. Since g and h are decreasing, we have $g = g^R$ and $h = h^R$. Therefore

$$\begin{aligned} f \sqcap (g^R \sqcup h^R) &= f \sqcap (g \sqcup h)^R = f^R \sqcap (g \sqcup h) \\ &= (f^R \sqcap g) \sqcup (f^R \sqcap h) = (f \sqcap g^R) \sqcup (f \sqcap h^R) \end{aligned}$$

The third equality is provided by Theorem 3.5.7 since f^R is decreasing, hence convex. Item 2 follows similarly. ■

The subalgebra C of convex functions is an interesting one. It contains the subalgebra $\mathrm{S}^{[2]}$ of intervals, which in turn contains the subalgebra S corresponding to the type-1 fuzzy sets. Moreover, when one considers the role of elements $f \in \mathrm{M}$ as giving "degrees of membership," it is not unnatural to restrict attention to such f that are convex. Indeed, the interval-valued fuzzy sets used in many type-2 applications associate a degree of membership uniformly distributed in an interval $[a, b]$. A sort of Gaussian centered in the middle of $[a, b]$ has natural appeal and is convex. While the subalgebra C has natural appeal, it has drawbacks. In particular, absorption fails in C as is seen in Equation (2.2) by taking any two distinct constant functions.

3.6 Convex normal functions

The set of functions that are both convex and normal is clearly a subalgebra of M, being the intersection of the subalgebra C of convex functions and the subalgebra N of normal functions. This subalgebra will be seen to enjoy many desirable properties. Additionally, the motivation for members of M as "degrees of membership" makes consideration of convex normal functions natural. Consideration of this subalgebra, and ones related to it, will be a prominent theme throughout this book.

Definition 3.6.1 *The symbol* L *denotes the set of functions that are both convex and normal.*

The absorption laws hold in L by the following Proposition.

Proposition 3.6.2 *If f is convex and g is normal, then*

$$f \sqcup (f \sqcap g) = f \sqcap (f \sqcup g) = f \tag{3.3}$$

Proof. By Equation (2.2)

$$f \sqcup (f \sqcap g) = f \sqcap (f \sqcup g) = (f \wedge g^{LR}) \vee (f^R \wedge f^L \wedge g)$$

Since f is convex and g is normal, we have

$$(f \wedge g^{LR}) \vee (f^R \wedge f^L \wedge g) = f \vee (f \wedge g) = f$$

Thus one absorption law holds. Since Birkhoff's equation holds in M, the other absorption law must also hold. ∎

The distributive laws hold by Theorem 3.5.7. From all the other properties of M, the subalgebra L of convex normal functions is a bounded distributive lattice, and * is an involution that satisfies De Morgan's laws with respect to $\sqcup$ and $\sqcap$. Thus we have the following. (See [24, 81, 108].)

Theorem 3.6.3 *The subalgebra* L *of* M *consisting of all convex normal functions is a De Morgan algebra.*

Both De Morgan algebras and Kleene algebras are, among other things, bounded, distributive lattices.

Theorem 3.6.4 *If A is a subalgebra of* M *that is a lattice with respect to $\sqcup$ and $\sqcap$ and that contains the subalgebra* S*, then the functions in A are normal and convex; that is, $A \subseteq$ L. Thus the subalgebra* L *of all convex normal functions is a maximal sublattice of* M.

Proof. Let $f \in A$. Since A is a lattice, the absorption laws hold, so for each $a \in [0,1]$,

$$1_a \sqcap (1_a \sqcup f) = 1_a \quad \text{and} \quad f \sqcap (f \sqcup 1_a) = f$$

From this and Equation (2.2), it follows that

$$1_a = 1_a \sqcap (1_a \sqcup f) = (1_a \wedge f^{LR}) \vee (1_a^L \wedge 1_a^R \wedge f)$$

so that $1_a(a) = 1 = f^{LR} \vee f(a)$. This, in turn, implies that $f^{LR} = 1$, so f is normal. Again by absorption and Equation (2.2),

$$f = f \sqcap (f \sqcup 1_a) = (f \wedge 1_a^{LR}) \vee (f^L \wedge f^R \wedge 1_a)$$

which implies that $f(a) = f(a) \vee (f^L(a) \wedge f^R(a)) = f^L(a) \wedge f^R(a)$. Since this holds for all $a \in [0,1]$, we have that $f = f^L \wedge f^R$; that is, f is convex. ∎

Definition 3.6.5 *For $f \in$ M we say the following:*

1. *f is a **lower function** if $x > 1/2$ implies $f(x) = 0$.*
2. *f is an **upper function** if $x < 1/2$ implies $f(x) = 0$.*

The symbol LU *denotes the set of all functions in* M *that are either lower functions or upper functions.*

Proposition 3.6.6 *The set* LU *is a subalgebra of* M.

Proof. The constants 1_0 and 1_1 are non-zero on a singleton, so belong to LU, and if f is a lower function, then f^* is upper, and if f is upper, then f^* is lower. Note that if f is lower, then f^R is lower, and if f is upper, then f^L is upper. It then follows from Theorem 1.4.5 that $f \sqcup g$ is upper if either of f, g is upper, and it is lower if both f, g are lower. Similarly, $f \sqcap g$ is lower if either f, g is lower, and is upper if both f, g are upper. ■

Definition 3.6.7 *Let* K *be the set of convex normal functions that are either lower or upper.*

Theorem 3.6.8 *The set* K *is a subalgebra of* M *and* K *is a Kleene algebra.*

Proof. Since K is the intersection of the subalgebra L of convex normal functions and the subalgebra LU of functions that are either lower or upper, K is a subalgebra of M. Also, since K is a subalgebra of M that is contained in L, K is a subalgebra of L. Since L is a De Morgan algebra, so also is K. It remains to show that Kleene's inequality holds in K. (See Definition 1.1.8.) One way to express the Kleene inequality is by the equation

$$(f \sqcap f^*) \sqcup (g \sqcup g^*) = (g \sqcup g^*) \tag{3.4}$$

and this is what we will prove. Since one of f and f^* is a lower function and the other is an upper function, we know that $h = f \sqcap f^*$ is a lower function and $k = g \sqcup g^*$ is an upper function. Also, by Theorem 1.4.5,

$$(h \sqcup k)(x) = (h(x) \vee k(x)) \wedge h^L(x) \wedge k^L(x)$$

Suppose $x > 1/2$. Since h is lower, $h(x) = 0$. Also since h is normal and $h(y) = 0$ for all $y \geq x$, then $h^L(x) = 1$. So

$$(h \sqcup k)(x) = k(x) \wedge k^L(x) = k(x)$$

And if $x < 1/2$, then since k is upper, $k^L(x) = 0$, so

$$(h \sqcup k)(x) \leq k^L(x) = 0 = k(x)$$

Suppose $x = 1/2$. Since h is normal and lower, $h^L(x) = 1$. And since k is upper, $k^L(x) = k(x)$. So

$$(h \sqcup k)(x) = (h(x) \vee k(x)) \wedge k(x) = k(x)$$

So for each x, we have $(h \sqcup k)(x) = k(x)$, establishing (3.4). ■

Theorem 3.6.9 *Suppose A is a subalgebra of* M *that contains the subalgebra* S *and is a Kleene algebra. Then $A \subseteq$* K. *Thus* K *is a maximal Kleene subalgebra of* M.

Proof. Suppose A is a Kleene algebra containing the subalgebra $\mathbf{S}$ of singletons. In particular, A is a lattice containing $\mathbf{S}$. Theorem 3.6.4 says that the elements of A are normal and convex. We only need to show that elements of A are either upper or lower functions. By Proposition 3.2.5, the partial orders $\sqsubseteq_{\sqcap}$ and $\sqsubseteq_{\sqcup}$ are the same on A. For simplicity, we will use the symbol $\sqsubseteq$ for both.

Let $f \in A$ and $a = 1/2$. Then $1_a \in A$ and $1_a = 1_a^*$. By the Kleene inequality,

$$1_a = 1_a \sqcap 1_a^* \sqsubseteq f \sqcup f^*$$

By Proposition 2.4.3, $1_a \sqsubseteq f \sqcup f^*$ gives

$$1_a^R \wedge (f \sqcup f^*) \leq 1_a \leq (f \sqcup f^*)^R$$

By (1.3),

$$f \sqcup f^* = (f \wedge f^{*L}) \vee (f^L \wedge f^*)$$

Since $1_a^R(x) = 1$ for $x \leq 1/2$, and $1_a(x) = 0$ for $x \neq 1/2$, the condition above implies that

$$f(x) \wedge f^{*L}(x) = 0 \quad \text{for all } x < 1/2$$

Suppose there are $x < 1/2 < y$ with $f(x) \neq 0$ and $f(y) \neq 0$. Note that $y' = 1 - y < 1/2$. By switching the roles of f and f^* if necessary, we may assume that $y' \leq x < 1/2$. Since $y' \leq x$ and $f^*(y') = f(y) \neq 0$, we have $f^{*L}(x) \neq 0$, and we assumed $f(x) \neq 0$. Then $f(x) \wedge f^{*L}(x) \neq 0$, a contradiction. Since there can be no $x < 1/2 < y$ with $f(x) \neq 0$ and $f(y) \neq 0$, f is either a lower function or an upper function. ■

3.7 Endmaximal functions

In [87], Nieminen considered the notions of endmaximal and of b-maximal (left-maximal) functions and showed the collection of such to have some nice algebraic properties. We present here some of those results, using mainly the pointwise formulas for $\sqcup$ and $\sqcap$ in computations.

Definition 3.7.1 *An element $f \in \mathbf{M}$ is **endmaximal** if $f^L = f^R$.*

An immediate consequence is that $f^L = f^R = f^{RL}$. The definition just says that the function assumes its maximum at its endpoints 0 and 1. First, notice that 1_0 and 1_1 are not endmaximal, so the set of endmaximal functions cannot be a subalgebra of $\mathbf{M}$ with its full type. But it is a subalgebra of $(\mathbf{M}, \sqcap, \sqcup, ^*)$.

Proposition 3.7.2 *The set of endmaximal functions is a subalgebra of $(\mathbf{M}, \sqcap, \sqcup, ^*)$.*

Proof. Since $f^{L*} = f^{*R}$ and $f^{R*} = f^{*L}$, it follows that the set of endmaximal functions is closed under *. Suppose that f and g are endmaximal. Then by Proposition 2.2.4,

$$(f \sqcup g)^L = f^L \sqcup g^L = f^{RL} \sqcup g^{RL} = f^R \sqcup g^R = (f \sqcup g)^R$$
$$(f \sqcap g)^R = f^R \sqcap g^R = f^{RL} \sqcap g^{RL} = f^L \sqcap g^L = (f \sqcap g)^L$$

Thus the set of endmaximal functions is closed under both $\sqcup$ and $\sqcap$. ■

In [87], it is stated that the set of normal endmaximal functions form a distributive subalgebra of $(\mathrm{M}, \sqcup, \sqcap)$. But it seems that normality is not needed. The following proposition follows immediately from Lemma 2.5.3 and the fact that $f^L = f^R = f^{LR}$.

Proposition 3.7.3 *The subalgebra of endmaximal functions is distributive, meaning that it satisfies both of the following distributive laws:*

1. $f \sqcap (g \sqcup h) = (f \sqcap g) \sqcup (f \sqcap h)$.
2. $f \sqcup (g \sqcap h) = (f \sqcup g) \sqcap (f \sqcup h)$.

We note that the algebra of endmaximal functions is not a lattice since it does not satisfy absorption.

In [87], it is shown that the endmaximal functions satisfy the modular identity $(x \wedge y) \vee (z \wedge y) = ((x \wedge y) \vee z) \wedge y$. This result follows easily from Proposition 3.7.3. The proof is identical to the proof of the modular laws from the distributive laws in the setting of lattices.

Proposition 3.7.4 *The endmaximal functions satisfy the following modular laws:*

1. $f \sqcap (g \sqcup (f \sqcap h)) = (f \sqcap g) \sqcup (f \sqcap h)$.
2. $f \sqcup (g \sqcap (f \sqcup h)) = (f \sqcup g) \sqcap (f \sqcup h)$.

Proof. For item 1, we use the distributive law from Proposition 3.7.3.

$$f \sqcap (g \sqcup (f \sqcap h)) = (f \sqcap g) \sqcup (f \sqcap f \sqcap h) = (f \sqcap g) \sqcup (f \sqcap h)$$

Item 2 is similar. ■

Functions that attain their supremum at the left endpoint 0 of the unit interval are discussed in [87]. The situation for functions that attain their supremum at the right endpoint 1 is similar.

Definition 3.7.5 *A function $f \in \mathrm{M}$ is **left-maximal** if $f^L = f^{LR}$.*

Left-maximal is also known as b-maximal. The constant 1_0 is left-maximal. However 1_1 is not left-maximal, so the left-maximal functions are not closed under negation. However, they are closed under $\sqcap$ and $\sqcup$, hence are a subalgebra of $(\mathrm{M}, \sqcap, \sqcup, 1_0)$.

Theorem 3.7.6 *If f and g are left-maximal, then so are $f \sqcup g$ and $f \sqcap g$.*

Proof. Suppose g and h are left-maximal. Then by Lemma 2.2.4

$$(f \sqcup g)^L = f^L \sqcup g^L = f^{LR} \sqcup g^{LR} = (f^L \sqcup g^L)^R = ((f \sqcup g)^L)^R = (f \sqcup g)^{LR}$$
$$(f \sqcap g)^L = f^L \sqcap g^L = f^{LR} \sqcap g^{LR} = (f^L \sqcap g^L)^R = ((f \sqcap g)^L)^R = (f \sqcap g)^{LR}$$

This establishes the claim. ■

One distributive law holds for left-maximal functions. The proof follows again from 2.5.3 and the fact that $f^L = f^{RL}$. The other distributive law fails since $f^R = f^{LR}$ does not hold.

Theorem 3.7.7 *If g and h are left-maximal, then $f \sqcap (g \sqcup h) = (f \sqcap g) \sqcup (f \sqcap h)$.*

A similar analysis could be made for right-maximal functions.

3.8 The algebra of sets

We have discussed members of M that are characteristic functions of points in I (Definition 3.2.1), and members of M that are characteristic functions of closed intervals in I (Definition 3.3.1). In each case, we obtained subalgebras of M. We extend this discussion.

Definition 3.8.1 *For a set $A \subseteq [0,1]$, the **characteristic function** of the set A is the function $1_A : [0,1] \to [0,1]$ that takes value 1 at each $x \in A$ and takes value 0 otherwise. Let* E *(for ensemble) be the set of all characteristic functions of subsets of* $[0,1]$.

The members of M that are characteristic functions of sets are exactly those that take on only the values 0 and 1.

Theorem 3.8.2 *The set* E *is a subalgebra of* M.

Proof. The constants 1_0 and 1_1 take on only the values 0 and 1. If f takes on only the values 0 and 1, then so does f^* since $f^*(x) = 1 - f(x)$. Suppose that f and g take only the values $0, 1$. Then by Theorem 1.4.5

$$f \sqcap g = (f \vee g) \wedge (f^L \wedge g^L)$$
$$f \sqcup g = (f \vee g) \wedge (f^R \wedge g^R)$$

Since f, g, f^L, g^L, f^R, g^R take only the values $0, 1$, so do $f \sqcap g$ and $f \sqcup g$. ■

In the proof of Theorem 3.8.2, we have used the following simple observation that will be of use later.

Proposition 3.8.3 *The set* E *is closed under the pointwise operations* $\wedge, \vee$, *and the operations* L, R.

We call the subalgebra E of M the **algebra of sets**. Each member f of E corresponds to a subset of $[0,1]$ via the mapping $f \mapsto f^{-1}(1)$. If we identify a function f with its corresponding set, the operations on E are as follows.

$$f \vee g = f \cup g \quad \text{and} \quad f \wedge g = f \cap g$$
$$f^L = \{x : x \geq \text{ some element of } f\}$$
$$f^R = \{x : x \leq \text{ some element of } f\}$$

Making this identification, the formulas for $\sqcup$ and $\sqcap$ become

$$f \sqcup g = (f \cup g) \cap f^L \cap g^L = (f \cap g^L) \cup (f^L \cap g) \tag{3.5}$$
$$f \sqcap g = (f \cup g) \cap f^R \cap g^R = (f \cap g^R) \cup (f^R \cap g)$$

3.9 Functions with finite support

We conclude this chapter with one final subalgebra, that of functions with finite support.

Definition 3.9.1 *For* $f \in \mathrm{M}$, *the* ***support*** *of* f *is* $\mathrm{Supp}(f) = \{x : f(x) \neq 0\}$. *A function* $f \in \mathrm{M}$ *has* ***finite support*** *if* $\mathrm{Supp}(f)$ *is a finite set. Let* F *be the set of all functions with finite support.*

Theorem 3.9.2 *The functions* F *with finite support are a subalgebra of* M.

Proof. The constants 1_0 and 1_1 belong to F since their support is a single element. If f is non-zero at only finitely many places, then so is f^*. Suppose f and g have finite support. By Theorem 1.4.5,

$$f \sqcap g = (f \vee g) \wedge (f^L \wedge g^L)$$
$$f \sqcup g = (f \vee g) \wedge (f^R \wedge g^R)$$

Thus the supports of both $f \sqcap g$ and $f \sqcup g$ are contained in $\mathrm{Supp}(f) \cup \mathrm{Supp}(g)$, hence they have finite support. ■

The algebra F has subalgebras that are of importance in Chapter 10. These are given in the following result, whose proof is left as an exercise (Exercise 29).

Proposition 3.9.3 *Let $m, n \geq 2$ be positive integers. The set of functions $f \in \mathrm{M}$ with $\mathrm{Supp}(f) \subseteq \{0, \frac{1}{n}, \frac{2}{n}, ..., \frac{n-1}{n}, 1\}$ taking values in $\{0, \frac{1}{m}, \frac{2}{m}, ..., \frac{m-1}{m}, 1\}$ is a subalgebra of* F.

3.10 Summary

It was shown that the algebra I used for type-1 fuzzy sets is isomorphic to the subalgebra S of M consisting of singletons; that is, characteristic functions of points. It was shown that the algebra $\mathrm{I}^{[2]}$ used for interval-valued fuzzy sets is isomorphic to the subalgebra $\mathrm{S}^{[2]}$ of M consisting of characteristic functions of closed intervals. It was shown that S is a maximal subchain of M and a Kleene subalgebra of M.

Other subalgebras of M were also given. These include the subalgebra N of normal functions, those whose supremum is 1, and the subalgebra C of convex functions, those f that are increasing and then decreasing. The algebra C was shown to be distributive, but it is not a lattice. The intersection of these subalgebras is the subalgebra L of convex normal functions. This was shown to be a De Morgan algebra. This is not only a maximal De Morgan subalgebra of M, but a maximal sublattice of M. Its subalgebra K of functions that are normal and convex and either lower or upper was shown to be a maximal Kleene subalgebra of M. The subalgebra L will be a primary feature in this book.

Also discussed were the endmaximal functions and left-maximal functions. These were shown to be subalgebras, with the endmaximal functions satisfying both distributive laws, but not being a sublattice. The subalgebra E of characteristic functions of sets was also discussed. This will play an important role later. A final subalgebra was F, the functions with finite support.

3.11 Exercises

1. Prove the intersection of subalgebras is a subalgebra.

2. Prove that every chain in a lattice is a subalgebra.

3. Show that every subset of a chain (considered as a lattice) is a subalgebra.

4. Give an example of an algebra A where the subalegbra generated by each

element $a \in A$ is infinite. (Hint: Let A have only one unary operation $s(x)$).

5. If $\alpha : A \to B$ is a homomorphism, S is a subalgebra of A, and T is a subalgebra of B, prove that $\alpha(S) = \{\alpha(s) : s \in S\}$ is a subalgebra of B, and that $\alpha^{-1}(T) = \{a \in A : \alpha(a) \in T\}$ is a subalgebra of A.

6. If $\alpha : A \to B$ is a one-to-one homomorphism, prove that the image $\alpha(A) = \{\alpha(a) : a \in A\}$ is a subalgebra of B, and α is an isomorphism from A to its image.

7. Consider a set A to be an algebra with no operations. What are the subalgebras of A?

8. A pointed set is a set A with a single distinguished element e. What are the subalgebras of a pointed set? What is a homomorphism between two pointed sets?

9. Prove item 2 of Proposition 3.2.2, which relates notation used for I to that used for M.

10. Prove Theorem 3.2.3, which establishes that the truth value algebra for type-1 fuzzy sets is isomorphic to a subalgebra of M.

11. For a subalgebra A of $(\mathrm{M}, \sqcap, \sqcup)$, prove that $f = f \sqcup (f \sqcap g)$ for any $f, g \in A$ if and only if the partial orderings $\sqsubseteq_{\sqcap}$ and $\sqsubseteq_{\sqcup}$ agree on A. (See Proposition 3.2.5.)

12. Prove that $a \leq b$ in I if and only if $1_a \sqsubseteq_{\sqcup} 1_b$ in M.

13. Prove that $1_a^L \wedge 1_b^L = (1_a \sqcup 1_b)^L$ and $1_a^R \vee 1_b^R = (1_a \sqcup 1_b)^R$ (Lemma 3.3.3).

14. Prove item 2 of Proposition 3.3.4 that if $f = 1_a^L \wedge 1_b^R$ and $g = 1_c^L \wedge 1_d^R$ then $f \sqcap g = (1_a \sqcap 1_c)^L \wedge (1_b \sqcap 1_d)^R$.

15. Prove Proposition 3.4.2 which lists four conditions, each equivalent to an element of M being normal.

16. Complete the proof of Proposition 3.4.3, showing that the set of normal elements is a subalgebra of M.

17. Prove that the set of functions that attain the value 1; that is, those that are strictly normal, is a subalgebra of M.

18. In Theorem 3.5.7, prove the second item; that is, $f \sqcap (g \sqcup h) = (f \sqcap g) \sqcup (f \sqcap h)$ for all $g, h \in \mathrm{M}$ if and only if f is convex.

19. Prove item 2 of Proposition 3.5.8, which says that if g, h are increasing, then $f \sqcup (g \sqcap h) = (f \sqcup g) \sqcap (f \sqcup h)$.

20. Complete the proof of Theorem 3.6.6, which says the set of all functions in M that are either lower or upper functions is a subalgebra of M.

21. Prove that if $f \in M$ is endmaximal, then $f^L = f^R = f^{RL}$.

22. Explain why 1_0 and 1_1 are not endmaximal.

23. Prove item 2 of Proposition 3.7.4, a modular law for endmaximal functions.

24. Prove Theorem 3.7.7 which states that if g and h are left maximal, then $f \sqcap (g \sqcup h) = (f \sqcap g) \sqcup (f \sqcap h)$.

25. Give an example of a function f and left maximal functions g and h that do not satisfy $f \sqcup (g \sqcap h) = (f \sqcup g) \sqcap (f \sqcup h)$.

26. Prove $1_A = \{1_a : a \in A\}$ is a subalgebra of M if A is a subalgebra of I.

27. Prove that the set E is closed under the pointwise operations $\wedge$ and $\vee$ and the operations L and R (Proposition 3.8.3).

28. Prove that the algebra $(\mathrm{E}, \wedge, \vee, 0, 1, \sqcap, \sqcup, ^*, L, R, 1_0, 1_1)$ is a subalgebra of the algebra $(\mathrm{M}, \wedge, \vee, 0, 1, \sqcap, \sqcup, ^*, L, R, 1_0, 1_1)$ where 0 and 1 are constant functions on M.

29. Let A be the set of functions $f \in \mathrm{M}$ with $\mathrm{Supp}(f) \subseteq \{0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1\}$ and $\mathrm{Image}(f) \subseteq \{0, \frac{1}{3}, \frac{2}{3}, 1\}$. Show that A is a subalgebra of F.

The following exercise is a more involved, and is suitable for a small project.

30. Decide which of the following are subalgebras of $(\mathrm{M}, \sqcap, \sqcup, ^*, 1_0, 1_1)$. For those that are not subalgebras, say under which operations they are closed.

    (a) The set of all constant functions in M.
    (b) The continuous functions in M.
    (c) The set of all step functions in M.
    (d) The set of all piecewise linear functions in M.
    (e) The set of all functions in M that take only finitely many values.

Chapter 4

Automorphisms

For any mathematical structure, its group of symmetries is an object of interest. For algebraic structures, these symmetries are called automorphisms. A description of the automorphism group of the truth value algebra M is the main goal of this chapter. In doing so, we gain further understanding of the structure of M. Another goal is the identification of subalgebras of M with the property that any automorphism of M takes these subalgebras onto themselves. We will find that the truth value algebras of type-1, and of interval-valued fuzzy sets are isomorphic to such subalgebras of M. This means intuitively that there are no other isomorphic copies of these subalgebras that sit in M in the same way. So in this precise mathematical sense, type-2 is a generalization of type-1 and of interval-valued fuzzy sets. These remarks hold both with and without negations included as part of the algebraic structures.

4.1 Preliminaries

We begin with some preliminary definitions.

Definition 4.1.1 *A* ***group*** *is an algebra* $(G, \circ, ^{-1}, e)$*, where* $\circ$ *is a binary operation,* $^{-1}$ *is a unary operation, and e is a nullary operation, satisfying the following properties for all* $x, y, z \in G$*:*

1. $(x \circ y) \circ z = x \circ (y \circ z)$ *(*$\circ$ *is* ***associative****).*
2. $x \circ e = e \circ x = x$ *(e is the* ***identity*** *of the group).*

3. *$x \circ x^{-1} = x^{-1} \circ x = e$ (x^{-1} is the* ***inverse*** *of x).*

A group is ***commutative*** *if it additionally satisfies $x \circ y = y \circ x$.*

The prototypical example of a group is that of all **permutations** of a set X. These permutations are the bijective mappings of X to itself. The group operation on these permutations is function composition, so for bijections $\alpha, \beta : X \to X$, $\alpha \circ \beta$ is the mapping given by $(\alpha \circ \beta)(x) = \alpha(\beta(x))$. The inverse operation is that of taking the inverse α^{-1} of a permutation α, and the identity element is the identity map. This idea has a natural extension to algebras, and this will be the primary focus of this chapter.

Definition 4.1.2 ***Automorphisms*** *of an algebra are one-to-one mappings of that algebra onto itself that preserve the operations of that algebra. For an algebra A,* $\mathrm{Aut}(A)$ *will denote the set of all automorphisms of A.*

In other words, an automorphism is an isomorphism from an algebra onto itself. The following theorem is a fundamental fact about automorphisms of an algebra. The proof of this theorem is left as an exercise.

Theorem 4.1.3 *For an algebra A, the set* $\mathrm{Aut}(A)$ *of all the automorphisms of A is a group under composition of maps. In particular, the inverse of an automorphism is an automorphism and the identity map of the algebra is the identity of the automorphism group.*

A set is a particularly simple type of algebra, one with no operations, and for a set X, its automorphism group is its group of permutations. A more interesting example comes from considering the group of automorphisms of a vector space such as $V = \mathbb{R}^2$. This group is isomorphic to the group $\mathrm{GL}(2, \mathbb{R})$ of all invertible 2×2 real matrices, where the group operation is matrix multiplication, the inverse is taking the matrix inverse, and the identity of the group is the identity matrix. Since there are matrices A, B with $AB \neq BA$, this group is not commutative. We next consider examples of automorphism groups of lattices. The following is very useful.

Proposition 4.1.4 *For a lattice $(L, \wedge, \vee)$ and an onto map $\alpha : L \to L$, the following are equivalent.*

1. *α is an automorphism of L.*
2. *For all $x, y \in L$ we have $x \leq y$ if and only if $\alpha(x) \leq \alpha(y)$.*

Proof. To see that (1) implies (2), suppose $x \leq y$. Then $x \wedge y = x$. Since α is an automorphism, $\alpha(x \wedge y) = \alpha(x) \wedge \alpha(y)$, hence $\alpha(x) \wedge \alpha(y) = \alpha(x)$, giving $\alpha(x) \leq \alpha(y)$. Conversely, if $\alpha(x) \leq \alpha(y)$, then $\alpha(x) \wedge \alpha(y) = \alpha(x)$, giving $\alpha(x \wedge y) = \alpha(x)$. Then since α is a bijection, $x \wedge y = x$, giving $x \leq y$.

To see that (2) implies (1), suppose $x \wedge y = z$. Then $z \leq x$ and $z \leq y$, giving that $\alpha(z) \leq \alpha(x)$ and $\alpha(y)$. Thus $\alpha(z)$ is a lower bound of $\alpha(x)$ and

$\alpha(y)$. Suppose v is another lower bound of $\alpha(x)$ and $\alpha(y)$. Since α is onto L, there is u with $\alpha(u) = v$. Then $\alpha(u) \leq \alpha(x)$ and $\alpha(y)$ implies by (2) that $u \leq x$ and y. Since z is the greatest lower bound of x and y, then $u \leq z$. Thus $v = \alpha(u) \leq \alpha(z)$. Thus $\alpha(z) = \alpha(x \wedge y)$ is the greatest lower bound of $\alpha(x)$ and $\alpha(y)$, showing that $\alpha(x \wedge y) = \alpha(x) \wedge \alpha(y)$. Showing that α preserves joins is similar. Also, (2) clearly implies that α is one-to-one. ■

The proof of the following result follows along similar lines to that above, and is left as an exercise. (See Exercise 5.)

Proposition 4.1.5 *Suppose L is a complete lattice, $A \subseteq L$, and α is an automorphism of L. Then the following hold:*

1. $\alpha(\bigvee A) = \bigvee\{\alpha(a) : a \in A\}$.
2. $\alpha(\bigwedge A) = \bigwedge\{\alpha(a) : a \in A\}$.

We turn next to the matter of computing automorphism groups of lattices. Using Proposition 4.1.4, it is usually not difficult to provide a supply of automorphisms. The more difficult task is in showing that there are no others. For this, the key observation is that if an element x of a lattice L satisfies some property, then for any automorphism α, the element $\alpha(x)$ will also satisfy that property. Here we are vague as to what constitutes a property. This can be formalized using first-order logic, or more generally types [22], but we will need only a few simple instances of this much more general idea. The first of these is a simple consequence of Proposition 4.1.4.

Proposition 4.1.6 *Let α be an automorphism of a lattice L.*

1. *If L has a least element* 0*, then* $\alpha(0) = 0$.
2. *If L has a largest element* 1*, then* $\alpha(1) = 1$.

We next describe the other properties we will use.

Definition 4.1.7 *For a lattice L, $x \in L$ is* ***meet irreducible*** *if $x = y \wedge z$ implies $x = y$ or $x = z$, and x is* ***join irreducible*** *if $x = y \vee z$ implies $x = y$ or $x = z$. We call x* ***irreducible*** *if it is both join irreducible and meet irreducible.*

Proposition 4.1.8 *The following hold for α an automorphism of a lattice L and $x \in L$:*

1. *x is meet irreducible if and only if $\alpha(x)$ is meet irreducible.*
2. *x is join irreducible if and only if $\alpha(x)$ is join irreducible.*
3. *x is irreducible if and only if $\alpha(x)$ is irreducible.*

Proof. We prove only the first statement. Suppose x is meet irreducible and that $\alpha(x) = u \wedge v$. Then there are y, z with $\alpha(y) = u$ and $\alpha(z) = v$. So $\alpha(x) = \alpha(y) \wedge \alpha(z) = \alpha(y \wedge z)$. Since α is one-one, we have $x = y \wedge z$. Since x is meet irreducible, we have $x = y$ or $x = z$. Thus $\alpha(x) = u$ or $\alpha(x) = v$. So $\alpha(x)$ is meet irreducible. For the converse, suppose $\alpha(x)$ is meet irreducible, and applying what we know using the automorphism α^{-1}, we get that $\alpha^{-1}\alpha(x) = x$ is meet irreducible. ∎

Consider the lattice L in Figure 4.1. This is an 8-element Boolean algebra. Its join irreducible elements are 0 and the elements a, b, c. Since every automorphism takes 0 to 0, and join irreducibles to join irreducibles, every automorphism induces a permutation of the set $\{a, b, c\}$. Further, every element of L is a join of these join irreducible elements. (This is always the case in a finite lattice.) So each automorphism of L is determined by its action on $\{a, b, c\}$. It is a simple matter to see that each permutation of $\{a, b, c\}$ does extend to an automorphism of L, so $\mathrm{Aut}(L)$ is isomorphic to the permutation group of the set $\{a, b, c\}$.

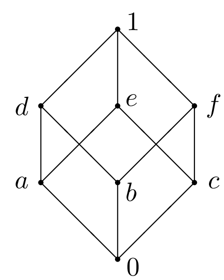


FIGURE 4.1: An 8-element Boolean algebra

We next consider general relationships between the automorphisms of an algebra A and those of algebras B that are related to A. We first consider the situation when B is a **reduct** of A, meaning that B has the same underlying set as A, and that each operation of B is an operation of A. In other words, when B is obtained from A by forgetting some of the operations of A. Then each automorphism of A is an automorphism of B since it is a bijection of the underlying set of B to itself and is compatible with all of the operations of A, and in particular is compatible with those of B. We use the term **subgroup** to mean a subalgebra of a group; that is, a subset of a group that is closed under its basic operations. (See Definition 3.1.2.) We have the following.

Proposition 4.1.9 *If B is a reduct of A, then the automorphism group* $\mathrm{Aut}(A)$ *is a subgroup of* $\mathrm{Aut}(B)$.

We next consider the situation when B is a subalgebra of A. It is generally not the case that each automorphism of B will extend to an automorphism of A, or that each automorphism of A will restrict to an automorphism of B.

However, when this later condition occurs, it is significant, as it intuitively means that B sits inside of A in a unique way.

Definition 4.1.10 *A subalgebra B of an algebra A is a **characteristic subalgebra** if every automorphism of A induces an automorphism of B.*

In this chapter, a number of subalgebras of M will be shown to be characteristic subalgebras of M, and some subalgebras of reducts of M will be shown to be characteristic subalgebras of reducts of M. These include the truth value algebras of sets, fuzzy sets, and interval-valued fuzzy sets.

4.2 Sets, fuzzy sets, and interval-valued fuzzy sets

In this section, we describe the automorphisms of the truth value algebras of sets, fuzzy sets, and interval-valued fuzzy sets. We also describe the automorphisms of the algebra of subsets of a set and of the algebra of fuzzy subsets of a set. The following is trivial.

Proposition 4.2.1 *The truth value algebra for sets is a 2-element Boolean algebra $\{0,1\}$ so has only the identity automorphism.*

We recall that for a set S, the algebra of subsets of S is the set $\mathrm{Map}(S,\{0,1\})$ with the pointwise operations $\wedge, \vee, ', 0, 1$. (See Definition 1.2.3.) The first step to determine the automorphisms of this algebra is to determine its join irreducible elements. Adapting the notation of Definition 3.2.1 in an obvious way, for each $a \in S$ we let 1_a be the function from S to $\{0,1\}$ that takes value 1 at a and is 0 otherwise. It is easily seen that these functions 1_a are join irreducible, and are all the non-zero join irreducibles in $\mathrm{Map}(S,\{0,1\})$. Since any automorphism of a lattice takes join irreducible elements to others, and 0 to itself, we have the following.

Lemma 4.2.2 *If σ is an automorphism of $\mathrm{Map}(S,\{0,1\})$, then for each $a \in S$ there is a unique $b \in S$ with $\sigma(1_a) = 1_b$.*

So each automorphism σ of $\mathrm{Map}(S,\{0,1\})$ gives a map $\beta : S \to S$ where $\beta(a) = b$ if $\sigma(1_a) = 1_b$. It is easily seen that β is a permutation of S. For a map $f : S \to \{0,1\}$, if $A = \mathrm{Supp}(f)$, then for each $a \in A$ we have $1_a \le f$, and further that $f = \bigvee\{1_a : a \in A\}$. By Proposition 4.1.5, for any automorphism σ of $\mathrm{Map}(S,\{0,1\})$ we have $\sigma(f) = \bigvee\{\sigma(1_a) : a \in A\}$. So automorphisms of $\mathrm{Map}(S,\{0,1\})$ are given by permutations of S, and it is easily seen that each permutation of S gives rise to an automorphism. This leads to the following.

Theorem 4.2.3 *The automorphism group of $\mathrm{Map}(S,\{0,1\})$ is isomorphic to the permutation group of S.*

We next turn our attention to the truth value algebra of fuzzy sets. (See Definition 1.2.5.) This is the unit interval I with the operations $\wedge, \vee$ of min and max, the constants 0 and 1, and the negation $x' = 1 - x$. We will consider automorphisms of this algebra both with and without negation. We note that there was no point in considering the truth value algebra for sets with and without negation since the answer was the same in both cases.

Proposition 4.2.4 *A map* $\alpha : \mathrm{I} \to \mathrm{I}$ *is an automorphism of* $(\mathrm{I}, \wedge, \vee)$ *if and only if any of the following equivalent conditions hold:*

1. *α is a strictly increasing mapping of* $[0,1]$ *onto itself.*
2. *α is a one-to-one increasing mapping of* $[0,1]$ *onto itself.*
3. *α is continuous, strictly increasing, and satisfies* $\alpha(0) = 0$ *and* $\alpha(1) = 1$.

Proof. The equivalence of (1) and (2) is obvious. For (2) implies (3), any increasing function from $[0,1]$ onto itself must satisfy $f(0) = 0$ and $f(1) = 1$. That it is continuous follows, since any discontinuity would be a jump discontinuity and lead to a function that is not onto. That (3) implies (1) is a basic result of analysis, namely that the continuous image of a connected set is connected. That these conditions are equivalent to α being an automorphism of the lattice I is a direct consequence of Proposition 4.1.4. ■

It follows from this result, and directly from Proposition 4.1.6, that any automorphism of $(\mathrm{I}, \wedge, \vee)$ preserves 0 and 1, hence is an automorphism of $(\mathrm{I}, \wedge, \vee, 0, 1)$. The same is not true of negation. For an automorphism α of I to preserve negation, it must satisfy $\alpha(1 - x) = 1 - \alpha(x)$. If we consider the graph of the function $\alpha : \mathrm{I} \to \mathrm{I}$, this means that if we reflect the graph of α in the line $x = 0.5$, and then in the line $y = 0.5$, we obtain again the graph of α. The situation is shown in Figure 4.2.

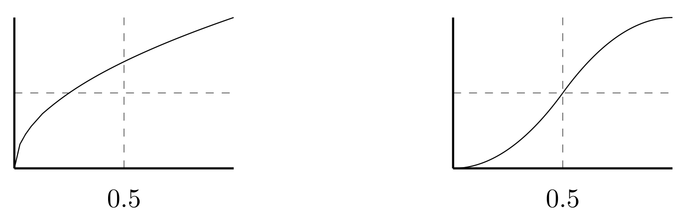


FIGURE 4.2: An automorphism of I that does not preserve negation at left, and one that does at right

Next we consider automorphisms of the algebra of fuzzy subsets of a set S. (See Definition 1.2.9.) This is the algebra $\mathrm{Map}(S, \mathrm{I})$ with its operations componentwise. This is an important topic for us since when specialized to

$S = \mathrm{I}$, the underlying set of this algebra is the same as that of M. Again, we consider the situation both with a negation on I, and hence on this algebra of fuzzy subsets, and without. The situation is similar to that of the algebra of sets, but slightly more nuanced. We begin with the following extension of earlier definitions that will be used here, and later.

Definition 4.2.5 *Suppose $A \subseteq S$ and $p \in \mathrm{I}$. The symbol p_A denotes the function from S to I with $p_A(a) = p$ if $a \in A$ and $p_A(a) = 0$ if $a \notin A$. In the case that A is the one-element set $\{a\}$, we write this as p_a.*

We note that this extends both the notion of a **characteristic function** 1_A of Definition 3.8.1, and that of 1_a from Definition 3.2.1. We shall call a function p_a a **point** and use the term **singleton** for a function 1_a. We note that each element of the lattice I is join irreducible since I is a chain. This leads to the following.

Lemma 4.2.6 *The join irreducible elements of $\mathrm{Map}(S, \mathrm{I})$ are exactly the points p_a, and the maximal join irreducibles in the pointwise order are the singletons 1_a where $a \in S$.*

Proof. Each join irreducible in this lattice must have its support be a single element, and so must be a point. Since each element of I is join irreducible, it follows that each point is join irreducible in $\mathrm{Map}(S, \mathrm{I})$. Among these, the maximal ones are the ones that take value 1 at a single element and are 0 elsewhere, and these are exactly the functions 1_a where $a \in S$. ■

By an argument that is now familiar (see Exercise 9), an automorphism σ of $\mathrm{Map}(S, \mathrm{I})$ must take maximal join irreducible elements to maximal join irreducible elements, hence σ induces a permutation β of S where $\beta(a) = b$ if $\sigma(1_a) = 1_b$. Further, if we use $\mathrm{I}_a = \{p_a : p \in \mathrm{I}\}$ for the set of points that lie under 1_a, it follows that σ must map I_a isomorphically onto I_b where $\beta(a) = b$. This provides the following.

Proposition 4.2.7 *For each automorphism σ of the lattice $\mathrm{Map}(S, \mathrm{I})$, there is a permutation β of S and a family α_a $(a \in S)$ of automorphisms of I such that for each $a \in S$ and $p \in \mathrm{I}$, the automorphism σ takes the point p_a to the point q_b, where $q = \alpha_a(p)$ and $b = \beta(a)$.*

This says that each automorphism σ of $\mathrm{Map}(S, \mathrm{I})$ gives an element β of the group $\mathrm{Perm}(S)$ and a family of elements $(\alpha_a)_S$ of the group $\mathrm{Aut}(\mathrm{I})$, hence to an element of the product group $\mathrm{Aut}(\mathrm{I})^S$. By Proposition 4.1.5, σ preserves arbitrary joins. So the following proposition shows that σ is determined by the data β and $(\alpha_a)_S$. The proof of this proposition is left as an exercise. (See Exercise 14 and also Exercise 15.)

Proposition 4.2.8 *For f an element of $\mathrm{Map}(S, \mathrm{I})$ we have*

$$f = \bigvee \{p_a : p_a \leq f\}$$

Thus each element of $\mathrm{Map}(S, \mathrm{I})$ is a join of join irreducibles.

Our final step in describing the automorphisms of $\mathrm{Map}(S, \mathrm{I})$ is to note that for any permutation β of S and any family $(\alpha_a)_S$ of automorphisms of I, there is an automorphism σ of $\mathrm{Map}(S, \mathrm{I})$ from which this data is obtained. Specifically, this automorphism σ is defined as follows. For any $f \in \mathrm{Map}(S, \mathrm{I})$ and any $b \in S$,

$$(\sigma(f))(b) = \alpha_a(f(a)) \tag{4.1}$$

where $a = \beta^{-1}(b)$. To show that σ is an automorphism, by Proposition 4.1.4 it is enough to show that σ is a bijection that satisfies $f \leq g$ if and only if $\sigma(f) \leq \sigma(g)$. This is more cumbersome than difficult, and is left to the reader. We thus have the following.

Theorem 4.2.9 *The automorphisms of the lattice* $\mathrm{Map}(S, \mathrm{I})$ *are in bijective correspondence with ordered pairs given by an element* β *of the permutation group* $\mathrm{Perm}(S)$ *and an element* $(\alpha_a)_S$ *of the group* $\mathrm{Aut}(\mathrm{I})^S$.

We have described the elements of the automorphism group of the lattice $\mathrm{Map}(S, \mathrm{I})$, but not the structure of this group. For readers with exposure to group theory, it is the semidirect product of $\mathrm{Perm}(S)$ and $\mathrm{Aut}(\mathrm{I})^S$. We will not need this fact, and leave further investigation to the interested reader.

We remark on the situation when we consider automorphisms of $\mathrm{Map}(S, \mathrm{I})$ that preserve the componentwise negation inherited from the negation on I. By Proposition 4.1.9, this automorphism group is a subgroup of the automorphism group of the lattice $\mathrm{Map}(S, \mathrm{I})$. In fact, it is easy to establish the following by examining the proof above.

Theorem 4.2.10 *The automorphisms of* $\mathrm{Map}(S, \mathrm{I})$ *that preserve the negation correspond to ordered pairs given by an element* β *of the permutation group* $\mathrm{Perm}(S)$ *and an element* $(\alpha_a)_S$ *of the group* $\mathrm{Aut}(\mathrm{I}, ')^S$, *where* $\mathrm{Aut}(\mathrm{I}, ')$ *is the group of automorphisms of* I *that preserve negation.*

We next turn our attention to automorphisms of the truth value algebra $\mathrm{I}^{[2]}$ of interval-valued fuzzy sets. (See Definition 1.2.10.) The elements of $\mathrm{I}^{[2]}$ are ordered pairs (a, b) of elements of I where $a \leq b$, and the operations $\wedge, \vee, 0, 1$ are componentwise. However, the negation $'$ is given by $(a, b)' = (b', a')$. A picture of $\mathrm{I}^{[2]}$ is given in Figure 4.3.

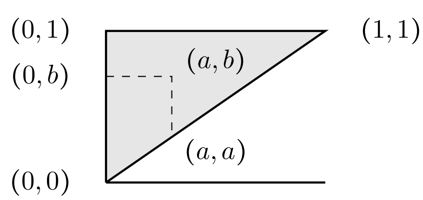


FIGURE 4.3: The lattice $\mathrm{I}^{[2]} = \{(a, b) : a \leq b\}$

Theorem 4.2.11 *For each automorphism α of* I*, the map $\gamma : \mathrm{I}^{[2]} \to \mathrm{I}^{[2]}$ given by $\gamma(a,b) = (\alpha(a), \alpha(b))$ is an automorphism of the lattice $\mathrm{I}^{[2]}$, and all of the automorphisms of $\mathrm{I}^{[2]}$ arise this way. So $\mathrm{Aut}(\mathrm{I}^{[2]})$ is isomorphic to $\mathrm{Aut}(\mathrm{I})$.*

Proof. Suppose that γ is an automorphism of $\mathrm{I}^{[2]}$. The join irreducibles in $\mathrm{I}^{[2]}$ are the elements $(0,a)$ along the vertical axis, and the elements (a,a) along the diagonal. The element $(0,1)$ is the unique join irreducible that has only the join irreducible $(1,1)$ above it. So $\gamma(0,1) = (0,1)$. It follows that γ maps the set $\{(0,a) : a \in \mathrm{I}\}$ bijectively to itself. So there is an automorphism α of I with $\gamma(0,a) = (0, \alpha(a))$ for each $a \in \mathrm{I}$. For each $a \in \mathrm{I}$, the element (a,a) is the least join irreducible that lies above $(0,a)$ and does not lie under $(0,1)$. So γ must map (a,a) to the least join irreducible that lies above $\gamma(0,a) = (0,\alpha(a))$ and does not lie under $\gamma(0,1) = (0,1)$. Thus $\gamma(a,a) = (\alpha(a), \alpha(a))$. So every automorphism γ is of the indicated form.

Conversely, if α is an automorphism of I, then since the lattice operations of $\mathrm{I}^{[2]}$ are componentwise, the map $\gamma(a,b) = (\alpha(a), \alpha(b))$ is an automorphism of the lattice $\mathrm{I}^{[2]}$. ■

We now consider the automorphisms of $\mathrm{I}^{[2]}$ that also preserve negation. Suppose $\gamma(a,b) = (\alpha(a), \alpha(b))$. Then γ preserving negation is equivalent to having $\gamma((a,b)') = (\gamma(a,b))'$ for all (a,b), and this is equivalent to having $(\alpha(b'), \alpha(a')) = (\alpha(b)', \alpha(a)')$ for all (a,b). This in turn is equivalent to α preserving negation. This provides the following.

Corollary 4.2.12 *The automorphisms of $\mathrm{I}^{[2]}$ that also preserve negation are exactly those coming from automorphisms of* I *that preserve negation. So $\mathrm{Aut}(\mathrm{I}^{[2]}, ')$ is isomorphic to $\mathrm{Aut}(\mathrm{I}, ')$.*

Finding the automorphism group of the algebra of interval-valued fuzzy subsets of a set S proceeds along the same path as for the algebra of fuzzy subsets of S. We leave this to the reader.

4.3 Automorphisms of $(\mathrm{M}, \wedge, \vee, L, R)$

In this section, we consider the truth value algebra M of type-2 fuzzy sets equipped with the pointwise operations $\wedge, \vee$ and L, R. Our eventual objective is to characterize the automorphisms of M with respect to the convolution operations $\sqcap$ and $\sqcup$, and the results obtained here will assist in that task. The key point is that $\mathrm{M} = \mathrm{Map}(\mathrm{I}, \mathrm{I})$, so the results of the previous section on the automorphisms of the lattice $\mathrm{Map}(S, \mathrm{I})$ apply when specialized to M. We begin with the following.

Proposition 4.3.1 *Each automorphism σ of $(\mathrm{M}, \wedge, \vee, L, R)$ preserves the convolution operations $\sqcap, \sqcup$ and the constants $1_0, 1_1$.*

Proof. For $f, g \in \mathrm{M}$, Theorem 1.4.5 gives $f \sqcup g = (f \vee g) \wedge f^L \wedge g^L$. Since σ preserves $\wedge, \vee, L, R$ we have $\sigma(f \sqcup g) = (\sigma(f) \vee \sigma(g)) \wedge \sigma(f)^L \wedge \sigma(g)^L$, and this shows that $\sigma(f \sqcup g) = \sigma(f) \sqcup \sigma(g)$. The calculation that σ preserves $\sqcap$ is similar. By Theorem 2.3.3, $1_0 \sqcup f = f$ for all $f \in \mathrm{M}$. Further, it is the unique element with this property, since $h \sqcup f = f$ for all $f \in M$ implies $1_0 = h \sqcup 1_0 = 1_0 \sqcup h = h$. So σ must map 1_0 to itself. A similar argument shows that σ must also map 1_1 to itself. ■

In a subsequent section, we will provide a converse of sorts to this result,namely, that each automorphism of $(\mathrm{M}, \sqcap, \sqcup)$ preserves the operations $\wedge, \vee, L, R$ and the constants $1_0, 1_1$. But we now concentrate on describing the automorphism group of $(\mathrm{M}, \wedge, \vee, L, R)$. We begin with the following result obtained by applying Proposition 4.2.7 to $\mathrm{M} = \mathrm{Map}(\mathrm{I}, \mathrm{I})$.

Proposition 4.3.2 *Suppose σ is an automorphism of $(\mathrm{M}, \wedge, \vee)$ that is given by the permutation β of I and the family $(\alpha_a)_{\mathrm{I}}$ of automorphisms of I as in Proposition 4.2.7. Then σ preserves the operations L and R if and only if the following conditions hold.*

1. *β is an automorphism of the lattice I.*
2. *The α_a where $a \in \mathrm{I}$ are all equal.*

Proof. Suppose that σ preserves L, R. Then by Proposition 4.3.1, σ also preserves $\sqcap$ and $\sqcup$. To verify the first condition, suppose $x, y \in \mathrm{I}$ with $x \leq y$. Then by Theorem 1.4.5, $1_x \sqcup 1_y = (1_x \vee 1_y) \wedge 1_x^L \wedge 1_y^L = 1_y$. Since σ preserves $\sqcup$ and $\sigma(1_x) = 1_{\beta(x)}$ and $\sigma(1_y) = 1_{\beta(y)}$, we have

$$1_{\beta(x)} \sqcup 1_{\beta(y)} = \sigma(1_x) \sqcup \sigma(1_y) = \sigma(1_x \sqcup 1_y) = \sigma(1_y) = 1_{\beta(y)}$$

Using Theorem 1.4.5 again, it follows that $\beta(x) \leq \beta(y)$. So β is an order-preserving permutation of I, hence an automorphism of I.

Still assuming that σ preserves L, R, we show the second condition. Let $p \in \mathrm{I}$ and let p_{I} be the constant function taking value p. (See Definition 4.2.5.) Then by Equation (4.1),

$$(\sigma(p_{\mathrm{I}}))(b) = \alpha_a(p_{\mathrm{I}}(a))$$

where $a = \beta^{-1}(b)$. But p_{I} is a constant function, so $p_{\mathrm{I}} = p_{\mathrm{I}}^{LR}$. Since σ preserves L and R, then $\sigma(p_{\mathrm{I}}) = \sigma(p_{\mathrm{I}})^{LR}$, hence $\sigma(p_{\mathrm{I}})$ is a constant function, say taking the value q. Since $p_{\mathrm{I}}(a) = p$ for all $a \in \mathrm{I}$, it follows that $\alpha_a(p) = q$ for all $a \in \mathrm{I}$. Thus all the α_a are equal.

We now consider the converse, and suppose that conditions (1) and (2) apply, and use α for the equal automorphisms α_a. We first show that if $g \in \mathrm{M}$ is increasing, then $\sigma(g)$ is increasing. Let $b_1, b_2 \in \mathrm{I}$ with $b_1 \leq b_2$ and suppose $\beta(a_i) = b_i$ for $i = 1, 2$. Then Equation 4.1 gives

$$(\sigma(g))(b_1) = \alpha(g(a_1)) \leq \alpha(g(a_2)) = (\sigma(g))(b_2)$$

Now to see that σ preserves L, suppose that $f \in \mathrm{M}$. Then f^L is the least increasing function above f. Then as σ is a lattice automorphism, it is order-preserving, and we have shown that it takes increasing functions to increasing functions. So $\sigma(f^L)$ is an increasing function above $\sigma(f)$. Suppose that g is another increasing function above $\sigma(f)$. Then applying our results to the automorphism σ^{-1} (whose associated permutation β^{-1} and automorphisms are α^{-1} satisfy conditions (1) and (2)), we have that $\sigma^{-1}(g)$ is an increasing function above f. Then, since f^L is the least increasing function above f, we have $f^L \leq \sigma^{-1}(g)$, and this implies $\sigma(f^L) \leq g$. Thus $\sigma(f^L)$ is the least increasing function above $\sigma(f)$ showing that $\sigma(f^L) = (\sigma(f))^L$. ■

Combining Proposition 4.3.2 with a reformulated version of Equation 4.1, we have the following.

Theorem 4.3.3 *The automorphisms σ of $(\mathrm{M}, \wedge, \vee, \sqcap, \sqcup, L, R)$ are in bijective correspondence with the ordered pairs (α, β) of automorphisms of* I *where the automorphism σ corresponding to the pair (α, β) is given by*

$$(\sigma(f))(x) = \alpha(f(\beta^{-1}(x)))$$

Written another way, $\sigma(f) = \alpha \circ f \circ \beta^{-1}$.

We can now describe the structure of the automorphism group of $(\mathrm{M}, \wedge, \vee, \sqcap, \sqcup, L, R)$. For groups G and H, the **product group** $G \times H$ is the group whose elements are all ordered pairs (g, h) with $g \in G$ and $h \in H$, and whose operations are componentwise, so $(g, h)^{-1} = (g^{-1}, h^{-1})$, $(g_1, h_1) \circ (g_2, h_2) = (g_1 \circ g_2, h_1 \circ h_2)$, and $e = (e, e)$. This is similar to the product of lattices that we have used many times.

Theorem 4.3.4 *The automorphism group of $(\mathrm{M}, \wedge, \vee, L, R)$ is isomorphic to the product group* $\mathrm{Aut}(\mathrm{I}) \times \mathrm{Aut}(\mathrm{I})$.

Proof. Consider the map Γ taking an automorphism σ of M to the pair (α, β) as given in Theorem 4.3.3. We have shown that this map is a bijection. Suppose that $\Gamma(\sigma_1) = (\alpha_1, \beta_2)$ and $\Gamma(\sigma_2) = (\alpha_2, \beta_2)$. Then for $f \in \mathrm{M}$, we have $(\sigma_1 \circ \sigma_2)(f) = \sigma_1(\sigma_2(f))$. From the description in Theorem 4.3.3, $\sigma_1(\sigma_2(f))$ is equal to $\alpha_1 \circ \alpha_2 \circ f \circ \beta_2^{-1} \circ \beta_1^{-1}$. Then by a basic property of inverse functions, $\beta_2^{-1} \circ \beta_1^{-1} = (\beta_1 \circ \beta_2)^{-1}$, so this expression becomes $(\alpha_1 \circ \alpha_2) \circ f \circ (\beta_1 \circ \beta_2)^{-1}$. Thus $\Gamma(\sigma_1 \circ \sigma_2)$ is equal to $(\alpha_1 \circ \alpha_2, \beta_1 \circ \beta_2)$. This shows that Γ preserves the binary group operation. This alone is enough to establish that Γ is a group isomorphism, but the reader may verify directly that Γ preserves the other group operations. ■

We next consider the matter of which automorphisms of $(\mathrm{M}, \wedge, \vee, L, R)$ also preserve the negation *. We recall that $f^*(x) = f(1 - x)$, or in other words, $f^*(x) = f(x')$ where $x' = 1 - x$ is the negation of I.

Proposition 4.3.5 *Suppose σ is an automorphism of $(\mathrm{M}, \wedge, \vee, L, R)$ with corresponding ordered pair (α, β) of automorphisms of* I. *Then σ preserves the negation * on* M *if and only if β preserves the negation $x' = 1 - x$ of* I.

Proof. By general properties of automorphisms, β preserves negation if and only if β^{-1} preserves negation. Suppose that β preserves negation. Then for any $f \in \mathrm{M}$ we have $f^*(\beta^{-1}(x)) = f(1 - \beta^{-1}(x)) = f(\beta^{-1}(1 - x))$. This then gives $(\sigma(f^*))(x) = \alpha(f^*(\beta^{-1}(x))) = \alpha(f(\beta^{-1}(1 - x))) = (\sigma(f))(1 - x)$. Thus $\sigma(f^*) = (\sigma(f))^*$, so σ preserves negation.

Suppose that σ preserves the negation *. Suppose $x \in \mathrm{I}$ and $y = \beta^{-1}(x)$. Noting that $\alpha(0) = 0$ and $\alpha(1) = 1$, we have $(\sigma(1_y))(x) = \alpha(1_y(\beta^{-1}(x))) = 1$. Since σ preserves *, then $1 = (\sigma(1_y))(x) = (\sigma(1_y^*))(1 - x)$. From the definition of σ, we have $(\sigma(1_y^*))(1 - x) = \alpha(1_y^*(\beta^{-1}(1 - x)))$. But $1_y^* = 1_{1-y}$. Then since α preserves 0 and 1, this expression simplifies to $1_{1-y}(\beta^{-1}(1 - x))$. Since $1_{1-y}(\beta^{-1}(1 - x)) = 1$, then $\beta^{-1}(1 - x) = 1 - y = 1 - \beta^{-1}(x)$. So β^{-1} preserves negation, and hence so does β. ■

Corollary 4.3.6 *The automorphism group of $(\mathrm{M}, \wedge, \vee, L, R, ^*)$ is isomorphic to the product* $\mathrm{Aut}(\mathrm{I}) \times \mathrm{Aut}(\mathrm{I}, ')$.

We conclude this section with one further result that, although not used in the sequel, may be of interest. It generalizes one direction of Proposition 4.3.5.

Proposition 4.3.7 *Suppose σ is an automorphism of $(\mathrm{M}, \wedge, \vee, L, R)$ with corresponding ordered pair (α, β) of automorphisms of* I. *Then for an n-ary operation $\circ$ of* I, *if the automorphism β preserves $\circ$, then σ preserves the convolution operation $\bullet$ of* M *derived from it.*

Proof. Again, β preserving $\circ$ is equivalent to β^{-1} preserving $\circ$. To make the proof easier to read, we write $\gamma = \beta^{-1}$ and prove this for the case when $\circ$ is binary. The general case is no more difficult. For $f, g \in \mathrm{M}$ we have

$$(\sigma(f \bullet g))(x) = \alpha((f \bullet g)(\gamma(x)))$$

Using the definition of convolution, and the fact that for any elements u, v with $u \circ v = \gamma(x)$ there are y, z with $\gamma(y) = u$ and $\gamma(z) = v$, we have

$$(f \bullet g)(\gamma(x)) = \bigvee \{f(\gamma(y)) \wedge g(\gamma(z)) : \gamma(y) \circ \gamma(z) = \gamma(x)\}$$

Note that α preserves arbitrary joins and finite meets. Also, γ preserves $\circ$, so $\gamma(y) \circ \gamma(z) = \gamma(x)$ if and only if $\gamma(y \circ z) = \gamma(x)$, which occurs if and only if $y \circ z = x$. Thus

$$(\sigma(f \bullet g))(x) = \bigvee \{\alpha(f(\gamma(y))) \wedge \alpha(g(\gamma(z))) : y \circ z = x\}$$

Then as $\alpha(f(\gamma(y)) = (\sigma(f))(y))$ and $\alpha(g(\gamma(z))) = (\sigma(f))(z)$, the definition of convolution gives $(\sigma(f \bullet g))(x) = (\sigma(f) \bullet \sigma(y))(x)$. ■

4.4 Characterizing certain elements of M

In this section we provide technical results that will be of importance in determining the automorphisms of $(\mathrm{M}, \sqcap, \sqcup)$. We consider the element $1_0 \vee 1_1$ of M that takes value 1 at 0 and 1, and is otherwise zero, and the constant function $\overline{1}$ taking value 1. (Note that $\overline{1}$ is equal to 1_{I}, but this is difficult to distinguish from the characteristic function 1_1 of the point 1, so we use the notation $\overline{1}$.) We give properties of these elements that determine them uniquely, and that we will later show are preserved by automorphisms.

Definition 4.4.1 *An element $h \in \mathrm{M}$ is **$\sqcup$-irreducible** if $h = f \sqcup g$ implies that $h = f$ or $h = g$, is **$\sqcap$-irreducible** if $h = f \sqcap g$ implies that $h = f$ or $h = g$, and is **irreducible** if it is both $\sqcup$-irreducible and $\sqcap$-irreducible.*

Proposition 4.4.2 *The element $1_0 \vee 1_1$ is irreducible.*

Proof. Suppose $1_0 \vee 1_1 = f \sqcup g$. By Theorem 1.4.5, $1_0 \vee 1_1 = (f \vee g) \wedge f^L \wedge g^L$. Since $f^L(0) = f(0)$ and $g^L(0) = g(0)$, it follows that both f and g take value 1 at 0. Then f^L and g^L are constantly 1, so $1_0 \vee 1_1 = f \vee g$. So both f, g are 0 on the interval $(0,1)$. Since $f \vee g$ takes value 1 at 1, one of these functions must take value 1 at 1. But then this function is equal to $1_0 \vee 1_1$. The argument that $1_0 \vee 1_1$ is $\sqcap$-irreducible is similar. ■

Our aim is to characterize $1_0 \vee 1_1$ among the irreducible functions. We begin with the following.

Lemma 4.4.3 *Let h be irreducible, and $a < b < c$ with h non-zero at both a and c. Set $u = \sup\{h(x) : x \le b\}$ and $v = \sup\{h(x) : b \le x\}$. Then*

1. *If $u \ge v$, then $h(0) = 1$ and h is identically 0 on the interval $(0, b]$.*
2. *If $u \le v$, then $h(1) = 1$ and h is identically 0 on the interval $[b, 1)$.*

Proof. We prove the first statement, the second is dual. Define

$$f(x) = \begin{cases} h(x) & \text{if } x \le b \\ 0 & \text{if } b < x \end{cases} \quad \text{and} \quad g(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } 0 < x \le b \\ h(x) & \text{if } b < x \end{cases}$$

Since $g(0) = 1$, $g^L = \overline{1}$. So by Theorem 1.4.5, $f \sqcup g = (f \vee g) \wedge f^L$. Since $f^L(0) = f(0) = h(0)$ and $g(0) = 1$, we have $(f \sqcup g)(0) = h(0)$. For $0 < x \le b$, we have $f \vee g = f(x) = h(x)$, hence $(f \sqcup g)(x) = h(x)$. For $b < x$, we have $f \vee g = g$, and $g(x) \le f^L(x)$ since $u \ge v$. So $(f \sqcup g)(x) = g(x) = h(x)$. We have shown that $h = f \sqcup g$. Since $h(c)$ is non-zero and $f(c) = 0$, we cannot have $h = f$. Therefore $h = g$, and the result follows. ■

Proposition 4.4.4 *If h is irreducible and is non-zero at at least two values, then either $h = 1_0 \vee p_1$ for some $p > 0$, or $h = p_0 \vee 1_1$ for some $p > 0$.*

Proof. If h is irreducible and non-zero at at least two values, then there are $a < b < c$ with h non-zero at a and c. We assume the first case of Lemma 4.4.3 applies. The argument if the second case applies is similar.

By Lemma 4.4.3, $h(0) = 1$ and h is zero on $(0, b]$. We claim there is at most one value in $(b, 1]$ at which h is non-zero. Suppose there are two, $c_1 < c_2$. Then we may apply Lemma 4.4.3 to h at the points $0 < c_1 < c_2$. Computing u and v for these points again gives $u \geq v$ since $h(0) = 1$. Therefore case 1 of Lemma 4.4.3 gives that h is zero on $(0, c_1]$, contrary to our assumption that h is non-zero at c_1. So there is a unique value c in the interval $(b, 1]$ at which h is non-zero. So there is $p > 0$ with $h = 1_0 \vee p_c$.

Using Theorem 1.4.5, it is easy to see that $h = (1_0 \vee p_1) \sqcap 1_c$. Since h is irreducible, then $h = 1_0 \vee p_1$. If the other case of Lemma 4.4.3 had applied, we would have obtained that $h = p_0 \vee 1_1$. ■

We have shown that the irreducibles whose support contains at least two elements are all of the form $1_0 \vee p_1$ or $p_0 \vee 1_1$ for some $p > 0$. It is routine to verify that these elements are actually irreducible. This is left as an exercise. It remains to characterize the elements whose support has at most one element. These are the constant function $\overline{0}$ taking value 0, and the points p_a. It is left as an exercise to show that the constant function $\overline{0}$ is irreducible, and that a point p_a is irreducible if and only if $p = 1$, that is, if it is a singleton 1_a. This gives the following.

Theorem 4.4.5 *The irreducible elements of* M *are exactly the elements* $1_0 \vee p_1$ *for some* $p > 0$*, and* $p_0 \vee 1_1$ *for some* $p > 0$*, and the singletons* 1_a *for some* $a \in \mathrm{I}$*, and the constant function taking value* 0.

We wish to characterize the function $1_0 \vee 1_1$ among the irreducibles, and we also wish to characterize the constant function $\overline{1}$. We will accomplish both tasks together. But first we need to make progress on characterizing $\overline{1}$. We begin with the following.

Definition 4.4.6 *A function $g \in \mathrm{M}$ is a* ***near unity function*** *if g is identically equal to 1 on the open interval* $(0, 1)$.

Lemma 4.4.7 *An element $g \in \mathrm{M}$ is a near unity function if and only if it has the following properties.*

1. *g is convex and normal.*
2. *For each $f \in \mathrm{M}$, both $f \sqcap g$ and $f \sqcup g$ are convex.*

Proof. We first show that a near unity function g has these properties. It is obviously convex and normal. For any $f \in \mathrm{M}$, Theorem 1.4.5 provides that

$f \sqcup g = (f \vee g) \wedge f^L \wedge g^L$. Since $f^L(0) = f(0)$ and $g^L(0) = g(0)$, at 0 this function takes value $f(0) \wedge g(0)$. On the interval $(0,1)$, this function takes value f^L. Thus on the interval $[0,1)$, the function is increasing, so no matter what value it takes at 1, it is convex. A similar argument shows that $f \sqcap g$ is also convex.

Suppose that g has the indicated properties. We show that it is a near unity function. Consider the function $1_0 \vee 1_1$ that takes value 1 at 0 and 1, and is 0 elsewhere. Theorem 1.4.5 and the observation that $(1_0 \vee 1_1)^L = \overline{1} = (1_0 \vee 1_1)^R$ provide

$$(1_0 \vee 1_1) \sqcup g = (1_0 \vee g \vee 1_1) \wedge g^L$$
$$(1_0 \vee 1_1) \sqcap g = (1_0 \vee g \vee 1_1) \wedge g^R$$

Since g is normal, $g^L(1) = 1$. Therefore $(1_0 \vee 1_1) \sqcup g$ takes value 1 at 1. Since this function is assumed to be convex, it must be increasing, and since it is equal to g on the interval $(0,1)$, it follows that g is increasing on $(0,1)$. A similar argument using $(1_0 \vee 1_1) \sqcap g$ shows that g is decreasing on $(0,1)$. Thus g takes a constant value p on $(0,1)$.

Suppose that $p < 1$. Since g is normal, it must take value 1 at either 0 or 1, or both. Suppose it takes value 1 at 0. The argument in the other case is similar. Consider the function f that takes a parabolic shape with values of 1 at both 0 and 1 and takes value p at its vertex at 0.5. Note that f is not convex. Theorem 1.4.5 gives $f \sqcup g = (f \vee g) \wedge f^L \wedge g^L$. Since f and g both take value 1 at 0, both f^L and g^L are equal to $\overline{1}$. So $f \sqcup g = f \vee g$ and this is equal to f. So $f \sqcup g$ is not convex, contrary to assumption. So $p = 1$, and this shows that g is a near unity function. ∎

We come to our key result.

Proposition 4.4.8 *Suppose f is an irreducible function that is not convex and g is a near unity function. Then $f = 1_0 \vee 1_1$ and $g = \overline{1}$ if and only if $f \sqcup g = g$ and $f \sqcap g = g$.*

Proof. By Proposition 4.4.4, $1_0 \vee 1_1$ is irreducible, and clearly it is not convex. Also $\overline{1}$ is obviously a near unity function. By Theorem 1.4.5 $(1_0 \vee 1_1) \sqcup \overline{1} = (1_0 \vee 1_1)^L = \overline{1}$ and $(1_0 \vee 1_1) \sqcap \overline{1} = (1_0 \vee 1_1)^R = \overline{1}$. So these functions satisfy the given conditions.

For the converse, suppose that f is irreducible and not convex, that g is a near unity function, and that $f \sqcup g = g$ and $f \sqcap g = g$. Since f is not convex, it must be non-zero at at least two points. So by Proposition 4.4.4 it is either $1_0 \vee p_1$ for some $p > 0$, or $p_0 \vee 1_1$ for some $p > 0$. We will assume that $f = 1_0 \vee p_1$ for some $p > 0$. The argument in the other case is similar.

Theorem 1.4.5 gives

$$(1_0 \vee p_1) \sqcup g = (1_0 \vee g \vee p_1) \wedge (1_0 \vee p_1)^L \wedge g^L$$
$$(1_0 \vee p_1) \sqcap g = (1_0 \vee g \vee p_1) \wedge (1_0 \vee p_1)^R \wedge g^R$$

If $p < 1$, then $(1_0 \vee p_1)^R$ is equal to p on $(0,1)$. So $(1_0 \vee p_1) \sqcap g$ is at most p on this interval, and g is equal to 1 on this interval since it is a near unity function. Thus we cannot have $(1_0 \vee p_1) \sqcap g = g$, contrary to assumption. So $p = 1$, and this shows that $f = 1_0 \vee 1_1$. Also the formulas above, applied when $p = 1$, give that $f \sqcup g = (1_0 \vee g \vee 1_1) \wedge g^L$ and $f \sqcap g = (1_0 \vee g \vee 1_1) \wedge g^R$. Since $f \sqcap g$ has value 1 at 1, we have $g(1) = 1$, and since $f \sqcap g$ takes value 1 at 0, we have $g(0) = 1$. Thus $g = \overline{1}$. ■

4.5 Automorphisms of $(\mathrm{M}, \sqcap, \sqcup)$

In this section, we characterize the automorphism group of $(\mathrm{M}, \sqcap, \sqcup)$. We will show that each such automorphism preserves the pointwise order $\leq$ and the operations L, R, hence is an automorphism of $(\mathrm{M}, \wedge, \vee, L, R)$. From the results of Section 4.3, it follows that the automorphism groups of these two algebras are equal, and are isomorphic to $\mathrm{Aut}(\mathrm{I}) \times \mathrm{Aut}(\mathrm{I})$. These results are specialized to the setting involving negation. The key technique is to find properties that characterize elements, or groups of elements, and that are preserved under automorphism. We begin with the following.

Proposition 4.5.1 *Let $\sigma \in \mathrm{Aut}(\mathrm{M}, \sqcap, \sqcup)$ and $f \in \mathrm{M}$. The following hold:*

1. *f is convex if and only if $\sigma(f)$ is convex.*
2. *f is normal if and only if $\sigma(f)$ is normal.*

Proof. (1) By Theorem 3.5.7, f is convex if and only if the distributive laws $f \sqcap (g \sqcup h) = (f \sqcap g) \sqcup (f \sqcap h)$ and $f \sqcup (g \sqcap h) = (f \sqcup g) \sqcap (f \sqcup h)$ hold for all $g, h \in \mathrm{M}$. Thus if f is convex, then $\sigma(f)$ is convex. The same result applies to σ^{-1}, so if $\sigma(f)$ is convex, then f is convex.

(2) Note that f is normal if and only if $f^L(1) = 1$. By Theorem 1.4.5 $f \sqcup 1_1 = (f \vee 1_1) \wedge f^L \wedge 1_1^L$, and it follows that f is normal if and only if $f \sqcup 1_1 = 1_1$. So if f is normal, then $\sigma(f)$ is normal, and the argument involving σ^{-1} provides the converse. ■

We now make use of the results of the previous section.

Lemma 4.5.2 *Let $\sigma \in \mathrm{Aut}(\mathrm{M}, \sqcap, \sqcup)$. The following hold:*

1. *$\sigma(1_0 \vee 1_1) = 1_0 \vee 1_1$.*
2. *$\sigma(\overline{1}) = \overline{1}$.*

Proof. Let $f = 1_0 \vee 1_1$ and $g = \overline{1}$. Proposition 4.4.8 shows that f is irreducible and not convex. Since irreducibility is preserved by an automorphism σ, we have that $\sigma(f)$ is irreducible, and by Proposition 4.5.1, $\sigma(f)$ is not convex. The element g is a near unity function. By Lemma 4.4.7, this means that it is convex and normal, and for all $h \in \mathrm{M}$ we have that $h \sqcap g$ and $h \sqcup g$ are convex. By Proposition 4.5.1, we then have that $\sigma(g)$ is convex and normal, and for each $h \in \mathrm{M}$ both $\sigma(h) \sqcup \sigma(g) = \sigma(g)$ and $\sigma(h) \sqcap \sigma(g) = \sigma(g)$. Thus, by Proposition 4.4.7, $\sigma(g)$ is a near unity function. By Proposition 4.4.8, we have $f \sqcup g = g$ and $f \sqcap g = g$. Thus $\sigma(f) \sqcup \sigma(g) = \sigma(g)$ and $\sigma(f) \sqcap \sigma(g) = \sigma(g)$. Then by Proposition 4.4.8, $\sigma(f) = 1_0 \vee 1_1$ and $\sigma(g) = \overline{1}$. ■

This result will have several key applications. The first is the following.

Proposition 4.5.3 *Let $\sigma \in \mathrm{Aut}(\mathrm{M}, \sqcap, \sqcup)$ and $f \in \mathrm{M}$. The following hold:*

1. $\sigma(f^L) = \sigma(f)^L$.
2. $\sigma(f^R) = \sigma(f)^R$.

Proof. Using Theorem 1.4.5, we have that $f \sqcup \overline{1} = f^L$ and $f \sqcap \overline{1} = f^R$. The result then follows from the fact that σ fixes $\overline{1}$. ■

Since a function f is increasing if and only if $f = f^L$, and decreasing if and only if $f = f^R$, we immediately have the following.

Corollary 4.5.4 *Let $\sigma \in \mathrm{Aut}(\mathrm{M}, \sqcap, \sqcup)$ and $f \in \mathrm{M}$. The following hold:*

1. *f is increasing if and only if $\sigma(f)$ is increasing.*
2. *f is decreasing if and only if $\sigma(f)$ is decreasing.*

Thus each automorphism of $(\mathrm{M}, \sqcap, \sqcup)$ preserves L and R. We will use this to obtain that it also preserves the pointwise order $\leq$, and therefore the operations $\wedge$ and $\vee$. Our first step is the following.

Proposition 4.5.5 *Let $\sigma \in \mathrm{Aut}(\mathrm{M}, \sqcap, \sqcup)$ and $g \in \mathrm{M}$ be convex. Then for all $f \in \mathrm{M}$, we have $f \leq g$ if and only if $\sigma(f) \leq \sigma(g)$.*

Proof. We first show this when g is increasing; that is, when $g = g^L$. In this case, using Theorem 1.4.5, we have

$$f \sqcup g = (f \vee g) \wedge f^L \wedge g^L = f^L \wedge g$$

Since g is increasing, $f \leq g$ if and only if $f^L \leq g$, which is equivalent to $f \sqcup g = f^L$. Therefore $f \leq g$ if and only if $\sigma(f) \sqcup \sigma(g) = \sigma(f^L) = \sigma(f)^L$, and as $\sigma(g)$ is increasing, this is equivalent to having $\sigma(f) \leq \sigma(g)$. A corresponding result for g decreasing is obtained using $f \sqcap g$.

Now suppose that g is convex, so $g = g^L \wedge g^R$. Thus $f \leq g$ if and only if $f \leq g^L$ and $f \leq g^R$. Then, using the result obtained for increasing functions and decreasing functions, $f \leq g$ if and only if $\sigma(f) \leq \sigma(g)^L$ and $\sigma(f) \leq \sigma(g)^R$.

This is equivalent to having $\sigma(f) \leq \sigma(g)^L \wedge \sigma(g)^R$, and since $\sigma(g)$ is convex, is equivalent to $\sigma(f) \leq \sigma(g)$. ∎

We recall that for any $p \in \mathrm{I}$ and any $a \in \mathrm{I}$, p_a is the function that takes value p at a and takes value 0 otherwise. Such functions that are non-zero at only a single place are called **points**. We shall treat the function that is constantly 0 as a point. By a **singleton** we mean a point that takes the value 1. So a singleton is a function of the form 1_a for some $a \in \mathrm{I}$.

Proposition 4.5.6 *The set* P *of points in* M *is a subalgebra of* M.

What is needed to prove this is that P is closed under the operations of M, which is the content of Exercise 20.

Proposition 4.5.7 *An element* $g \in \mathrm{M}$ *is a point if and only if every* f *with* $f \leq g$ *is convex.*

Proof. Clearly every point is convex, and if a function f lies beneath a point, then it is also a point, hence is convex. For the converse, suppose that a function g is not a point. Then there are $a, b \in \mathrm{I}$ with g being non-zero at both a, b. Then the function f defined to be equal to g at a, b and zero elsewhere clearly lies beneath g. But f is not convex. ∎

Proposition 4.5.8 *Each* $\sigma \in \mathrm{Aut}(\mathrm{M}, \sqcap, \sqcup)$ *maps the points of* M *bijectively onto the points of* M.

Proof. Suppose g is a point. To see that $\sigma(g)$ is a point, suppose that $h \leq \sigma(g)$. Then there is f with $\sigma(f) = h$. So $\sigma(f) \leq \sigma(g)$. Since g is a point, it is convex, so by Proposition 4.5.1, $\sigma(g)$ is convex and by Proposition 4.5.5, $f \leq g$. Since g is a point, by Proposition 4.5.7, f is convex. Thus by Proposition 4.5.1, $\sigma(f) = h$ is convex. Thus every element beneath $\sigma(g)$ is convex, and so by Proposition 4.5.7, $\sigma(g)$ is a point. Applying the same argument to σ^{-1}, if $\sigma(g)$ is a point, then so is g. ∎

Corollary 4.5.9 *The algebra* P *is a characteristic subalgebra of* M.

Corollary 4.5.10 *Each* $\sigma \in \mathrm{Aut}(\mathrm{M}, \sqcap, \sqcup)$ *maps the set of singletons of* M *bijectively onto itself.*

Proof. Each such σ maps points bijectively to points and preserves normality. The singletons are exactly the normal points. ∎

We turn next to a series of technical results that will be used to attain our primary aim that each automorphism σ is order preserving.

Lemma 4.5.11 *Suppose* $\sigma \in \mathrm{Aut}(\mathrm{M}, \sqcap, \sqcup)$ *and* $a \in \mathrm{I}$. *The following hold:*

1. $1_0 \vee 1_a = (1_0 \vee 1_1) \sqcap 1_a$.

2. $\sigma(1_0 \vee 1_a) = 1_0 \vee \sigma(1_a)$.

Proof. (1) Since $(1_0 \vee 1_1)^R = \overline{1}$, Theorem 1.4.5 gives

$$(1_0 \vee 1_1) \sqcap 1_a = (1_0 \vee 1_a \vee 1_1) \wedge 1_a^R = 1_0 \vee 1_a$$

(2) By (1) and Lemma 4.5.2, $\sigma(1_0 \vee 1_a) = \sigma(1_0 \vee 1_1) \sqcap \sigma(1_a) = (1_0 \vee 1_1) \sqcap \sigma(1_a)$. Proposition 4.5.10 gives that $\sigma(1_a)$ is a singleton, so another application of item (1) gives that this expression is equal to $1_0 \vee \sigma(1_a)$. ∎

Lemma 4.5.12 *Suppose* $\sigma \in \mathrm{Aut}(\mathrm{M}, \sqcap, \sqcup)$ *and* $f, g \in \mathrm{M}$.

1. $f(0) = 1$ *if and only if* $f^L = \overline{1}$.
2. $f(0) = 1$ *if and only if* $(\sigma(f))(0) = 1$.
3. *If* $f(0) = 1$ *and* $g(0) = 1$, *then* $f \sqcup g = f \vee g$.
4. *If* $f(0) = 1$ *and* $g(0) = 1$, *then* $f \leq g$ *if and only if* $\sigma(f) \leq \sigma(g)$.

Proof. Item (1) follows directly from the definition of f^L. Item (2) follows from (1) since $\sigma(\overline{1}) = \overline{1}$. Item (3) is immediate from Theorem 1.4.5 since $f(0) = 1$ implies that $f^L = \overline{1}$ and $g(0) = 1$ implies $g^L = \overline{1}$. With the assumptions in Item (4), (3) gives that $f \leq g$ if and only if $f \sqcup g = g$, which is equivalent to $\sigma(f) \sqcup \sigma(g) = \sigma(g)$. But by (2), both $\sigma(f)$ and $\sigma(g)$ take value 1 at 0, so by (3), our condition is equivalent to $\sigma(f) \leq \sigma(g)$. ∎

Lemma 4.5.13 *If* $\sigma \in \mathrm{Aut}(M, \sqcap, \sqcup)$ *and* p_a *is a point, then* $\sigma(1_0 \vee p_a) = 1_0 \vee \sigma(p_a)$.

Proof. Since 1_a is a singleton, by Corollary 4.5.10, $\sigma(1_a) = 1_b$ for some $b \in \mathrm{I}$. Next, by part (4) of Lemma 4.5.12 we have that $\sigma(1_0 \vee p_a) \leq \sigma(1_0 \vee 1_a)$, and by Lemma 4.5.11, $\sigma(1_0 \vee 1_a) = 1_0 \vee \sigma(1_a)$. Thus $\sigma(1_0 \vee p_a) \leq 1_0 \vee 1_b$. By part (2) of Lemma 4.5.12, we have that $\sigma(1_0 \vee p_a)$ takes value 1 at 0, and it follows that $\sigma(1_0 \vee p_a) = 1_0 \vee q_b$ for some $q \in \mathrm{I}$. We have shown that $\sigma(1_0 \vee p_a) = 1_0 \vee q_b$ for some q, b where $\sigma(1_a) = 1_b$.

Since $p_a \leq (1_0 \vee p_a)^R$ and this latter element is decreasing, hence convex, by Proposition 4.5.5, $\sigma(p_a) \leq \sigma((1_0 \vee p_a)^R) = (\sigma(1_0 \vee p_a))^R = (1_0 \vee q_b)^R$. Since $p_a \leq 1_a$ and 1_a is convex, by Proposition 4.5.5, $\sigma(p_a) \leq \sigma(1_a) = 1_b$. It follows that $\sigma(p_a) \leq q_b$. We have shown that $1_0 \vee \sigma(p_a) \leq \sigma(1_0 \vee p_a)$.

Now apply the result obtained to the automorphism σ^{-1} and the function $\sigma(1_0 \vee p_a) = 1_0 \vee q_b$. This gives $1_0 \vee \sigma^{-1}(q_b) \leq \sigma^{-1}(1_0 \vee q_b) = 1_0 \vee p_a$. Thus $\sigma^{-1}(q_b) \leq p_a$. Since p_a is convex, Proposition 4.5.5 gives $q_b \leq \sigma(p_a)$. From above, $\sigma(p_a) \leq q_b$, so $\sigma(p_a) = q_b$. Therefore $\sigma(1_0 \vee p_a) = 1_0 \vee \sigma(p_a)$. ∎

Proposition 4.5.14 *If* $\sigma \in \mathrm{Aut}(\mathrm{M}, \sqcap, \sqcup)$, p_a *is a point, and* $f \in \mathrm{M}$, *then* $p_a \leq f$ *if and only if* $\sigma(p_a) \leq \sigma(f)$.

Proof. Suppose $p_a \leq f$. Then since $(1_0 \vee p_a)^L = \overline{1}$, by Theorem 1.4.5

$$(1_0 \vee p_a) \sqcup f = (1_0 \vee p_a \vee f) \wedge f^L$$

Then since $p_a \leq f$ and $f^L(0) = f(0)$, it follows that $(1_0 \vee p_a) \sqcup f = f$. Using this and Lemma 4.5.13, we have

$$\sigma(f) = \sigma(1_0 \vee p_a) \sqcup \sigma(f) = (1_0 \vee \sigma(p_a)) \sqcup \sigma(f)$$

Then since $(1_0 \vee \sigma(p_a))^L = \overline{1}$, Theorem 1.4.5 gives

$$\sigma(f) = (1_0 \vee \sigma(p_a) \vee \sigma(f)) \wedge \sigma(f)^L$$

The distributive law for $\wedge, \vee$ with $\sigma(f)^L = \sigma(f^L)$ and $\sigma(f) \leq \sigma(f)^L$ gives

$$\sigma(f) = ((1_0 \vee \sigma(p_a)) \wedge \sigma(f^L)) \vee \sigma(f)$$

An obvious simplification gives

$$\sigma(f) = (\sigma(p_a) \wedge \sigma(f^L)) \vee \sigma(f)$$

Now $p_a \leq f \leq f^L$, and f^L is increasing, hence convex. So Proposition 4.5.5 gives $\sigma(p_a) \leq \sigma(f^L)$. We then obtain

$$\sigma(f) = \sigma(p_a) \vee \sigma(f)$$

Thus $\sigma(p_a) \leq \sigma(f)$. The converse is obtained by applying our result with σ^{-1} to show that $\sigma(p_a) \leq \sigma(f)$ implies that $p_a \leq f$. ■

To apply this result, Proposition 4.5.8 and Proposition 4.5.14 show that the points that lie beneath $\sigma(f)$ are exactly the $\sigma(p_a)$ where $p_a \leq f$. So by Proposition 4.2.8,

$$\sigma(f) = \bigvee\{\sigma(p_a) : p_a \leq f\}$$

If $f \leq g$, then $p_a \leq f$ implies that $p_a \leq g$. Proposition 4.5.14 then gives $\sigma(p_a) \leq \sigma(g)$. Thus $\sigma(g)$ is an upper bound of $\{\sigma(p_a) : p_a \leq f\}$, and since $\sigma(f)$ is the least upper bound of this set, we have $\sigma(f) \leq \sigma(g)$. Applying this result with σ^{-1} gives the converse. We have shown the following.

Proposition 4.5.15 *If* $\sigma \in \mathrm{Aut}(\mathrm{M}, \sqcap, \sqcup)$, *then for* $f, g \in \mathrm{M}$ *we have* $f \leq g$ *if and only if* $\sigma(f) \leq \sigma(g)$.

By Proposition 4.1.4, it follows that every automorphism of $(\mathrm{M}, \sqcap, \sqcup)$ also preserves $\wedge$ and $\vee$, and we have shown that they also preserve L and R. From our earlier results we then have the following.

Theorem 4.5.16 *The following automorphism groups are equal to each other:*

1. $\mathrm{Aut}(\mathrm{M}, \sqcap, \sqcup)$,
2. $\mathrm{Aut}(\mathrm{M}, \sqcap, \sqcup, 1_0, 1_1)$,
3. $\mathrm{Aut}(\mathrm{M}, \wedge, \vee, L, R)$,
4. $\mathrm{Aut}(\mathrm{M}, \sqcap, \sqcup, \wedge, \vee, L, R, 1_0, 1_1)$.

In Theorem 4.3.3 we saw that the automorphisms σ of $(\mathrm{M}, \wedge, \vee, L, R)$ correspond to ordered pairs (α, β) of automorphisms of I, with the correspondence given by

$$\sigma(f) = \alpha \circ f \circ \beta^{-1}$$

In view of Theorem 4.5.16, this holds also for automorphisms of $(\mathrm{M}, \sqcap, \sqcup)$. As a consequence of this, and of Theorem 4.3.4, we have the following.

Theorem 4.5.17 *The automorphism group of* $(\mathrm{M}, \sqcap, \sqcup)$ *is isomorphic to the product* $\mathrm{Aut}(\mathrm{I}) \times \mathrm{Aut}(\mathrm{I})$.

Proposition 4.3.5 characterized the automorphisms of $(\mathrm{M}, \wedge, \vee, L, R)$ that also preserve the negation * of M. In view of the results above, this immediately gives the following.

Theorem 4.5.18 *The automorphisms of* $(\mathrm{M}, \sqcap, \sqcup, ^*, 1_0, 1_1)$ *correspond to ordered pairs* (α, β) *of automorphisms of* I *where* β *preserves negation. Thus* $\mathrm{Aut}(\mathrm{M}, \sqcap, \sqcup, ^*, 1_0, 1_1)$ *is isomorphic to* $\mathrm{Aut}(\mathrm{I}) \times \mathrm{Aut}(\mathrm{I}, ')$.

Additionally, Proposition 4.3.7 gives the following.

Proposition 4.5.19 *If* σ *is an automorphism of* $(\mathrm{M}, \sqcap, \sqcup)$ *corresponding to the pair* (α, β)*, then* σ *preserves the convolution of each operation of* I *that is preserved by* β.

4.6 Characteristic subalgebras of M

Recall that a subalgebra of B of an algebra A is a characteristic subalgebra of A if every automorphism of A restricts to give an automorphism of B. (See Definition 4.1.10.) In the process of finding a description of the automorphism group of M, it was shown that several subalgebras of M were characteristic subalgebras. We note that these are characteristic subalgebras of $(\mathrm{M}, \sqcap, \sqcup)$ and also of $(\mathrm{M}, \sqcap, \sqcup, 1_0, 1_1, ^*)$. These include the following:

- The subalgebra N of normal functions (Proposition 4.5.1).
- The subalgebra C of convex functions (Proposition 4.5.1).
- The subalgebra L of convex, normal functions (Proposition 4.5.1).
- The subalgebra S of singletons (Corollary 4.5.10).
- The subalgebra P of points (Corollary 4.5.9).

We remark that the subalgebra S of singletons is the realization of the truth value algebra of fuzzy sets within M. (See Theorem 3.2.3.) Using Theorem 4.5.17, it is easy to establish that other subalgebras of interest are characteristic.

Corollary 4.6.1 *The subalgebra of characteristic functions of closed intervals, the realization of* $\mathrm{S}^{[2]}$ *within* M*, is a characteristic subalgebra of* M*.*

Corollary 4.6.2 *The subalgebra* E *of characteristic functions of sets is a characteristic subalgebra of* M*.*

There are many other characteristic subalgebras. Some of these are explored in the exercises.

4.7 Summary

We determined the automorphism groups of the truth value algebra I of fuzzy sets, and the truth value algebra $\mathrm{I}^{[2]}$ of interval-valued fuzzy sets. We also determined the automorphisms of the algebra $\mathrm{Map}(S, \mathrm{I})$ of fuzzy subsets of a set. This result about the automorphisms of the fuzzy subsets of I was used to obtain the automorphism group of the algebra M of type-2 fuzzy sets with the convolution operations $\sqcap$ and $\sqcup$.

We showed that the group $\mathrm{Aut}(\mathrm{M})$ is isomorphic to $\mathrm{Aut}(\mathrm{I}) \times \mathrm{Aut}(\mathrm{I})$. This correspondence associates to an ordered pair (α, β) of automorphisms of I the automorphism σ of M that takes $f \in \mathrm{M}$ to $\alpha \circ f \circ \beta^{-1}$. The automorphism group of M with the convolution operations and negation is isomorphic to $\mathrm{Aut}(\mathrm{I}) \times \mathrm{Aut}(\mathrm{I}, ')$ where $\mathrm{Aut}(\mathrm{I}, ')$ is the group of automorphisms of I that also preserve its negation. An automorphism of M corresponding to the pair (α, β) was shown to preserve the convolution of every operation of I with which β is compatible.

Many subalgebras of M were shown to be characteristic subalgebras. These include the subalgebras S of singletons that is isomorphic to the truth value algebra of fuzzy sets, and the subalgebra of characteristic functions of closed intervals that is isomorphic to the truth value algebra $\mathrm{I}^{[2]}$ of interval-valued fuzzy sets. Also characteristic are the subalgebras N of normal function, C of convex functions, and L of convex normal functions, among others.

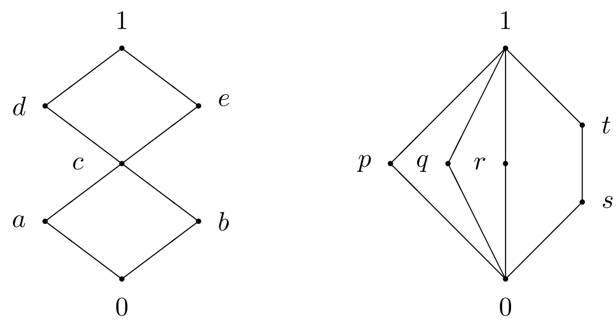
The subalgebra L of convex normal functions will play an important role. We would like to know its automorphism group, but do not. Since L is a characteristic subalgebra of M, the restriction map is a homomorphism from Aut(M) to Aut(L), and this is easily seen to be one-to-one. But the possibility remains that there are automorphisms of L not obtained this way. We note that L is a lattice, and that the elements of L that are both join and meet irreducible are those of the form $a_{\mathrm{I}} \vee 1_0$ and $a_{\mathrm{I}} \vee 1_1$ for some constant function a_{I}. Any automorphism of L must map these functions to themselves. For further details, see [110] and [111].

4.8 Exercises

1. Prove Theorem 4.1.3.
2. How many elements are in the group of permutations of a set X with n elements?
3. Prove that the set $\mathrm{SL}(2,\mathbb{R})$ of all 2×2 matrices of determinant 1 is a subgroup of the group $\mathrm{SL}(2,\mathbb{R})$ of all 2×2 invertible matrices.
4. Prove that the intersection of two characteristic subalgebras is a characteristic subalgebra.
5. Suppose that L is a complete lattice, that $S \subseteq L$, and that α is an automorphism of L. Prove that $\alpha(\bigvee S) = \bigvee\{\alpha(s) : s \in S\}$.
6. Suppose that $(S,\vee)$ is a join semilattice with associated partial ordering $\leq$. If α is a bijection of S to itself, prove that α is an automorphism if and only if it satisfies $x \leq y \Leftrightarrow \alpha(x) \leq \alpha(y)$.
7. Suppose that $(S,\vee)$ is a join semilattice. Define $x \in S$ to be join irreducible if $x = y \vee z$ implies $x = y$ or $x = z$. Prove that if α is an automorphism of S, then x is join irreducible if and only if $\alpha(x)$ is join irreducible.
8. Consider the lattice L in Figure 4.1. Show that each permutation of $\{a,b,c\}$ extends to an automorphism of L.
9. Suppose that L is a lattice and that α is an automorphism of L. Prove that if x is a maximal member of the set of join irreducible elements of L, then so is $\alpha(x)$.

10. Suppose L is a lattice with least element 0. An element $a \in L$ is called an **atom** of L if $a \neq 0$, and if $b \in L$ satisfies $0 \leq b \leq a$, then either $b = 0$ or $b = a$. Prove that if $x \in L$ and α is an automorphism of L, then x is an atom if and only if $\alpha(x)$ is an atom.

11. Describe the automorphism groups of each of the following lattices.



12. Give all characteristic subalgebras of the lattices in Exercise 11.

13. Prove that in a finite lattice, each element is the join of the join irreducibles beneath it.

14. Prove Proposition 4.2.8, that if $f \in \mathrm{Map}(S, \mathrm{I})$, then f is the join of the join irreducibles beneath it.

15. Consider the lattice $\mathbb{R} \times \mathbb{R}$ with a new bottom element $-\infty$ and new top element $+\infty$ inserted. Show that $\pm\infty$ are the only join irreducible elements of this lattice. Conclude that there is a complete lattice where it is not the case that every element is the join of join irreducibles.

16. Suppose φ is a one-to-one mapping of $[0,1]^{[0,1]}$ onto itself.

    (a) If $\varphi(f \sqcup g) = \varphi(f) \sqcup \varphi(g)$, show that $\varphi(1_0) = 1_0$.

    (b) If $\varphi(f \sqcap g) = \varphi(f) \sqcap \varphi(g)$, show that $\varphi(1_1) = 1_1$.

17. Show that for any $p > 0$, the elements $1_0 \vee p_1$ and $p_0 \vee 1_1$ are irreducible. See Theorem 4.4.5.

18. Show that the constant function $\overline{0}$ taking value 0 is irreducible.

19. Show that a point p_a is irreducible if and only if $p = 1$.

20. Show that P is closed under the usual operations of M by showing that for points p_a and q_b in M, $p_a \sqcap q_b = (p \wedge q)_{a \wedge b}$, $p_a \sqcup q_b = (p \wedge q)_{a \vee b}$, $p_a \sqcap 0 = p_a \sqcup 0 = 0$, and $p_a^* = p_{1-a}$, and noting that $1_0 \in \mathrm{P}$ and $1_1 \in \mathrm{P}$.

21. If $(L, \wedge, \vee)$ is a lattice and α is a bijection of L to itself that preserves $\wedge$, prove that α also preserves $\vee$, hence is an automorphism of $(L, \wedge, \vee)$. Give an example to show that this is not the case for a bisemilattice $(S, \wedge, \vee)$. Explain why this is the case.

22. Show that the subalgebra Q of constant functions is a characteristic subalgebra of $(\mathrm{M}, \sqcap, \sqcup)$.

23. Show that the subalgebra F of functions of finite support is a characteristic subalgebra of $(\mathrm{M}, \sqcap, \sqcup)$.

24. Show that the subalgebra of endmaximal functions is a characteristic subalgebra of $(\mathrm{M}, \sqcap, \sqcup)$.

25. Find a subalgebra of $(\mathrm{M}, \sqcap, \sqcup)$ that is not characteristic. (Hint: it can be quite small).

Chapter 5

T-norms and T-conorms

Additional operations on $\mathrm{I} = ([0,1], \vee, \wedge, 0, 1)$ give additional operations on fuzzy sets. The operations $\wedge$ and $\vee$ on I are generally interpreted as AND and OR operations for fuzzy logic and generalized to triangular norms (t-norms) and triangular conorms (t-conorms). Triangular norms and conorms were first developed by K. Menger [80] for application in the field of probabilistic metric spaces. The current axioms used to define t-norm and t-conorm were given by B. Schweizer and A. Sklar [96, 97, 98]. These operations have found applications is several fields, including fuzzy set theory, where they play a significant role in both theory and practice. See, for example, [68] and [86]. In this chapter, these operations are extended to interval-valued and type-2 t-norms and t-conorms [29, 52, 62, 64, 65, 108].

5.1 Preliminaries

In this section, we give background that places the results of this chapter in a wider context. While we restrict matters to the commutative setting since that is what will apply here, the non-commutative versions are also well studied. We begin with the following.

Definition 5.1.1 *A* ***commutative monoid*** *is an algebra* $(A, \cdot, e)$ *that has a binary operation* $\cdot$ *and a constant* e *and satisfies the following:*

1. $x \cdot y = y \cdot x$ *(commutative).*
2. $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ *(associative).*

3. *$x \cdot e = x = e \cdot x$ (identity).*

There are many examples of commutative monoids. Any commutative group $(G, +, -, 0)$ (Definition 4.1.1) has as a reduct, a monoid $(G, +, 0)$, and any meet semilattice $(S, \wedge, 1)$ that has a largest element 1 is a commutative monoid (Definition 2.1.4). Our primary interest in monoids will be in conjunction with partial orderings (Definition 1.1.3).

Definition 5.1.2 *A **partially ordered commutative monoid** $(P, \leq, \cdot, e)$ is a system satisfying the following:*

1. *$(P, \leq)$ is a partially ordered set.*
2. *$(P, \cdot, e)$ is a commutative monoid.*
3. *$x \leq y$ implies that $x \cdot z \leq y \cdot z$.*

Certain partially ordered commutative monoids will be the focus of this chapter. These can have various additional properties that are important.

Definition 5.1.3 *A **lattice-ordered commutative monoid** is an algebra $(L, \wedge, \vee, \cdot, e)$ satisfying the following:*

1. *$(L, \wedge, \vee)$ is a lattice.*
2. *$(L, \cdot, e)$ is a commutative monoid.*
3. *$x \cdot (y \vee z) = (x \cdot y) \vee (x \cdot z)$.*

The term "lattice-ordered commutative monoid" is not used in a standard way in fuzzy theory. At times the lattice is also assumed to be distributive, and at other times it is also assumed that $x \cdot (y \wedge z) = (x \cdot y) \wedge (x \cdot z)$. The usage we follow is the original [7]. These differences are of minor concern as the instances that apply here will be distributive and satisfy the condition involving meet. There is a further refinement that is important.

Definition 5.1.4 *A partially ordered commutative monoid $(P, \leq, \cdot, e)$ is **residuated** if for each x, z there is a largest element y with $x \cdot y \leq z$.*

The relevance of residuation to our studies stems from its connection to logic. Suppose that P is the set of propositions of some logic, and partially order P by setting $p \leq q$ if p is a stronger assertion than q; that is, if the truth of p necessitates the truth of q. Then conjunction $\wedge$ gives a monoidal structure on P whose identity element is the proposition True. When residuated, the weakest proposition r such that $p \wedge r$ is stronger than q is the implication $p \Rightarrow q$. A bounded lattice whose meet operation is residuated is called a **Heyting algebra**. For more details see [26]. The following is obvious.

Lemma 5.1.5 *A partially ordered monoid $(P, \leq, \cdot, e)$ is residuated if and only if for each $x, z \in P$, the following hold.*

1. *The set* $\{y : x \cdot y \leq z\}$ *has a least upper bound.*

2. $x \cdot (\bigvee\{y : x \cdot y \leq z\}) \leq z$.

In the following chapter, we will see how certain lattices carry intrinsic topologies, and that these intrinsic topologies are related to residuation. For now, we concern ourselves with the familiar situation of the lattice I and its usual topology. We will need the following technical result later in this section. It is a simple adaption of the familiar Intermediate Value Theorem from first year calculus [94].

Proposition 5.1.6 *Suppose* $\diamond : \mathrm{I} \times \mathrm{I} \to \mathrm{I}$ *is continuous and* $x_1, x_2, y_1, y_2 \in \mathrm{I}$ *with* $x_1 \leq x_2$, $y_1 \leq y_2$ *and* $x_1 \diamond y_1 \leq x_2 \diamond y_2$. *Then for any* $z \in \mathrm{I}$ *with* $x_1 \diamond y_1 \leq z \leq x_2 \diamond y_2$, *there are* $x, y \in \mathrm{I}$ *with* $x_1 \leq x \leq x_2$ *and* $y_1 \leq y \leq y_2$ *such that* $x \diamond y = z$.

Proof. Define $g : \mathrm{I} \to \mathrm{I} \times \mathrm{I}$ by setting $g(\lambda) = (x_1, y_1) + \lambda[(x_2, y_2) - (x_1, y_1)]$. So g is a parameterization of the line segment connecting (x_1, y_1) to (x_2, y_2). Let $f(\lambda) = \diamond(g(\lambda))$. Since $\diamond$ and g are continuous, so is f. Also $f(0) = x_1 \diamond y_1$ and $f(1) = x_2 \diamond y_2$. So $f(0) \leq z \leq f(1)$. By the Intermediate Value Theorem there is some $0 \leq \lambda \leq 1$ with $f(\lambda) = z$. Then $g(\lambda) = (x, y)$ for some x, y with $x_1 \leq x \leq x_2$, $y_1 \leq y \leq y_2$ and $x \diamond y = z$. ■

There is a converse of sorts to this result that is analogous to the fact that an increasing function $f : \mathrm{I} \to \mathrm{I}$ is continuous if and only if it satisfies the Intermediate Value Theorem. We will not need this result, but it is of interest in connection with t-norms and t-conorms.

5.2 Triangular norms and conorms

We begin with a discussion of type-1 t-norms and t-conorms.

Definition 5.2.1 *A* ***t-norm*** *is a binary operation on the unit interval* I, *in other words a map* $\triangle : \mathrm{I} \times \mathrm{I} \to \mathrm{I}$, *satisfying the following:*

1. $x \triangle y = y \triangle x$ *(commutative).*

2. $x \triangle (y \triangle z) = (x \triangle y) \triangle z$ *(associative).*

3. *If* $y \leq z$ *then* $x \triangle y \leq x \triangle z$ *(increasing in each variable).*

4. $x \triangle 1 = 1 \triangle x = x$ *(*1 *is an identity).*

Definition 5.2.2 *A* ***t-conorm*** *is a binary operation* $\triangledown : \mathrm{I} \times \mathrm{I} \to \mathrm{I}$ *on* I *that satisfies the same conditions as a t-norm except it has* 0 *as an identity.*

In the literature, a t-norm and t-conorm are often written as $T(x,y) = x \vartriangle y$ and $S(x,y) = x \triangledown y$, respectively.

Proposition 5.2.3 *If $\vartriangle$ is a t-norm, then there is a t-conorm $\triangledown$, called its* ***dual****, defined by $x \triangledown y = (x' \vartriangle y')'$. Conversely, given a t-conorm $\triangledown$, there is a t-norm $\vartriangle$, called its* ***dual****, defined by $x \vartriangle y = (x' \triangledown y')'$.*

The most common t-norms are **minimum**: $x \vartriangle y = x \wedge y$, **algebraic product**: $x \vartriangle y = xy$, and **bounded product**, also known as **Łukasiewicz**: $x \vartriangle y = (x + y - 1) \vee 0$. The corresponding dual t-conorms are **maximum**: $x \triangledown y = x \vee y$, **algebraic sum**: $x \triangledown y = x + y - xy$, and **bounded sum**, also known as **Łukasiewicz**: $x \vartriangle y = (x+y) \wedge 1$. Notice that each of these operations is continuous. In fact, all continuous t-norms can be constructed from these [70]. See Exercises 12 and 13 for an example of a noncontinuous t-norm and noncontinuous t-conorm.

Proposition 5.2.4 *If $\vartriangle$ is a t-norm and $\triangledown$ is a t-conorm, then both $(\mathrm{I}, \vartriangle, 1)$ and $(\mathrm{I}, \triangledown, 0)$ are lattice-ordered monoids that additionally satisfy the following:*

1. $x \vartriangle (y \wedge z) = (x \vartriangle y) \wedge (x \vartriangle z)$.
2. $x \triangledown (y \wedge z) = (x \triangledown y) \wedge (x \triangledown z)$.

The proof of this result is a simple consequence of the fact that any order-preserving n-ary operation on I preserves both finite joins and finite meets. It is left as an exercise (Exercise 8).

Proposition 5.2.5 *If $\vartriangle$ is a continuous t-norm, then the partially ordered monoid $(\mathrm{I}, \vartriangle, 1)$ is residuated.*

Proof. We apply Lemma 5.1.5. Suppose that $x, z \in \mathrm{I}$, and consider the set $S = \{y : x \vartriangle y \leq z\}$. Since I is complete, S has a least upper bound w. It is a basic fact about suprema in the interval that there is an increasing sequence y_n of elements of S that converges to w. Then since $\vartriangle$ is continuous, we have

$$x \vartriangle w = x \vartriangle \left(\lim_{n \to \infty} y_n \right) = \lim_{n \to \infty} (x \vartriangle y_n)$$

Since each $y_n \in S$, we have that each $x \vartriangle y_n \leq z$, so the limit $x \vartriangle w \leq z$. ■

Thus the t-norms of minimum, algebraic product, and bounded product are residuated. An examination of the proof of Proposition 5.2.5 shows that a lesser condition than continuity is used. We do not need that $\vartriangle$ preserves limits of all sequences, just of increasing sequences. So $\vartriangle$ is residuated if for each $x \in \mathrm{I}$, the function $x \vartriangle (\cdot) : \mathrm{I} \to \mathrm{I}$ is continuous from the left.

For operations on I, the properties of being order preserving, preserving finite joins, and preserving finite meets, are all equivalent. This is not the case for more general lattices, and allows various possibilities for extending the notion of t-norms and t-conorms to the more general setting of a bounded lattice L. One might consider a binary operation $\triangle$ that yields a partially ordered monoid, a lattice-ordered monoid, or a lattice-ordered monoid where $\triangle$ additionally preserves finite meets.

Definition 5.2.6 *Suppose that L is a bounded lattice and $\triangle : L \times L \to L$ is such that $(L, \triangle, 1)$ is a commutative monoid. Then $\triangle$ is a* ***t-norm*** *if it satisfies the first condition, a* ***join preserving t-norm*** *if it satisfies the first two conditions, and* ***a lattice-ordered t-norm*** *if it satisfies all three of the following conditions:*

1. $x_1 \leq x_2$ *and* $y_1 \leq y_2$ *imply* $x_1 \triangle y_1 \leq x_2 \triangle y_2$.
2. $x \triangle (y \vee z) = (x \triangle y) \vee (x \triangle z)$.
3. $x \triangle (y \wedge z) = (x \triangle y) \wedge (x \triangle z)$.

The definitions of a ***t-conorm****, a* ***join preserving t-conorm****, and a* ***lattice-ordered t-conorm*** *are obtained by replacing 1 as a unit with 0 as a unit.*

For I, or for any chain, the three conditions coincide (Exercise 8). In any lattice the second or third condition implies the first, but no other implications among these conditions hold. We turn next to apply these notions.

5.3 T-norms and t-conorms on intervals

Here we consider t-norms and t-conorms on the truth value algebra $\mathrm{I}^{[2]}$ for interval-valued fuzzy sets. (See Definition 1.2.10.) These are understood in terms of Definition 5.2.6. We recall that $\mathrm{I}^{[2]} = \{(a, b) : a, b \in \mathrm{I} \text{ and } a \leq b\}$, and that the lattice operations $\wedge$ and $\vee$ on $\mathrm{I}^{[2]}$ are componentwise.

Proposition 5.3.1 *Let $\triangle_1$ and $\triangle_2$ be t-norms on* I *such that $x \triangle_1 y \leq x \triangle_2 y$ for all $x, y \in \mathrm{I}$, and define*

$$(x, y)\, \hat{\triangle}\, (z, w) = (x \triangle_1 z, y \triangle_2 w)$$

for all $(x, y), (z, w) \in \mathrm{I}^{[2]}$. Then $\hat{\triangle}$ is a lattice-ordered t-norm on $\mathrm{I}^{[2]}$.

Proof. The condition that $x \triangle_1 y \leq x \triangle_2 y$ implies that the operation $\hat{\triangle}$ is well defined. Indeed, if (x, y) and (z, w) are elements of $\mathrm{I}^{[2]}$, then $x \leq y$ and $z \leq w$. Then $x \triangle_1 z \leq y \triangle_1 w \leq y \triangle_2 w$. So $(x, y)\, \hat{\triangle}\, (z, w)$ does belong to $\mathrm{I}^{[2]}$. Since the lattice operations of $\mathrm{I}^{[2]}$ are componentwise, the operation $\hat{\triangle}$ satisfies the monoid axioms, and is compatible with both join and meet, hence gives a lattice-ordered t-norm. ■

Definition 5.3.2 *For t-norms $\triangle_1$ and $\triangle_2$ with $x \triangle_1 y \leq x \triangle_2 y$ for each x, y, denote the resulting t-norm on $\mathrm{I}^{[2]}$ by $\triangle_1 \times \triangle_2$ and denote $\triangle \times \triangle$ by $\triangle^2$.*

A t-norm on $\mathrm{I}^{[2]}$ obtained as $\triangle_1 \times \triangle_2$ for some $\triangle_1$ and $\triangle_2$ is called **t-representable**. Not all t-norms on $\mathrm{I}^{[2]}$ are representable. The following example is given in [17], as are a number of others.

Proposition 5.3.3 *Let $\triangle$ be a t-norm on* I. *Then for each $t \in [0,1]$ with $t \neq 1$, there is a non-representable t-norm on $\mathrm{I}^{[2]}$ defined by*

$$(x,y) \,\hat{\triangle}\, (z,w) = (x \triangle z,\ (t \triangle y \triangle w) \vee (x \triangle w) \vee (y \triangle z))$$

It is known that every join preserving t-norm on the unit square $\mathrm{I} \times \mathrm{I}$ is representable. However, there are join preserving t-norms on $\mathrm{I}^{[2]}$ that are not representable, and there is no satisfactory characterization of representable t-norms on $\mathrm{I}^{[2]}$. See [17] for a complete account. However, we do have the following intrinsic characterization of what is likely the most useful class of t-norms on this lattice [29].

Theorem 5.3.4 *A lattice-ordered t-norm $\hat{\triangle}$ on $\mathrm{I}^{[2]}$ is representable as $\triangle^2$ for some t-norm $\triangle$ on* I *if and only if it satisfies the following conditions:*

1. *$D \,\hat{\triangle}\, D \subseteq D$, where $D = \{(x,x) : x \in \mathrm{I}\}$.*
2. *$(0,1) \,\hat{\triangle}\, (x,y) = (0,y)$.*

Proof. Suppose $\hat{\triangle}$ is given by $(x,y) \,\hat{\triangle}\, (z,w) = (x \triangle z, y \triangle w)$ for some t-norm $\triangle$ on I. Then by Proposition 5.3.1, $\hat{\triangle}$ is a lattice-ordered t-norm, and it is easily seen that it satisfies properties 1 and 2.

Suppose that $\hat{\triangle}$ is a lattice-ordered t-norm on $\mathrm{I}^{[2]}$ satisfying properties 1 and 2. Since $\hat{\triangle}$ carries D^2 into D, we have for each $a, b \in \mathrm{I}$,

$$(a,a) \,\hat{\triangle}\, (b,b) = (c,c)$$

for some $c \in \mathrm{I}$. Define $a \triangle b = c$. We need to show that $\triangle$ is a t-norm. The commutative property of $\triangle$ is immediate, since $(a,a) \,\hat{\triangle}\, (b,b) = (b,b) \,\hat{\triangle}\, (a,a)$. For associativity, note $(a,a) \,\hat{\triangle}\, (b,b) = (a \triangle b, a \triangle b)$. So

$$((a,a) \,\hat{\triangle}\, (b,b)) \,\hat{\triangle}\, (c,c) = ((a \triangle b) \triangle c, (a \triangle b) \triangle c)$$

and similarly,

$$(a,a) \,\hat{\triangle}\, ((b,b) \,\hat{\triangle}\, (c,c)) = (a \triangle (b \triangle c), a \triangle (b \triangle c))$$

So the associativity of $\hat{\triangle}$ yields that of $\triangle$. Also, $(a,a) \,\hat{\triangle}\, (1,1) = (a,a)$ gives $a \triangle 1 = a$. If $a \leq b$, then $(a,a) \leq (b,b)$ implies that $(a,a) \,\hat{\triangle}\, (c,c) \leq (b,b) \,\hat{\triangle}\, (c,c)$ or $(a \triangle c, a \triangle c) \leq (b \triangle c, b \triangle c)$, which means that $a \triangle c \leq b \triangle c$. Thus $\triangle$ is a t-norm.

Now since $\hat{\vartriangle}$ is a lattice-ordered t-norm,

$$\begin{aligned}(a,b)\,\hat{\vartriangle}\,(c,d) &= (a,b)\,\hat{\vartriangle}\,((c,c)\vee(0,d))\\ &= ((a,b)\,\hat{\vartriangle}\,(c,c))\vee((a,b)\,\hat{\vartriangle}\,(0,d))\\ &= ((a,b)\,\hat{\vartriangle}\,(c,c))\vee(0,e)\end{aligned}$$

for some $e \in \mathrm{I}$. Therefore the first component of $(a,b)\,\hat{\vartriangle}\,(c,d)$ does not depend on d, and similarly does not depend on b. Also, by property 2, $(a,b)\,\hat{\vartriangle}\,(c,d)$ has second component the same as

$$(a,b)\,\hat{\vartriangle}\,(c,d)\,\hat{\vartriangle}\,(0,1) = (a,b)\,\hat{\vartriangle}\,(0,1)\,\hat{\vartriangle}\,(c,d)\,\hat{\vartriangle}\,(0,1) = (0,b)\,\hat{\vartriangle}\,(0,d)$$

Thus the second component of $(a,b)\,\hat{\vartriangle}\,(c,d)$ does not depend on a or c. So $\hat{\vartriangle}$ acts componentwise. From $(a,a)\,\hat{\vartriangle}\,(c,c) = (a \vartriangle c, a \vartriangle c)$ and $(b,b)\,\hat{\vartriangle}\,(d,d) = (b \vartriangle d, b \vartriangle d)$, it follows that $(a,b)\,\hat{\vartriangle}\,(c,d) = (a \vartriangle c, b \vartriangle d)$, and the proof is complete. ■

In the next section, we look at convolutions with respect to $\vee$ and $\wedge$ of two classes of binary operations on I, namely, the class of t-norms, and their duals, t-conorms.

5.4 Convolutions of t-norms and t-conorms

In M, the notion of "increasing" is ambiguous, so the definition of t-norm cannot be generalized literally. One way to generalize to type-2, is to look at convolutions with respect to $\vee$ and $\wedge$ of type-1 t-norms and their duals. We will refer to these as **type-2 t-norms**, and **type-2 t-conorms**. Note that the minimum t-norm $\wedge$ and maximum t-conorm $\vee$ have already been dealt with in this way, resulting in the operations $\sqcap$ and $\sqcup$, respectively. The other two basic t-norms, product xy and Łukasiewicz $(x+y-1)\vee 0$, will give new binary operations on M.

Proposition 5.4.1 *Let $\vartriangle$ be a t-norm on* I*, and $\triangledown$ be its dual conorm with respect to the negation $'$. Let $\blacktriangle$ and $\blacktriangledown$ be the convolutions of these operations*

$$\begin{aligned}(f \blacktriangle g)(x) &= \bigvee\{f(y)\wedge g(z) : y \vartriangle z = x\}\\ (f \blacktriangledown g)(x) &= \bigvee\{f(y)\wedge g(z) : y \triangledown z = x\}\end{aligned}$$

Then for $f, g, h \in \mathrm{M}$, the following hold:

1. *$\blacktriangle$ is commutative and associative.*
2. *$f \blacktriangle 1_1 = f$.*

3. $f \blacktriangle (g \vee h) = (f \blacktriangle g) \vee (f \blacktriangle h)$.

4. *If* $g \leq h$, *then* $(f \blacktriangle g) \leq (f \blacktriangle h)$.

5. $\blacktriangledown$ *is commutative and associative.*

6. $f \blacktriangledown 1_0 = f$.

7. $f \blacktriangledown (g \vee h) = (f \blacktriangledown g) \vee (f \blacktriangledown h)$.

8. *If* $g \leq h$, *then* $(f \blacktriangledown g) \leq (f \blacktriangledown h)$.

9. $(f \blacktriangle g)^* = f^* \blacktriangledown g^*$ *and* $(f \blacktriangledown g)^* = f^* \blacktriangle g^*$.

This was given in [108], and is left as recommended exercises (Exercises 17 and 18).

As we have seen, an isomorphic copy of the algebra $\mathrm{I} = ([0,1], \vee, \wedge, ', 0, 1)$ is a subalgebra of M, namely the characteristic functions 1_a for $a \in \mathrm{I}$. The formula

$$(1_a \blacktriangle 1_b)(x) = \bigvee \{1_a(y) \wedge 1_b(z) : y \triangle z = x\}$$

says that $1_a \blacktriangle 1_b$ is the characteristic function of $a \triangle b$, as it should be. It is clear that the t-norm $\blacktriangle$ acts on the subalgebra of characteristic functions in exactly the same way as the t-norm $\triangle$ acts on the algebra I. This establishes the following.

Theorem 5.4.2 *The mapping* $a \to 1_a$ *is an isomorphism from the algebra* $(\mathrm{I}, \triangle)$ *into the algebra* $(\mathrm{M}, \blacktriangle)$ *whose image is the singletons* 1_a, *where* $a \in \mathrm{I}$.

We next consider the subalgebra of M consisting of characteristic functions of closed intervals. By Theorem 3.3.5, there is an isomorphism from $\mathrm{I}^{[2]}$ onto the subalgebra of M consisting of characteristic functions of closed intervals mapping the element (a, b) to the characteristic function $1_{[a,b]}$ of the closed interval $[a, b]$. We show that for continuous t-norms, this isomorphism is compatible with t-norms and their convolutions.

Theorem 5.4.3 *Let* $\triangle$ *be a continuous t-norm on* I, $\blacktriangle$ *be its convolution, and* $\triangle^2$ *be the t-norm on* $\mathrm{I}^{[2]}$ *given by* $(a, b) \triangle^2 (c, d) = (a \triangle c, b \triangle d)$. *Then the mapping* $(a, b) \to 1_{[a,b]}$ *is an isomorphism from* $(\mathrm{I}^{[2]}, \triangle^2)$ *into* $(\mathrm{M}, \blacktriangle)$.

Proof. Consider the formula

$$(1_{[a,b]} \blacktriangle 1_{[c,d]})(x) = \bigvee \{(1_{[a,b]})(y) \wedge (1_{[c,d]})(z) : y \triangle z = x\}$$

We see that this function takes only the values 0 and 1, and it takes value 1 at exactly those x for which $x = y \triangle z$ for some $y \in [a, b]$ and $z \in [c, d]$. So the smallest value at which this function attains value 1 is $a \triangle c$, and the largest value at which it attains value 1 is $b \triangle d$. Since $\triangle$ is continuous, Proposition 5.1.6 shows that if $a \vartriangle c \leq x \leq b \vartriangle d$, then there is some $y \in [a, b]$ and $z \in [c, d]$ with $x = y \vartriangle z$. So this function attains value 1 exactly on the interval $[a \vartriangle c, b \vartriangle d]$. ■

A third subalgebra of M of interest is L, the algebra of convex normal functions. We will look further at type-2 t-norms that come from continuous t-norms before considering t-norms for the subalgebra L.

5.5 Convolutions of continuous t-norms and t-conorms

Here we collect various properties of the convolution of a continuous t-norm and of a continuous t-conorm.

Proposition 5.5.1 *Let $\vartriangle$ be a continuous t-norm with convolution $\blacktriangle$, and let $\triangledown$ be its dual t-conorm with convolution $\blacktriangledown$. Then for all $f, g \in$ M, the following hold:*

1. $(f \blacktriangle g)^R = f^R \blacktriangle g^R$.
2. $(f \blacktriangle g)^L = f^L \blacktriangle g^L$.
3. $f \blacktriangle \overline{1} = f^R$.
4. $f \blacktriangle 1_1 = f$.
5. $(f \blacktriangledown g)^R = f^R \blacktriangledown g^R$.
6. $(f \blacktriangledown g)^L = f^L \blacktriangledown g^L$.
7. $f \blacktriangledown \overline{1} = f^L$.
8. $f \blacktriangledown 1_0 = f$.

Proof. We prove the first four items. The proofs of the others are nearly identical.

To prove item 1, note first that $y \geq u$ and $z \geq v$ for some u and v with $u \vartriangle v = x$ clearly implies that $y \vartriangle z \geq x$. Since $\vartriangle$ is continuous, Proposition 5.1.6 provides the converse. Now by the definition of $\blacktriangle$,

$$(f^R \blacktriangle g^R)(x) = \bigvee \{f^R(u) \wedge g^R(v) : u \vartriangle v = x\}$$

Note that $f^R(u) \wedge g^R(v) = \vee\{f(y) \wedge g(z) : y \geq u \text{ and } z \geq v\}$ by the meet continuity of I (Proposition 1.4.4). Combining this with the expression above and the remarks about $y \vartriangle z \geq x$ then gives

$$(f^R \blacktriangle g^R)(x) = \bigvee\{f(y) \wedge g(z) : y \triangle z \geq x\}$$

This latter expression is $\bigvee\{(f \blacktriangle g)(w) : w \geq x\}$, which equals $(f \blacktriangle g)^R(x)$.

For item 2, we use the fact that $f^L = f^{*R*}$, item 1, and Proposition 5.4.1, which says that $(f \blacktriangle g)^* = f^* \blacktriangledown g^*$ and $(f \blacktriangledown g)^* = f^* \blacktriangle g^*$. Then

$$(f \blacktriangle g)^L = (f^* \blacktriangledown g^*)^{R*} = (f^{*R} \blacktriangledown g^{*R})^* = f^L \blacktriangle g^L$$

For item 3, since $\overline{1}(z) = 1$ for each z, $(f \blacktriangle \overline{1})(x) = \bigvee\{f(y) : y \triangle z = x\}$. Since $\triangle$ is continuous, Proposition 5.1.6 says that for any $y \geq x$ there is $z \geq x$ with $y \triangle z = x$. Thus $(f \blacktriangle \overline{1})(x) = \bigvee\{f(y) : y \geq x\} = f^R(x)$.

For item 4, $(f \blacktriangle 1_1)(x) = \bigvee\{f(y) \wedge 1_1(z) : y \triangle z = x\}$. Since $1_1(z)$ is zero except for $z = 1$, and the equation $y \triangle 1 = x$ has only the solution $y = x$, it follows that $(f \blacktriangle 1_1)(x) = f(x)$. ∎

Remark 5.5.2 The assumption of continuity is necessary for some of these properties. Consider the example for $f \blacktriangle \overline{1} = f^R$, due to Hernández et al. [52], of the noncontinuous t-norm and the function

$$x \triangle y = \begin{cases} x \wedge y & \text{if } x \vee y = 1 \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad f(x) = \begin{cases} 0.3 & \text{if } x \neq 0.5 \\ 1 & \text{if } x = 0.5 \end{cases}$$

In this case, $(f \blacktriangle \overline{1})(0.1) = \bigvee\{(f(y) : y \triangle z = 0.1\} = f(0.1) \vee f(1) = 0.3$, and this differs from $f^R(0.1) = 1$.

We have noted that every t-norm and every t-conorm on I is lattice-ordered; that is, it preserves the operations $\wedge$ and $\vee$ of meet and join. One might hope that the convolution of a t-norm or t-conorm is also lattice-ordered in the sense that it preserves $\sqcap$ and $\sqcup$. This is not the case generally, as the following theorem shows. However, it does apply if we restrict to convolutions of continuous t-norms and t-conorms, and also restrict our domain to a subalgebra of M. This result will be of key importance in the following.

Theorem 5.5.3 *Let $\triangle$ be a continuous t-norm with convolution $\blacktriangle$, and let $\triangledown$ be its dual t-conorm with convolution $\blacktriangledown$. Then for $f \in$ M, the following hold for all $g, h \in$ M if and only if f is convex.*

1. $f \blacktriangle (g \sqcap h) = (f \blacktriangle g) \sqcap (f \blacktriangle h)$.
2. $f \blacktriangle (g \sqcup h) = (f \blacktriangle g) \sqcup (f \blacktriangle h)$.
3. $f \blacktriangledown (g \sqcap h) = (f \blacktriangledown g) \sqcap (f \blacktriangledown h)$.
4. $f \blacktriangledown (g \sqcup h) = (f \blacktriangledown g) \sqcup (f \blacktriangledown h)$.

Proof. We first outline the structure of the proof. We will prove that item 2 holds for every convex function f and every $g, h \in \mathrm{M}$. The proof that item 4 holds for every convex f and every $g, h \in \mathrm{M}$ is identical. Then the following identities can be used to show that items 1 and 3 hold for every convex f and every $g, h \in \mathrm{M}$.

$$(f \sqcup g)^* = f^* \sqcap g^* \qquad (f \sqcap g)^* = f^* \sqcup g^*$$
$$(f \blacktriangle g)^* = f^* \blacktriangledown g^* \qquad (f \blacktriangledown g)^* = f^* \blacktriangle g^*$$

To see this for item 1, note first that since f is convex, so is f^*. Then using item 4, $f^* \blacktriangledown (g^* \sqcup h^*) = (f^* \blacktriangledown g^*) \sqcup (f^* \blacktriangledown h^*)$. Applying the identities above then gives that $(f \blacktriangle (g \sqcap h))^* = ((f \blacktriangle g) \sqcap (f \blacktriangle h))^*$, and item 1 follows. Obtaining item 3 from item 2 is similar.

This will establish that for each convex f and each $g, h \in \mathrm{M}$, items (1)–(4) hold. For the converse, we will show that if $f \in \mathrm{M}$, and for each $g, h \in \mathrm{M}$ that item 2 holds for this f, then f is convex.

Suppose that f is convex and $g, h \in \mathrm{M}$. We first note that

$$(f \blacktriangle (g \sqcup h))(x) = \bigvee \{f(y) \wedge (g \sqcup h)(z) : y \vartriangle z = x\}$$
$$(g \sqcup h)(z) = \bigvee \{g(u) \wedge h(v) : u \vee v = z\}$$

It follows from meet continuity of I (Proposition 1.4.4) that $(f \blacktriangle (g \sqcup h))(x)$ is equal to

$$\bigvee \{f(y) \wedge g(u) \wedge h(v) : y \vartriangle (u \vee v) = x\} \tag{5.1}$$

A similar calculation shows that $((f \blacktriangle g) \sqcup (f \blacktriangle h))(x)$ is equal to

$$\bigvee \{f(p) \wedge g(q) \wedge f(s) \wedge h(t) : (p \vartriangle q) \vee (s \vartriangle t) = x\} \tag{5.2}$$

To establish item 2, we must show that (5.1) equals (5.2).

Suppose that $y \vartriangle (u \vee v) = x$. Then $(y \vartriangle u) \vee (y \vartriangle v) = x$, and it follows that $(5.1) \leq (5.2)$. For the reverse inequality, suppose that $(p \vartriangle q) \vee (s \vartriangle t) = x$. We want y such that both of the following hold since this will show that the given element in (5.2) lies beneath an element of (5.1).

$$y \vartriangle (q \vee t) = (y \vartriangle q) \vee (y \vartriangle t) = x \tag{5.3}$$

$$f(y) \wedge g(q) \wedge h(t) \geq f(p) \wedge g(q) \wedge f(s) \wedge h(t) \tag{5.4}$$

If $p \vartriangle q = s \vartriangle t = x$, let $y = p \wedge s$. Then $(y \vartriangle q) \vee (y \vartriangle t) = x$, providing (5.3). Also, $f(y) = f(p)$ or $f(y) = f(s)$, and in either case providing (5.4).

Otherwise, we may as well assume that $p \triangle q < x$ and $s \triangle t = x$. If $s \triangle q \leq x$, then

$$(s \triangle q) \vee (s \triangle t) = s \triangle t = x$$

and, taking $y = s$, we have (5.3) and (5.4). On the other hand, if $s \triangle q > x$, then $s \triangle q > s \triangle t$ implies $q > t$. Thus we have

$$p \triangle q < x < s \triangle q$$

so by Proposition 5.1.6, there is a y with $p < y < s$ and $y \triangle q = x$. Then $t < q$ implies that $y \triangle t \leq y \triangle q = x$. This gets (5.3). Since f is convex and $p < y < s$, we have $f(y) \geq f(p) \wedge f(s)$, and this gets (5.4). It follows that (5.1) $\geq$ (5.2), and hence (5.1) = (5.2) when f is convex.

It remains only to show that if $f \in \mathrm{M}$ and item 2 holds for this f and for all $g, h \in \mathrm{M}$, then f is convex. For such f we have

$$f \blacktriangle (1_1 \sqcup \overline{1}) = (f \blacktriangle 1_1) \sqcup (f \blacktriangle \overline{1})$$

The left side is

$$f \blacktriangle (1_1 \sqcup \overline{1}) = f \blacktriangle 1_1 = f$$

Proposition 5.5.1 and Theorem 1.4.5 provide that the right side is

$$(f \blacktriangle 1_1) \sqcup (f \blacktriangle \overline{1}) = f \sqcup f^R = (f \vee f^R) \wedge (f^L \wedge f^{RL}) = f^R \wedge f^L$$

Thus $f = f^L \wedge f^R$, and hence is convex. ■

On the unit interval, t-norms are increasing in each variable. For convex functions, type-2 t-norms behave in a similar way with respect to the orders $\sqsubseteq_{\sqcap}$ and $\sqsubseteq_{\sqcup}$.

Corollary 5.5.4 *Assume* $\blacktriangle$ *and* $\blacktriangledown$ *are convolutions of continuous t-norms or continuous t-conorms. If* f *is convex and* $g \sqsubseteq_{\sqcap} h$*, then*

$$f \blacktriangle g \sqsubseteq_{\sqcap} f \blacktriangle h \qquad \textit{and} \qquad f \blacktriangledown g \sqsubseteq_{\sqcap} f \blacktriangledown h$$

If f *is convex and* $g \sqsubseteq_{\sqcup} h$*, then*

$$f \blacktriangle g \sqsubseteq_{\sqcup} f \blacktriangle h \qquad \textit{and} \qquad f \blacktriangledown g \sqsubseteq_{\sqcup} f \blacktriangledown h$$

Proof. Since $g \sqsubseteq_{\sqcap} h$, we have $g = g \sqcap h$. Thus $f \blacktriangle g = f \blacktriangle (g \sqcap h) = (f \blacktriangle g) \sqcap (f \blacktriangle h)$, whence $f \blacktriangle g \sqsubseteq_{\sqcap} f \blacktriangle h$. The other parts follow similarly. ■

5.6 T-norms and t-conorms on L

In this section, we show that the type-2 t-norms and t-conorms on M that are given by convolutions of continuous t-norms and continuous t-conorms on I induce lattice-ordered t-norms and t-conorms on the subalgebra L of convex normal functions.

Proposition 5.6.1 *Let* $\blacktriangle$ *and* $\blacktriangledown$ *be convolutions of a continuous t-norm and a continuous t-conorm. If* $f, g \in \mathrm{L}$, *then so are* $f \blacktriangle g$ *and* $f \blacktriangledown g$.

Proof. Suppose $\triangle$ is continuous and f, k are convex. By Theorem 5.5.3,

$$f \blacktriangle (g \sqcap h) = (f \blacktriangle g) \sqcap (f \blacktriangle h)$$

for all g, h if and only if f is convex. Thus we have

$$\begin{aligned}(f \blacktriangle k) \blacktriangle (g \sqcap h) &= f \blacktriangle (k \blacktriangle (g \sqcap h)) \\ &= f \blacktriangle ((k \blacktriangle g) \sqcap (k \blacktriangle h)) \\ &= (f \blacktriangle (k \blacktriangle g)) \sqcap (f \blacktriangle (k \blacktriangle h)) \\ &= ((f \blacktriangle k) \blacktriangle g) \sqcap ((f \blacktriangle k) \blacktriangle h)\end{aligned}$$

Therefore, $f \blacktriangle k$ is convex. The proof that $f \blacktriangledown k$ is convex is similar.

Now f is normal if and only if $f^{RL} = \overline{1}$ (the constant function). Then by Proposition 5.5.1, if $\triangle$ is a continuous t-norm,

$$(f \blacktriangle g)^{RL} = f^{RL} \blacktriangle g^{RL}$$

Then since $\overline{1} \blacktriangle \overline{1} = \overline{1}$, if $\triangle$ is continuous and f and g are normal, so is $f \blacktriangle g$. And by a similar argument, so is $f \blacktriangledown g$. ■

This result, together with Theorem 5.5.3, yield the following.

Theorem 5.6.2 *Let* $\blacktriangle$ *and* $\blacktriangledown$ *be convolutions of a continuous t-norm and a continuous t-conorm. Then* $\blacktriangle$ *and* $\blacktriangledown$ *are a lattice-ordered t-norm and lattice-ordered t-conorm on* L.

We remark that related results are obtained in [65, 82]. They consider a more restrictive setting, that of convex normal functions that are additionally upper semicontinous. In this setting, they show that the convolution of a continuous t-norm gives a lattice-ordered t-norm on the subalgebra considered, and further, that this resulting t-norm is residuated. Matters related to this will be the focus of the following chapter.

5.7 Summary

We discussed t-norms and t-conorms on the truth value algebra I of fuzzy sets and the truth value algebra $\mathrm{I}^{[2]}$ of interval-valued fuzzy sets. For a t-norm $\triangle$ on I, we described properties of its convolution to an operation $\blacktriangle$ on M, especially in the case when $\triangle$ is continuous. It was shown that the restriction of the convolution $\blacktriangle$ to the algebra of singletons is isomorphic to $(\mathrm{I}, \triangle)$, and that the restriction of $\blacktriangle$ to the subalgebra of characteristic functions of closed intervals is isomorphic to $(\mathrm{I}^{[2]}, \triangle^2)$ when $\triangle$ is continuous.

Properties of the convolution $\blacktriangle$ of a continuous t-norm $\triangle$ were developed in the context of convex functions. In particular, when $\triangle$ is continuous, then its convolution $\blacktriangle$ gives a lattice-ordered t-norm on the subalgebra L of convex normal functions.

5.8 Exercises

1. If L is a bounded lattice and $\vee$ its join, prove that $(L, \vee, 0)$ is a lattice-ordered monoid.

2. If L is a bounded lattice and $\wedge$ its meet, prove that $(L, \wedge, 1)$ is a lattice-ordered monoid if and only if L is distributive.

3. Suppose that L is a complete distributive lattice that is meet continuous; that is, it satisfies $x \wedge \bigvee y_i = \bigvee (x \wedge y_i)$. Prove that $(L, \wedge, 1)$ is a residuated lattice-ordered monoid.

4. Let X be a set, and $\mathcal{P}(X)$ be its **power set**; that is, the collection of all subsets of X. Prove that the operation $\cap$ of intersection on $\mathcal{P}(X)$ is residuated; that is, that for each $A, C \subseteq X$ there is a largest subset B of X with $A \cap B \subseteq C$. Show that this set B is given by $\neg A \cup C$, where $\neg A$ denotes the complement of A in X.

5. For the real numbers $\mathbb{R}$, prove that $(\mathbb{R}, +, 0)$ is a lattice-ordered monoid.

6. Describe how the plane $\mathbb{R} \times \mathbb{R}$ can be considered as a lattice-ordered monoid under coordinatewise addition.

7. Suppose that $(L, \cdot, e)$ is a lattice-ordered monoid and $\cdot$ is a group operation. Prove that if L has more than one element, then it has neither a largest nor a least element.

8. Let $\circ$ be a binary operation on I; that is, $\circ : \mathrm{I} \times \mathrm{I} \to \mathrm{I}$. Show that the following statements are equivalent.

   (a) $\circ$ is increasing in each argument.

   (b) For all $x, y, z \in \mathrm{I}$, $x \circ (y \vee z) = (x \circ y) \vee (x \circ z)$.

   (c) For all $x, y, z \in \mathrm{I}$, $x \circ (y \wedge z) = (x \circ y) \wedge (x \circ z)$.

9. Prove that each of the operations $x \triangle y = x \wedge y$, $x \triangle y = xy$, and $x \triangle y = (x + y - 1) \vee 0$ is a t-norm.

10. Prove that each of the operations $x \triangledown y = x \vee y$, $x \triangledown y = x + y - xy$, and $x \triangledown y = (x + y) \wedge 1$ is a t-conorm.

11. Prove that minimum is the largest t-norm and maximum is the smallest t-conorm; that is, $x \vartriangle y \leq x \wedge y$ for any t-norm $\vartriangle$ on I, and $x \triangledown y \geq x \vee y$ for any t-conorm $\triangledown$ on I.

12. Prove that the **drastic product** defined by

$$x \vartriangle_0 y = \begin{cases} x \wedge y & \text{if} \quad x \vee y = 1 \\ 0 & \text{if} \quad x \vee y < 1 \end{cases}$$

    is a t-norm and is the smallest t-norm on I.

13. Prove that the **drastic sum** defined by

$$x \triangledown_1 y = \begin{cases} x \vee y & \text{if} \quad x \wedge y = 0 \\ 1 & \text{if} \quad x \wedge y > 0 \end{cases}$$

    is a t-conorm and is the largest t-conorm on I.

14. Prove that maximum is dual to minimum, algebraic sum is dual to algebraic product, and bounded sum is dual to bounded product; that is,

    (a) $(x \wedge y)' = x' \vee y'$.
    (b) $(xy)' = x' + y' - x'y'$.
    (c) $((x + y - 1) \vee 0)' = (x' + y') \wedge 1$.

15. Prove that the drastic sum is dual to the drastic product.

16. Show that for the lattice I, the condition that a t-norm $\vartriangle$ is increasing in each argument can be replaced by the condition

$$x \vartriangle (y \vee z) = (x \vartriangle y) \vee (x \vartriangle z)$$

    Give an example of a lattice for which these conditions are not equivalent.

17. Prove the properties of type-2 t-norms in Proposition 5.4.1.

18. Prove the properties of type-2 t-conorms in Proposition 5.4.1.

19. Find an example of a t-norm that is not residuated.

20. Prove the remaining three parts of Corollary 5.5.4.

Chapter 6

Convex Normal Functions

The subalgebra L of convex normal functions has many desirable features. It is a De Morgan algebra that contains the subalgebras of fuzzy sets and interval-valued fuzzy sets, and the convolutions of continuous t-norms and t-conorms extend to lattice-ordered t-norms and t-conorms on L. Further, the restriction to convex normal functions is natural for applications. The focus of this chapter is on constructing an algebra from L that retains all of these desirable features, and adds even more. This new algebra is also a De Morgan algebra that contains the subalgebras of fuzzy sets and interval-valued fuzzy sets. It also behaves well with the convolutions of continuous t-norms and t-conorms and has natural motivation for applications. It is further a completely distributive lattice that has a compact metric space topology given by integration, and under this topology its operations are continuous. This algebra was first studied in [64, 65, 81, 82] as the algebra of upper semicontinuous convex normal functions. The results described here are contained in [44, 45].

6.1 Preliminaries

Portions of these preliminaries deal with common topics in analysis. A good resource for these is the standard undergraduate text [94]. For a thorough discussion of the application of analytic and topological methods in lattice theory, see [7, 32, 33].

Definition 6.1.1 *A **metric** on a set X is a mapping $d : X \times X \to [0, \infty)$ that satisfies the following properties.*

1. $d(x, y) = d(y, x)$.
2. $d(x, z) \leq d(x, y) + d(y, z)$.
3. $d(x, y) = 0$ *if and only if* $x = y$.

*A **quasi-metric** weakens the third condition to $d(x, x) = 0$. A **metric space** is a set together with a metric.*

The real numbers $\mathbb{R}$ is a standard example of a metric space where the metric is the ordinary distance $d(x, y) = |x - y|$. In any metric space there are notions of limits and continuity, just as in first-year calculus, replacing $|x - y| < \epsilon$ with $d(x, y) < \epsilon$. We illustrate with the notion of uniform continuity for metric spaces. The definition of continuity for metric spaces is left as an exercise (Exercise 1).

Definition 6.1.2 *For metric spaces (X, d) and (Y, d'), a function $f : X \to Y$ is **uniformly continuous** if for each $\epsilon > 0$ there is a $\delta > 0$, so that for any $x, y \in X$, if $d(x, y) < \delta$, then $d'(f(x), f(y)) < \epsilon$.*

In dealing with limits and continuity, a key notion is that of an **open set**. For the reals, a set $A \subseteq \mathbb{R}$ is open if for each $x \in A$ there is an $\epsilon > 0$ with the open interval $(x - \epsilon, x + \epsilon) \subseteq A$. The intersection of finitely many open sets is open, and the union of arbitrarily many open sets is open (Exercise 5).

Definition 6.1.3 *A **topology** on a set X is a collection τ of subsets of X that contains the empty set, the set X, and is closed under finite intersections and arbitrary unions. A set X with a topology τ is a **topological space** and the members of τ are called its **open subsets**.*

In a metric space, a set A is open if for each $x \in A$ there is some $\epsilon > 0$ so that $d(x, y) < \epsilon$ implies that $y \in A$. So every metric space is a topological space, but not conversely. There are generalizations of the notions of limits and continuity to arbitrary topological spaces. For example, a function between topological spaces is **continuous** if the inverse image of each open set is open. The topological spaces encountered here will primarily be metric spaces, but familiarity with the general terminology is an asset.

Definition 6.1.4 *A topological space X is **Hausdorff** if for each pair of distinct points x, y there are disjoint open sets with one containing x and the other y. It is **compact** if whenever a collection of open sets has its union all of X, then some finite subcollection of the sets has their union all of X.*

There are many relationships between the analytic notions of metric and topological spaces and lattices. Of importance here will be the situation when a set has both a metric or topological structure as well as a lattice structure.

We mention that there are other deep connections between topology and lattice theory. For instance, lattice theory is used to study topology through the subject of point-free topology [32] (Exercise 6), and topology is used to study lattices through the subject of Stone duality [7].

Definition 6.1.5 *A* ***topological lattice*** *is a lattice L with a topology for which the lattice operations $\wedge, \vee : L \times L \to L$ are continuous.*

An example of a topological lattice is the reals $\mathbb{R}$ with the usual lattice structure and topology (Exercise 7). There are general methods for constructing topological lattices. We will describe two such.

Definition 6.1.6 *A* ***valuation*** *on a lattice L is a map $v : L \to \mathbb{R}$ that satisfies the following conditions:*

1. *$x \leq y$ implies that $v(x) \leq v(y)$.*
2. *$v(x) + v(y) = v(x \vee y) + v(x \wedge y)$.*
3. *$x < y$ implies that $v(x) < v(y)$.*

If v satisfies only the first two conditions, it is called a ***quasi-valuation****.*

Examples of lattices with a valuation include the lattice $\mathbb{R}$ where $v(x) = x$ for each $x \in \mathbb{R}$, and the lattice of subspaces of a finite-dimensional vector space V where $v(S) = \dim S$ is the dimension of the subspace S. Note that for subspaces S and T, $\dim S + \dim T = \dim(S \vee T) + \dim(S \wedge T)$ is a familiar formula from linear algebra (Exercise 8). The key result about lattices with valuation is the following [7].

Theorem 6.1.7 *Let L be a lattice with valuation v and define*

$$d(x, y) = v(x \vee y) - v(x \wedge y)$$

Then d is a metric on L, and with the metric space topology, $\wedge$ and $\vee$ are uniformly continuous.

There is an extension to this theorem that will be of use here. First, recall that an **equivalence relation** θ on a set X is a relation that is reflexive ($x\,\theta\,x$), symmetric ($x\,\theta\,y$ implies $y\,\theta\,x$), and transitive ($x\,\theta\,y$ and $y\,\theta\,z$ implies $x\,\theta\,z$). For an equivalence relation θ on X, the **equivalence class** x/θ of $x \in X$ is the set of all elements that are θ-related to x. The **quotient** of X by θ is the set $X/\theta = \{x/\theta : x \in X\}$ of all equivalence classes of elements of X.

Given a quasi-metric d on a set X, we can define an equivalence relation θ on X by setting $x\,\theta\,y$ if $d(x, y) = 0$ (Exercise 9). Further, on the quotient set

X/θ, we can define a function D by setting $D(x/\theta, y/\theta) = d(x, y)$. Then this function D is a metric on the set X/θ. When applied to quasi-valuations, this yields the following [7].

Theorem 6.1.8 *Let L be a lattice with a quasi-valuation v. Then there is a quasi-metric on L given by $d(x, y) = v(x \vee y) - v(x \wedge y)$, and its associated equivalence relation θ is a congruence on L. The map $D(x/\theta, y/\theta) = d(x, y)$ is then a metric on L/θ.*

We turn attention to matters related to completeness and infinite joins and meets.

Definition 6.1.9 *Let L be a complete lattice and S be a sublattice of L. Then S is a* ***complete sublattice*** *of L if for each $A \subseteq S$, the join and meet of A as taken in L belongs to S.*

If S is a complete sublattice of L, then S itself is a complete lattice and joins and meets in S are computed as in L. We note that a sublattice of L that is complete as a lattice in its own right may not be a complete sublattice (Exercise 10).

In Definition 1.4.4, we have seen the condition of meet continuity that gives the distributivity of finite meet over arbitrary join. Complete lattices that satisfy this condition have their meet operation residuated, and are Heyting algebras. There are an unlimited number of variants of the distributive law involving the use of infinite join and meet. The strongest of all is the following.

Definition 6.1.10 *A lattice L is* ***completely distributive*** *if for each set J and each indexed family $(x_{j,k})$ $k \in K_j$ of elements of L, we have*

$$\bigwedge_{j\in J} \bigvee_{k\in K_j} x_{j,k} = \bigvee_{\alpha\in\prod_J K_j} \bigwedge_{j\in J} x_{j,\alpha(j)}$$

To illustrate the meaning, consider

$$(x_{1,1} \vee x_{1,2} \vee x_{1,3}) \wedge (x_{2,1} \vee x_{2,2}) \wedge (x_{3,1} \vee x_{3,2} \vee x_{3,3} \vee x_{3,4})$$

This would be the left side of the equation above with $J = \{1, 2, 3\}$, and the sets $K_1 = \{1, 2, 3\}$, $K_2 = \{1, 2\}$ and $K_3 = \{1, 2, 3, 4\}$. Expanding this with the distributive law, it would eventually be expressed as a join of terms, such as $x_{1,3} \wedge x_{2,1} \wedge x_{3,1}$, which is a meet consisting of one element of each of the three groups of terms whose join we are taking. The expression $x_{1,3} \wedge x_{2,1} \wedge x_{3,1}$ is $x_{1,\alpha(1)} \wedge x_{2,\alpha(2)} \wedge x_{3,\alpha(3)}$ for some α that chooses one element from each K_j. There are 24 such choice functions α, and they are the elements of $\prod_J K_j$.

Examples of completely distributive lattices include all finite non-empty distributive lattices, complete chains, and the power set lattice of any set (Exercise 11). The following describes how new completely distributive lattices can be built from old. Its proof is left as an exercise (Exercise 12).

Proposition 6.1.11 *The product of arbitrarily many completely distributive lattices is completely distributive, and a complete sublattice of a completely distributive lattice is completely distributive.*

Complete distributivity is connected to the subject of continuous lattices. Continuous lattices in turn are connected to analytic notions such as variants of continuous functions, and also to the subject of topological lattices. Continuous lattices occur in many branches of mathematics and computer science, and there is a very large literature on the subject [32, 33]. We begin with the following.

Definition 6.1.12 *A subset D of a lattice L is* ***directed*** *if for any $x, y \in D$, there is $z \in D$ with $x, y \leq z$. The join of a directed set is called a* ***directed join****.*

The key notion for continuous lattices is that of an element being way below another.

Definition 6.1.13 *If a, b are elements of a complete lattice L, then a is* ***way below*** *b, written $a \ll b$, if for every directed set D with $b \leq \bigvee D$ there is $d \in D$ with $a \leq d$.*

The reader should verify that in the unit interval I, we have $a \ll b$ if and only if $a < b$ (Exercise 13).

Definition 6.1.14 *A complete lattice L is a* ***continuous lattice*** *if for each $b \in L$, $b = \bigvee\{a : a \ll b\}$.*

A weakening of complete distributivity characterizes continuous lattices among complete lattices [33, Theorem 2.3]. This weaker condition is a version of the one in Definition 6.1.10 where the joins are restricted to directed families. Thus, we have the following [33, Corollarry 2.5].

Proposition 6.1.15 *Every completely distributive lattice is continuous.*

There is a source of continuous lattices that has bearing on the matters we consider. To describe this, we need the following conditions that weaken the usual notion of continuity.

Definition 6.1.16 *Let f be a function $\mathbb{R} \to \mathbb{R}$.*

1. *f is* ***lower semicontinuous*** *if $\{x : f(x) \leq \alpha\}$ is closed for each α.*
2. *f is* ***upper semicontinuous*** *if $\{x : f(x) \geq \alpha\}$ is closed for each α.*

A function is continuous if and only if it is both lower semicontinuous and upper semicontinuous. In terminology common to fuzzy theory, an upper semicontinuous function is one whose **α-cuts** $\{x : f(x) \geq \alpha\}$ are closed. We abbreviate these as LSC and USC, respectively. See [94] for a complete account of these notions.

Proposition 6.1.17 *Let f_1, f_2 and f_j $(j \in J)$ be functions in* M*:*

1. *If f_1, f_2 are* LSC*, then $f_1 \wedge f_2$ is* LSC*.*
2. *If each f_j $(j \in J)$ is* LSC*, then $\bigvee_J f_j$ is* LSC*.*
3. *If f_1, f_2 are* USC*, then $f_1 \vee f_2$ is* USC*.*
4. *If each f_j $(j \in J)$ is* USC*, then $\bigwedge_J f_j$ is* USC*.*

Here, all joins and meets are taken pointwise.

The proof is a simple consequence of the fact that the union of two closed sets is closed and the intersection of arbitrarily many closed sets is closed. This proof is left to the reader (Exercises 15 and 16). This proposition has as a consequence that both the set of LSC functions in M and the set of USC functions in M are complete lattices. In fact, the set of LSC functions in M is a continuous lattice. This is part of the following more general result [33, Proposition 1.21.2].

Proposition 6.1.18 *If X is a compact space, then the set* LSC(X) *of lower semicontinous functions on X is a continuous lattice.*

There are connections between continuous lattices and topological lattices that are important here. Several definitions are required. For a poset P, a subset $A \subseteq P$ is an **upset** if $x \in A$ and $x \leq y$ implies that $y \in A$, and is a **downset** if $x \in A$ and $x \geq y$ implies that $y \in A$. For each $a \in P$, we have the **principal upset** $a\!\uparrow = \{x : a \leq x\}$ and the **principal downset** $a\!\downarrow = \{x : x \leq a\}$.

Definition 6.1.19 *An upset A of a complete lattice L is* ***Scott open*** *if for each directed set D, if $\bigvee D \in A$, then some $d \in D$ belongs to A.*

The intersection of two Scott open sets is Scott open, and an arbitrary union of Scott open sets is Scott open (Exercise 17). So the Scott open sets form a topology on L known as the **Scott topology**. The smallest topology on L that has as open sets the complements of the principal upsets $a\!\uparrow$ for all $a \in L$ and all Scott open subsets of L is the **Lawson topology** of L. The following is given in [32, Theorem 1.10, p. 146].

Theorem 6.1.20 *For any continuous lattice L, the Lawson topology on L is a compact Hausdorff topology.*

A map $f : L \to M$ between complete lattices **preserves arbitrary meets** if $f(\bigwedge A) = \bigwedge f(A)$ for each $A \subseteq L$. The definitions of preserving finite meets, and of preserving arbitrary and finite joins are similar. So also is the definition of **preserving directed joins**, but for clarity, this means that $f(\bigvee D) = \bigvee f(D)$ for each directed subset D of L. The following result is given in [32, Theorem 1.8, p. 145].

Theorem 6.1.21 *Suppose that $f : L \to M$ is a map between complete lattices that preserves finite meets. Then f is continuous with respect to the Lawson topologies of L and M if and only if it preserves arbitrary meets and directed joins.*

There is a further result that will be of importance here. The **interval topology** on a complete lattice L is the smallest topology on L that has the complements of all principal upsets $a\uparrow$ and the complements of all principal downsets $a\downarrow$ open. The interval topology on the reals is the usual one, but in general, the interval topology on a lattice is not well behaved. However, for completely distributive lattices, the situation is different.

Theorem 6.1.22 *For a completely distributive lattice L, the interval topology is equal to the Lawson topology, and this is the unique compact Hausdorff topology on L that makes L a topological lattice.*

Proof. Suppose that L is a completely distributive lattice. Papert-Strauss showed that the interval topology on L is a compact Hausdorff topology, and that with this topology, L is a topological lattice [90]. It then follows from [32, Ex. 1.22, p. 151] that the interval and Lawson topologies on L agree. It remains to show uniqueness.

Suppose that τ is a compact Hausdorff topology on L that makes L a topological lattice. Then for each $a \in L$, the functions $f(x) = x \wedge a$ and $g(x) = x \vee a$ are continuous. Since τ is Hausdorff, $\{a\}$ is a closed subset of L. So the inverse images $f^{-1}(a)$ and $g^{-1}(a)$ are closed sets in τ. But $f^{-1}(a) = a\uparrow$ and $g^{-1}(a) = a\downarrow$. Since $a\uparrow$ and $a\downarrow$ are closed in τ, their complements are open. So τ contains the interval topology. A basic result of topological spaces says that if a compact Hausdorff topology on a set contains the open sets of another compact Hausdorff topology on the same set, then the topologies must be equal [66]. So τ is equal to the interval topology. ■

In this chapter, we produce a completely distributive lattice from the convex normal functions L. This completely distributive lattice will be constructed in several isomorphic ways. First, it will be constructed as a subalgebra of L consisting of USC functions. Then it will be constructed as the metric space quotient of a lattice of functions with a quasi-valuation given by integration. It will be shown that the metric space topology on this quotient lattice is a compact Hausdorff topology that makes it a topological lattice. Thus, this metric space topology is equal to its interval topology, as well as to its Lawson topology. The agreement of these topologies is of use in studying t-norms on this lattice since continuity with respect to the metric is related to preservation of joins and hence to residuation.

Our first step requires much less theory. It is a simple matter of considering the partial ordering of L in a new, and much simpler way.

6.2 Straightening the order

In earlier chapters, the description of the orders $\sqsubseteq_{\sqcap}$ and $\sqsubseteq_{\sqcup}$ were simplified to become tractable. It was also shown that on the subalgebra $\mathbf{L}$ of convex normal functions, these orders agree, and give a lattice ordering on $\mathbf{L}$ that has $\sqcap$ and $\sqcup$ as its meet and join. When restricted to $\mathbf{L}$, the description of these equal orders simplifies much further, and this greatly simplifies the study of $\mathbf{L}$. That is the focus of this section.

Definition 6.2.1 *Let $\sqsubseteq$ be the restriction of the orders $\sqsubseteq_{\sqcap}$ and $\sqsubseteq_{\sqcup}$ to* $\mathbf{L}$.

Proposition 2.4.3 gives conditions for $f \sqsubseteq_{\sqcap} g$ and $f \sqsubseteq_{\sqcup} g$. Specializing these to $\mathbf{L}$, where $\sqsubseteq_{\sqcap}$ and $\sqsubseteq_{\sqcup}$ are equal to $\sqsubseteq$, easily yields the following.

Proposition 6.2.2 *For $f, g \in \mathbf{L}$, these are equivalent.*

1. $f \sqsubseteq g$.
2. $g^L \leq f^L$ *and* $f^R \leq g^R$.

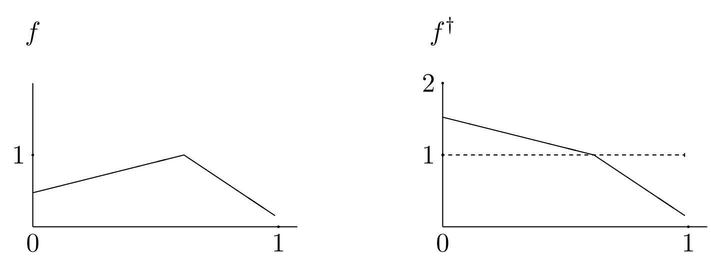
Definition 6.2.3 *The symbol $\mathbf{I}^{\dagger}$ denotes the closed interval* $[0,2]$.

It will be helpful to view $\mathbf{I}^{\dagger}$ as being the unit interval $\mathbf{I}$ with a copy of the dual of $\mathbf{I}$ placed on top of it, where the top element of $\mathbf{I}$ and the bottom element of the dual copy of $\mathbf{I}$ are identified. We now make precise the idea of "straightening out" a convex function f.

Definition 6.2.4 *For $f : \mathbf{I} \to \mathbf{I}$, define $f^{\dagger} : \mathbf{I} \to \mathbf{I}^{\dagger}$ by setting*

$$f^{\dagger}(x) = \begin{cases} 2 - f(x) & \text{if } f(x) = f^L(x) \\ f(x) & \text{otherwise} \end{cases}$$

While defined for any function, we only consider $f^{\dagger}$ in the case that f is convex and normal. Roughly, $f^{\dagger}$ is produced by taking the mirror image of the increasing portion of f about the line $y = 1$, and leaving the remainder of f alone. The following diagram illustrates the situation.



While we consider the convolution ordering $\sqsubseteq$ on L, we shall consider the ordinary pointwise ordering of functions $\leq$ for functions from I to $\mathrm{I}^{\dagger}$, thus **straightening the order**. Our key result follows below. In its proof, and elsewhere, we use repeatedly two consequences of convexity and normality—for each x in I, at least one of $f^L(x)$ and $f^R(x)$ equals $f(x)$, and at least one of $f^L(x)$ and $f^R(x)$ equals 1.

Theorem 6.2.5 *For $f, g \in \mathrm{L}$, $f \sqsubseteq g$ if and only if $f^{\dagger} \leq g^{\dagger}$.*

Proof. Assume $f \sqsubseteq g$. Proposition 6.2.2 then gives $g^L \leq f^L$ and $f^R \leq g^R$. We will show $f^{\dagger}(x) \leq g^{\dagger}(x)$. Consider the possibilities for x. First, suppose $g(x) < g^L(x)$. Then $g^{\dagger}(x) = g(x) = g^R(x) \geq f^R(x) \geq f(x)$ and the strict inequality $f(x) \leq g(x) < g^L(x) \leq f^L(x)$ implies $f^{\dagger}(x) = f(x)$. It follows that $f^{\dagger}(x) \leq g^{\dagger}(x)$. Now suppose that $g(x) = g^L(x)$. If $f(x) = f^L(x)$ is also true, then $g(x) \leq f(x)$, so $f^{\dagger}(x) = 2 - f(x) \leq 2 - g(x) = g^{\dagger}(x)$. If $f(x) < f^L(x)$, then $f^{\dagger}(x) = f(x) \leq 1 \leq 2 - g(x) = g^{\dagger}(x)$.

For the converse, assume $f^{\dagger} \leq g^{\dagger}$. To show $f \sqsubseteq g$, by Proposition 6.2.2 it is enough to show $g^L \leq f^L$ and $f^R \leq g^R$. Again, consider possibilities for x. First suppose $g(x) < g^L(x)$. Then $g^{\dagger}(x) = g(x) = g^R(x) < 1$ and $g^L(x) = 1$. Also, $f^{\dagger}(x) \leq g^{\dagger}(x) < 1$ implies $f^{\dagger}(x) = f(x) = f^R(x) < f^L(x)$. Thus $f^R(x) \leq g^R(x)$ and $f^L(x) = g^L(x) = 1$. Now suppose that $g(x) = g^L(x)$, so $g^{\dagger}(x) = 2 - g(x)$. If we also have $f(x) = f^L(x)$, then $g^{\dagger}(x) = 2 - g(x) \geq f^{\dagger}(x) = 2 - f(x)$ implies $g^L(x) = g(x) \leq f(x) = f^L(x)$. Also, in this case, $f^R(x) = g^R(x) = 1$. Finally, suppose that $g(x) = g^L(x) \leq g^R(x) = 1$ and $f(x) < f^L(x) = 1$. Then $g^L(x) \leq f^L(x) = 1$ and $f^R(x) \leq g^R(x) = 1$. ■

We next describe the functions that arise as $f^{\dagger}$ for some convex normal f. We recall that a point x is **in the closure** of a subset A of the reals if for each $\epsilon > 0$ the interval $(x - \epsilon, x + \epsilon)$ contains a point of A.

Proposition 6.2.6 *For $g : \mathrm{I} \to \mathrm{I}^{\dagger}$ these are equivalent.*

1. *$g = f^{\dagger}$ for some $f \in \mathrm{L}$.*
2. *g is decreasing and 1 is in the closure of the image of g.*

Proof. To see that the first condition implies the second, suppose that $f \in \mathrm{L}$. One sees that $\mathrm{J} = \{x : f(x) = f^L(x)\}$ is an initial segment of I, that f is increasing on J, and that f is decreasing on $\mathrm{I} - \mathrm{J} = \{x : x \in \mathrm{I} \text{ and } x \notin \mathrm{J}\}$. As $f^{\dagger} = 2 - f$ on J, $f^{\dagger}$ is decreasing on J, and as $f^{\dagger} = f$ on $\mathrm{I} - \mathrm{J}$, $f^{\dagger}$ is decreasing on $\mathrm{I} - \mathrm{J}$. Then as $f^{\dagger} \geq 1$ on J and $f^{\dagger} \leq 1$ on $\mathrm{I} - \mathrm{J}$, it follows that $f^{\dagger}$ is decreasing on I. As 1 is the supremum of f, for $\epsilon > 0$ there is x with $f(x)$ within distance ϵ of 1. Then both $f(x)$ and $2 - f(x)$ lie within ϵ of 1, hence $f^{\dagger}(x)$ lies within ϵ of 1. So 1 is in the closure of the image of $f^{\dagger}$.

For the converse, assume that g is decreasing and that 1 is in the closure of the image of g. Set $f(x) = \min\{g(x), 2 - g(x)\}$. As f is the pointwise meet of a decreasing and increasing function, it is convex. It cannot be that both

$g(x)$ and $2 - g(x)$ are greater than 1, so $0 \le f(x) \le 1$. Since 1 is in the closure of the image of g, for $\epsilon > 0$, there is an x with $g(x)$ within ϵ of 1. Then $g(x)$ and $2 - g(x)$ are within ϵ of 1, hence so is $f(x)$. It follows that f is normal, hence $f \in \mathrm{L}$. Let $\mathrm{J} = \{x : 2 - g(x) \le g(x)\}$. Then J is an initial segment of I, $f(x) = 2 - g(x)$ on J, and as $2 - g(x)$ is increasing, we have $f = f^L$ on J. Thus $f^\dagger(x) = 2 - (2 - g(x)) = g(x)$ on J. For $x \in \mathrm{I} - \mathrm{J}$ we have $f(x) = g(x) < 2 - g(x)$. In particular, $f(x) < 1$. As f is decreasing on $\mathrm{I} - \mathrm{J}$ and the supremum of f is 1, there is some $y < x$ with $f(y) > f(x)$. Then $f(x) \ne f^L(x)$, so $f^\dagger(x) = f(x) = g(x)$. ■

The results of this section allow us to work completely with the familiar pointwise order. We will put this to good use in the following sections.

6.3 Realizing L as an algebra of decreasing functions

In this section we construct an isomorphic copy of L as an algebra of decreasing functions from I to $\mathrm{I}^\dagger$. Since this isomorphic copy is ordered by the pointwise order, it is simpler to work with.

Definition 6.3.1 *Let* D *be the set of all decreasing functions from* I *to* $\mathrm{I}^\dagger$ *that have* 1 *in the closure of their image.*

The collection of all functions from I to $\mathrm{I}^\dagger$ is a lattice under pointwise join and meet. It is the product of I copies of the lattice $\mathrm{I}^\dagger$. It is easy to see that the collection of all decreasing functions from I to $\mathrm{I}^\dagger$ is a sublattice of this lattice. We show that D is also a sublattice under these pointwise operations.

Proposition 6.3.2 *The set* D *is a sublattice of the lattice of all decreasing functions from* I *to* $\mathrm{I}^\dagger$.

Proof. Suppose $f, g \in \mathrm{D}$. We will show $f \wedge g \in \mathrm{D}$ where $\wedge$ is the pointwise meet. This only requires us to show 1 is in the closure of the image of $f \wedge g$. Given $\epsilon > 0$ there are x, y with $f(x)$ and $g(y)$ within ϵ of 1. Suppose $x \le y$. Then $1 - \epsilon < f(x) < 1 + \epsilon$ and since g is decreasing, $1 - \epsilon < g(y) \le g(x)$. It follows that $1 - \epsilon < (f \wedge g)(x) < 1 + \epsilon$. Similarly, $f \vee g$ also has 1 in the closure of its image. So D is a sublattice. ■

Theorem 6.2.5 and Proposition 6.2.6 give the following.

Theorem 6.3.3 *The lattice* L *is isomorphic to* D.

The set L_1 of **convex strictly normal functions** is the members of L that take value 1. If $f(x) = 1$ and $g(y) = 1$, then since $f \sqcup g = (f \vee g) \wedge f^L \wedge g^L$, it follows that $f \sqcup g$ has value 1 at $\max\{x, y\}$. Similarly, $f \sqcap g$ attains value 1 at $\min\{x, y\}$. Thus L_1 is a subalgebra of L. The following is immediate.

Proposition 6.3.4 *The lattice* L_1 *is isomorphic to the sublattice* D_1 *of* D *consisting of those decreasing functions from* I *to* $\mathrm{I}^\dagger$ *that take the value* 1.

There will be a further sublattice of L considered in a later section. First, we use the description of L in terms of decreasing functions to address its completeness.

6.4 Completeness of L

In the previous section, Theorem 6.3.3 shows that L is isomorphic to the sublattice D of the complete lattice $(\mathrm{I}^\dagger)^\mathrm{I}$. We will show that L is complete by showing that D is complete. However, we will see that D is not a complete sublattice of $(\mathrm{I}^\dagger)^\mathrm{I}$ since infinite joins and meets in D are not always pointwise. We will see that L and D do not satisfy the infinite distributive law of meet continuity.

Definition 6.4.1 *For* $a \in \mathrm{I}$, 1_a *and* $1'_a$ *are the functions from* I *to* $\mathrm{I}^\dagger$ *where* 1_a *takes value* 1 *at* a *and* 0 *otherwise, while* $1'_a$ *takes value* 1 *at* a *and value* 2 *otherwise.*

Theorem 6.4.2 *Let* $(f_j)_J$ *be a family in* D, *and set* $a = \sup\{x : \bigvee_J f_j(x) \geq 1\}$ *and* $b = \sup\{x : \bigwedge_J f_j(x) \geq 1\}$. *Then the following hold:*

1. *If* $\bigvee_J f_j$ *belongs to* D, *this is the join of this family in* D. *Otherwise, the join of this family in* D *is* $\bigvee_J f_j \vee 1_a$.
2. *If* $\bigwedge_J f_j$ *belongs to* D, *this is the meet of this family in* D. *Otherwise, the meet of this family in* D *is* $\bigwedge_J f_j \wedge 1'_b$.

Proof. Let $f = \bigvee_J f_j$. If $f \in \mathrm{D}$, then it is surely the least upper bound of this family in D. If f does not belong to D, then as f is decreasing, this can only be because 1 is not in the closure of the image of f. So there is some $\epsilon > 0$ bounding the image of f away from 1. By the definition of a, we have $f(x) \geq 1$ for all $x < a$, and $f(x) < 1$ for all $x > a$. Thus $f(x) \geq 1 + \epsilon$ for all $x < a$ and $f(x) \leq 1 - \epsilon$ for all $x > a$.

We claim $f(a) \leq 1 - \epsilon$. Otherwise, $f(a) \geq 1 + \epsilon$. So there would be f_j with $f_j(a) \geq 1 + \epsilon/2$. But as f_j is decreasing, this would yield $f_j(x) \geq 1 + \epsilon/2$ for all $x \leq a$, and since $f_j \leq f$ we would have $f_j(x) \leq 1 - \epsilon$ for all $x > a$. This would contradict 1 being in the closure of the image of f_j. Thus $f(a) \leq 1 - \epsilon$.

It follows that $f \vee 1_a$ is decreasing and takes value 1 at a, hence belongs to D, and is clearly an upper bound of the family $(f_j)_J$. Suppose $g \in \mathrm{D}$ is another upper bound of this family. Then $g \geq f$, so $g(x) \geq 1 + \epsilon$ for all $x < a$. Since 1 is in the closure of the image of g, it must be in the closure of the image of the restriction of g to the interval $[a, 1]$. Then since g is decreasing, we must have $g(a) \geq 1$. Thus $g \geq f \vee 1_a$. This shows that $f \vee 1_a$ is the least upper bound of this family in D. The proof of the second statement is similar. ∎

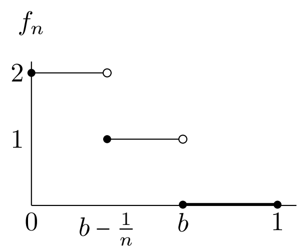
A complete distributive lattice is meet-continuous if $x \wedge \bigvee y_j = \bigvee (x \wedge y_j)$. It is well known that a complete distributive lattice is a Heyting algebra if and only if it is meet-continuous [3]. We show that the complete distributive lattice L of convex normal functions is not meet-continuous. It follows that it is not a Heyting algebra. Again, it is more convenient to work through the isomorphic copy D.

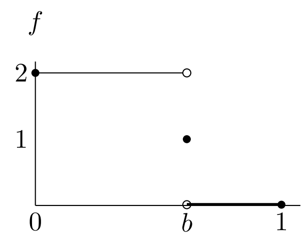
Proposition 6.4.3 *Neither* D *nor* L *satisfies* $x \wedge \bigvee y_j = \bigvee (x \wedge y_j)$.

Proof. Suppose $0 < b < 1$ and define a family of functions f_n by setting

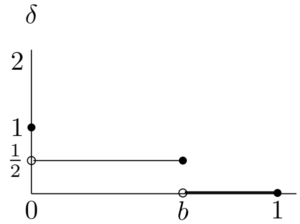
$$f_n(x) = \begin{cases} 2 & \text{if } x < b - \frac{1}{n} \\ 1 & \text{if } b - \frac{1}{n} \leq x < b \\ 0 & \text{if } b \leq x \end{cases}$$

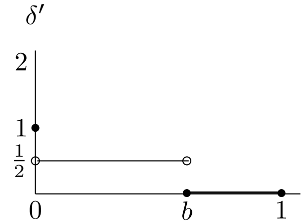
Here we only define f_n for n large enough to ensure $b - \frac{1}{n} > 0$. Note that these functions are decreasing and take the value 1, hence belong to D. Since the pointwise join $\bigvee f_n$ takes only the values 0 and 2, it does not belong to D. Therefore $f = \bigvee f_n \vee 1_b$ is the join of the family f_n in D. These functions are shown below.





Consider the functions δ and δ' shown below. Both are decreasing and take the value 1, hence belong to D.





It is clear that $\delta \leq f$, hence $\delta \wedge f = \delta$, while $\delta \wedge f_n = \delta'$ for each n. Writing $\bigcup f_n$ for the join of the family f_n in D, we have $\delta \wedge \bigcup f_n = \delta > \delta' = \bigcup \delta \wedge f_n$. So neither D, nor the isomorphic lattice L, is meet continuous. ∎

The situation is the same for the lattice L_1 of convex strictly normal functions. There are formulas for infinite joins and meets in the isomorphic lattice D_1 as in Theorem 6.4.2, and with these, the failure of meet continuity in D_1 and L_1 follows exactly as in Proposition 6.4.3. The details are not difficult, and are a recommended exercise. Complete proofs are found in [45].

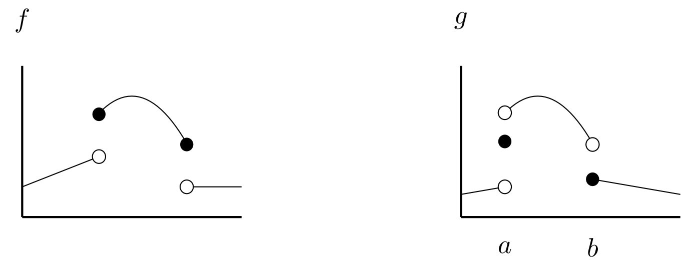
The failure of meet continuity in L is more than just a curiosity. It precludes the use of L in the role we have intended in Chapter 9 in all but a finite setting. Fortunately, there is a natural algebra closely related to L that satisfies not only the infinite distributive law of meet continuity, but is completely distributive. This algebra will be the focus of the remainder of this chapter, and will be a central ingredient of Chapter 9.

6.5 Convex normal upper semicontinuous functions

There is another subalgebra of L that plays an important role, that of the convex normal upper semicontinuous functions. This subalgebra was first studied by Kawaguchi and Miyakoshi [64], where it was shown to be a complete meet continuous lattice. See also [65, 81, 82]. Later, in [45], this lattice was shown to be a continuous lattice in the sense of [32, 33]. This section provides the basics of this algebra. In the following sections, we construct an algebra that is isomorphic to it, and that we feel is more naturally motivated.

Definition 6.5.1 *The symbol* L_u *denotes the set of all convex normal functions that are upper semicontinuous.*

When restricted to convex functions, upper semicontinuity can be described in terms that are easy to work with in practice. Recall that a function f has a **jump discontinuity** at b if the limit as x approaches b from the left exists, and the limit as x approaches b from the right exists, but the two limits are different. The only kind of discontinuity of a function that is increasing, or decreasing, is a jump discontinuity, and the same is true of a convex function. So a convex function is USC if at each point of discontinuity, the function takes the value of the larger one-sided limit.



In the diagrams above, the function f at left is USC, while the function g at right fails to be USC because of its behavior at both a and b.

Proposition 6.5.2 *For a function $g \in \mathrm{D}$, these conditions are equivalent:*

1. $g = f^\dagger$ *for some* $f \in \mathrm{L}_u$.
2. $g^{-1}[\alpha, 2-\alpha]$ *is closed for each* $0 \leq \alpha \leq 1$.
3. $g \vee \overline{1}$ *is* LSC *and* $g \wedge \overline{1}$ *is* USC.

Further, these conditions imply that g attains the value 1.

Proof. To prove item (1) implies item (2), let $g = f^\dagger$ where f is USC. So $\{x : \alpha \leq f(x)\}$ is closed. We show this set equals $g^{-1}[\alpha, 2-\alpha]$. If $f(x) = f^L(x)$, then as $1 \leq 2 - f(x) = f^\dagger(x)$ we have $\alpha \leq f(x)$ if and only if $\alpha \leq f^\dagger(x) \leq 2 - \alpha$. If $f(x) \neq f^L(x)$, then as $f(x) = f^\dagger(x)$ and $1 \leq 2 - \alpha$ we have $\alpha \leq f(x)$ if and only if $\alpha \leq f^\dagger(x) \leq 2 - \alpha$.

To prove item (2) implies item (3), assume that $g^{-1}[\alpha, 2-\alpha]$ is closed for each $0 \leq \alpha \leq 1$. To show $g \vee \overline{1}$ is LSC it is enough to show $(g \vee \overline{1})^{-1}[0, 2-\alpha]$ is closed for each $0 \leq \alpha \leq 1$. As $g \in \mathrm{D}$, we have 1 is in the closure of the image of g, so $X = g^{-1}[\alpha, 2-\alpha]$ is non-empty. Say $x \in X$. Then because $g \vee \overline{1}$ is decreasing, we have $(g \vee \overline{1})^{-1}[0, 2-\alpha] = g^{-1}[\alpha, 2-\alpha] \cup [x, 1]$, hence is closed. The argument showing that $g \wedge \overline{1}$ is USC is similar.

To prove item (3) implies item (1), assume that $g \vee \overline{1}$ is LSC and that $g \wedge \overline{1}$ is USC. Since $g \in \mathrm{D}$, it follows from Proposition 6.2.6 that $g = f^\dagger$ for some $f \in \mathrm{L}$, and this f must be given by $f(x) = \min\{g(x), 2 - g(x)\}$. For $0 \leq \alpha \leq 1$ we have that $\alpha \leq f(x)$ if and only if $\alpha \leq g(x)$ and $\alpha \leq 2 - g(x)$, which is equivalent to $\alpha \leq g(x) \leq 2 - \alpha$. So $\{x : \alpha \leq f(x)\}$ is equal to the intersection of $\{x : \alpha \leq (g \wedge \overline{1})(x)\}$ and $\{x : (g \vee \overline{1})(x) \leq 2 - \alpha\}$. Our assumptions give that both of these sets are closed, hence so is their intersection. So f is USC so belongs to L_u.

To see that g attains the value 1, note that for $0 \leq \alpha < 1$, the collection of sets $g^{-1}[\alpha, 2-\alpha]$ forms a decreasing family of closed sets. Since 1 is in the closure of the image of g, we have that each of these sets is non-empty. Because I is compact, the intersection of this family of sets is non-empty, providing some x with $g(x) = 1$. ∎

Definition 6.5.3 *A function $g \in \mathrm{D}$ that satisfies the equivalent conditions of Proposition 6.5.2 is a* ***band semicontinuous*** *function. The symbol* D_u *denotes the collection of all such* g.

Proposition 6.5.4 $(\mathrm{L}_u, \sqsubseteq)$ *is isomorphic to* $(\mathrm{D}_u, \leq)$.

Proof. D_u is the image of L_u under the order embedding $f \rightsquigarrow f^{\dagger}$. ■

As the union and intersection of two closed sets is closed, it follows that the pointwise join and meet of two LSC functions is LSC and the pointwise join and meet of two USC functions is USC. We make use of this in the following.

Proposition 6.5.5 D_u *is a sublattice of* D.

Proof. Suppose $f, g \in \mathrm{D}_u$. Note that $(f \vee g) \vee \overline{1} = (f \vee \overline{1}) \vee (g \vee \overline{1})$ and $(f \vee g) \wedge \overline{1} = (f \wedge \overline{1}) \vee (g \wedge \overline{1})$. By part 3 of Proposition 6.5.2, it follows that $(f \vee g) \vee \overline{1}$ is the join of two LSC functions, hence is LSC, and $(f \vee g) \wedge \overline{1}$ is the join of two USC functions, hence is USC. By Proposition 6.5.2, it follows that $f \vee g$ belongs to D_u. The argument showing $f \wedge g$ belongs to D_u is similar. ■

The following is then immediate.

Corollary 6.5.6 L_u *is a sublattice of* L.

Further properties of the lattice L_u can be developed directly as in [64], or via decreasing functions and the pointwise order as in [45]. We shall instead develop the properties of L_u by realizing it isomorphically in yet another form. This is the focus of the next section.

6.6 Agreement convexly almost everywhere

There are many circumstances where changing the value of a function at a single point, or a small number of points, is of no consequence to the result. A classic example is taking the integral of a function. However, a direct application of this idea to members of L is hopeless—the bounds 1_0 and 1_1 of the lattice L agree at all but two points. The notion that is useful is agreement of the "straightened" versions of these function almost everywhere, which is termed *convexly almost everywhere*. In this section, we develop this notion and show that it leads to a lattice that is isomorphic to the lattice L_u.

Definition 6.6.1 *Let* X *be the set of decreasing functions from* $[0,1]$ *to* $[0,2]$.

Since the pointwise meet and join of decreasing functions are decreasing, X is a sublattice of the completely distributive lattice $[0,2]^{[0,1]}$ that is closed under arbitrary meets and joins. Thus X is a complete sublattice of $[0,2]^{[0,1]}$ in the sense of Definition 6.1.9. Then Proposition 6.1.11 gives the following.

Proposition 6.6.2 X *is a complete, completely distributive lattice.*

There are several notions of "size" of a subset of I commonly encountered. First, a set is **countable** if it is either finite, or in bijective correspondence with the natural numbers. A set is **dense** in I if its closure is all of I, meaning that for each $x \in \mathrm{I}$ and each $\epsilon > 0$, the interval $(x - \epsilon, x + \epsilon)$ intersects the set. Finally, there is a notion of the Lebesgue **measure** of a set [94] that associates to a set a number that indicates its size. For an interval, this number is its length, but for more complicated sets, matters are more intricate. Fortunately in our setting, the measures that arise can be described in much simpler terms. This is the content of the following lemma. Here, functions f and g are said to agree **almost everywhere** if $\{x : f(x) \neq g(x)\}$ has measure zero. This is written f and g agree a.e.

Lemma 6.6.3 *For $f, g \in \mathrm{X}$, these conditions are equivalent:*

1. *f and g agree a.e.*
2. *f and g agree on a dense set.*
3. *f and g agree except at countably many points.*

Proof. $3 \Rightarrow 1 \Rightarrow 2$ is trivial. For $2 \Rightarrow 3$, define $h(x) = |f(x) - g(x)|$ and let C be the set of all points where both f and g are continuous. Then h is continuous at each point of C. Since f, g are decreasing, their discontinuities are all jump discontinuities, and they can have only countably many of them. So $[0,1] \smallsetminus C$ is countable. We claim $h = 0$ on C, which will establish the result. If not, there is $x \in C$ with $h(x) = \epsilon > 0$. By continuity, there is an interval around x with $h > \epsilon/2$ on this interval. But then f and g agree at no points of the interval, contrary to their agreeing on a dense set. ∎

Definition 6.6.4 *Let Θ be the relation on* X *defined by $f \Theta g$ if $f = g$ a.e.*

It is well known, and easily seen, that Θ is a congruence on the lattice X. In our context, this follows from the basic fact that the union of finitely many countable sets is countable. The proof is left as an exercise (Exercise 18). We also show that Θ is compatible with arbitrary meets and joins in X.

Lemma 6.6.5 *If f_j $(j \in J)$ is a family of elements of* X*, the following hold:*

1. *$(\bigvee f_j)/\Theta$ is the least upper bound of the family f_j/Θ $(j \in J)$.*
2. *$(\bigwedge f_j)/\Theta$ is the greatest lower bound of the family f_j/Θ $(j \in J)$.*

Thus X/Θ is complete, $(\bigvee f_j)/\Theta = \bigvee(f_j/\Theta)$, and $(\bigwedge f_j)/\Theta = \bigwedge(f_j/\Theta)$.

Proof. Let $f = \bigvee f_j$. For each $j \in J$, we have that $f_j \leq f$ everywhere, so $f_j/\Theta \leq f/\Theta$. Thus f/Θ is an upper bound of this family. Since we are in a lattice, to show this element is the least upper bound, it is enough to show that if $g \in \mathrm{X}$ is such that $g/\Theta < f/\Theta$, then g/Θ is not an upper bound of this family.

For such g, we may assume $g < f$ by considering $g' = f \wedge g$ if necessary. Then as $g/\Theta < f/\Theta$, we cannot have g and f agree on a dense set, so there is an open interval $(x - \epsilon, x + \epsilon)$ on which $g < f$. There must be points in this interval where g is continuous, and it does no harm to assume that g is continuous at the point x.

As $g < f$ and $f = \bigvee f_j$, there are $j \in J$ and $\lambda > 0$ with $f_j(x) = g(x) + \lambda$. By continuity, there is $0 < \epsilon' < \epsilon$ with $g(y) < g(x) + \lambda/2$ for all $y \in (x - \epsilon', x + \epsilon')$. As f_j is decreasing, for all $y \in (x - \epsilon', x)$ we have $g(y) < g(x) + \lambda/2 < f_j(x) \leq f_j(y)$. So it is not the case that $f_j/\Theta \leq g/\Theta$, showing that g/Θ is not an upper bound of this family. ■

For the following, see Definitions 6.1.10 and 6.1.14.

Corollary 6.6.6 *The lattice* X/Θ *is complete and completely distributive, hence is also a continuous lattice.*

Proof. Proposition 6.6.2 gives that X is completely distributive, so it follows from Lemma 6.6.5 that X/Θ is complete and completely distributive. That both are continuous lattices is given by Proposition 6.1.15. ■

Definition 6.6.7 *For* $f \in \mathrm{X}$ *and* $a \in \mathrm{I}$*, denote one-sided limits as follows:*

1. $f(a^-) = \lim\limits_{x \to a^-} f(x)$.
2. $f(a^+) = \lim\limits_{x \to a^+} f(x)$.

We next provide a result that describes when a function $f \in \mathrm{X}$ belongs to D_u. Informally, it says that at each jump discontinuity, the value f attains must be the one as close as possible to the line $y = 1$. The proof is similar to that of the well-known fact that a decreasing function is upper semicontinuous if and only if it is continuous from the left, and we omit it.

Lemma 6.6.8 *For* $f \in \mathrm{X}$ *we have* $f \in D_u$ *if and only if the following hold:*

1. *If* $f(0^+) \geq 1$*, then* $f(0) = f(0^+)$*, otherwise* $f(0) = 1$.
2. *If* $f(1^-) \leq 1$*, then* $f(1) = f(1^-)$*, otherwise* $f(1) = 1$.
3. *If* $f(a^-) \leq 1$*, then* $f(a) = f(a^-)$.
4. *If* $f(a^+) \geq 1$*, then* $f(a) = f(a^+)$.
5. *If* $f(a^-) > 1$ *and* $f(a^+) < 1$*, then* $f(a) = 1$.

Proposition 6.6.9 *For each $f \in \mathrm{X}$, there is a unique $f^{\ddagger} \in \mathrm{D}_u$ that agrees with f a.e. Further, $\ddagger : \mathrm{X} \to \mathrm{D}_u$ is an idempotent lattice endomorphism; that is, a* ***retraction****.*

Proof. To produce such $f^{\ddagger}$, one modifies the values of f at $0, 1$ and any of the countably many jump discontinuities to comply with the conditions of the above lemma. Namely, if $f(0^+) \geq 1$, we set $f^{\ddagger}(0) = f(0^+)$ and otherwise set $f^{\ddagger}(0) = 1$, and so forth. The resulting $f^{\ddagger}$ is seen to be decreasing. So $f^{\ddagger}(0^+)$, $f^{\ddagger}(1^-)$, $f^{\ddagger}(a^-)$, and $f^{\ddagger}(a^+)$ exist for all $0 < a < 1$. As $f^{\ddagger}$ agrees with f at all but countably many points, these values agree with $f(0^+)$, $f(1^-)$, $f(a^-)$, and $f(a^+)$ for all $0 < a < 1$. It follows that $f^{\ddagger}$ satisfies the conditions of the above lemma, hence belongs to D_u, and by construction $f^{\ddagger}$ agrees with f a.e.

For uniqueness, suppose g is a function in D_u that agrees with f a.e. Then by Lemma 6.6.3, we have that f and g agree on a dense set, and this implies that $f(0^+) = g(0^+)$, $f(1^-) = g(1^-)$, $f(a^-) = g(a^-)$, and $f(a^+) = g(a^+)$ for each $0 < a < 1$. As $g \in \mathrm{D}_u$, by the above lemma its values are determined by the values of the $g(0^+)$, $g(1^-)$, $g(a^-)$, and $g(a^+)$, hence g is determined by f.

For the further comments, idempotence is obvious because $f^{\ddagger}$ is a member of D_u that agrees with itself a.e., hence $f^{\ddagger\ddagger} = f^{\ddagger}$. To see that $\ddagger$ preserves finite meets, note first that finite meets in D_u are given componentwise. So for $f, g \in \mathrm{X}$, we have $f^{\ddagger} \wedge g^{\ddagger}$ belongs to D_u, and since $f^{\ddagger}$ agrees with f a.e. and $g^{\ddagger}$ agrees with g a.e., we have $f^{\ddagger} \wedge g^{\ddagger}$ agrees with $f \wedge g$ a.e. Thus $(f \wedge g)^{\ddagger} = f^{\ddagger} \wedge g^{\ddagger}$. That $\ddagger$ preserves finite joins follows from symmetry. ■

Theorem 6.6.10 *The lattice D_u, hence also L_u, is isomorphic to X/Θ.*

Proof. This follows immediately from Proposition 6.6.9 since $\ddagger : \mathrm{X} \to \mathrm{D}_u$ is a surjective homomorphism whose kernel is Θ. ■

The isomorphism between L_u and X/Θ can be made explicit. Recall the "straightening function" $\dagger : \mathrm{L} \to \mathrm{D}$ that provides an isomorphism between L and D (see Definition 6.2.4).

Definition 6.6.11 *Functions $f, g \in \mathrm{L}$ agree* ***convexly almost everywhere*** *(c.a.e.) if their "straightened versions" $f^{\dagger}$ and $g^{\dagger}$ agree a.e. Let Φ be the relation on L given by $f \Phi g$ if f and g agree c.a.e.*

We now collect results.

Theorem 6.6.12 *The lattices L_u, D_u, X/Θ, D/Θ and L/Φ are isomorphic, and they are completely distributive lattices and continuous lattices.*

Proof. Proposition 6.5.4 shows that L_u and D_u are isomorphic as posets, and Corollary 6.5.6 shows they are lattices. Theorem 6.6.10 shows that these two lattices are isomorphic to X/Θ. Since D is a sublattice of X and each equivalence class of the congruence Θ of X contains a member of D, it follows that

D /Θ is isomorphic to X /Θ. Finally, Theorem 6.3.3 shows that † is an isomorphism from L to D. The relation Φ on L is defined so that it is carried by the isomorphism † to the relation Θ on D. Therefore L /Φ is isomorphic to D /Θ, and hence to the others. By Corollary 6.6.6, X /Θ is a completely distributive lattice and a continuous lattice, and as these lattices are isomorphic, they all have these properties. ■

This result shows that L_u is a continuous lattice by showing that it has the stronger property of being completely distributive. The definition of a continuous lattice, Definition 6.1.14, is that of a lattice where each element is the join of the elements way below it. In [45] it was shown directly that L_u is a continuous lattice by showing that each element is the join of those way below it. We paraphrase the results of [45] below.

Proposition 6.6.13 *In L_u and D_u, each element is the join of a family of 3-valued step functions that are way below it.*

Throughout, we have considered our structures only in terms of the lattice operations. The bounds of L are the constants 1_0 and 1_1. Since these are USC, L_u is a bounded sublattice of L. By Theorem 6.6.12, the other lattices must be bounded as well. The bounds of D_u are $1_0^{\dagger}$, which equals 1_0, and $1_1^{\dagger}$, which equals 2 everywhere except at 1, and equals 1 at 1. The bounds of X are the constant functions $\overline{0}$ and $\overline{2}$, and the bounds of X /Θ are $\overline{0}/\Theta$ and $\overline{2}/\Theta$.

The lattice L has a negation * that makes it a De Morgan algebra. Specifically, $f^*(x) = f(1-x)$ takes the mirror image of f in the line $x = 0.5$. It is easy to see that L_u is a subalgebra with respect to this negation as well, so is also a De Morgan algebra. The other isomorphic lattices will also carry a De Morgan negation. In fact, this negation can be seen to come from a natural De Morgan structure on the most basic structure X.

Definition 6.6.14 *For $f \in \mathrm{X}$, $f^*(x) = 2 - f(1-x)$.*

Note that the negation * on X is built from a negation on I and one on $\mathrm{I}^{\dagger}$. It takes the mirror image in the line $x = 0.5$ and $y = 1$. It is easily seen that this negation is compatible with the congruence Θ, so X /Θ is a De Morgan algebra, and that the previously described isomorphisms are isomorphisms also with this negation. Similar comments hold for D /Θ and D_u. We then have the following.

Theorem 6.6.15 *Each of L_u, D_u, X /Θ and L /Φ carries the structure of a De Morgan algebra, and as De Morgan algebras, they are isomorphic.*

We turn next to additional properties of these algebras. In developing these properties it is most natural to work in the setting of the most basic of them, the algebra X /Θ.

6.7 Metric and topological properties

In this section, it is shown that X/Θ naturally carries a compact metric space structure under which it is a topological De Morgan algebra. The reader should review the preliminaries, and to begin with, Definition 6.1.6.

Definition 6.7.1 *Define* $v : \mathrm{X} \to \mathbb{R}$ *by* $v(f) = \int_0^1 f(x)dx$.

A bit of basic analysis provides the following.

Proposition 6.7.2 *The map v is an quasi-valuation on* X.

For $f, g \in \mathrm{X}$, set $d(f,g) = v(f \vee g) - v(f \wedge g) = \int_0^1 |f(x) - g(x)|dx$. Note that $d(f,g) = 0$ if and only if f and g agree a.e., so $d(f,g) = 0$ if and only if $f \,\Theta\, g$. Then by Theorems 6.1.7 and 6.1.8, we have the following.

Theorem 6.7.3 X/Θ *is a metric space under the metric*

$$D(f/\Theta, g/\Theta) = \int_0^1 |f(x) - g(x)|dx$$

and under this metric, meet and join are uniformly continuous.

We will show this topology on X/Θ is compact, but first a lemma.

Lemma 6.7.4 *For $f \in \mathrm{X}$ and $\epsilon > 0$ there is a natural number n and $\delta > 0$ so that for any $g \in \mathrm{X}$ with $|g(i/n) - f(i/n)| < \delta$ for each $i = 0, \dots, n$, we have $d(f,g) < \epsilon$.*

Proof. Choose n so that $1/n < \epsilon/4$, and let $\delta = \epsilon/2$. For $i = 0, \dots, n$, let $x_i = i/n$ and $y_i = f(x_i)$, and for $i = 1, \dots, n$, let J_i be the interval $[x_{i-1}, x_i]$. Consider the behavior of f and g on the interval J_i. As f is decreasing, we have $y_i \le f \le y_{i-1}$ on J_i. Since $g(x_{i-1})$ is within δ of y_{i-1}, $g(x_i)$ is within δ of y_i, and g is decreasing, we have $y_i - \delta < g < y_{i-1} + \delta$ on J_i. So $|f - g| < y_{i-1} - y_i + \delta$ on J_i. Thus

$$\int_0^1 |f - g|dx < \frac{1}{n}(y_0 - y_1 + \delta) + \frac{1}{n}(y_1 - y_2 + \delta) + \cdots + \frac{1}{n}(y_{n-1} - y_n + \delta)$$

So $d(f,g) < \frac{1}{n}(y_0 - y_n) + \delta$, and since y_0, y_n lie between 0 and 2, $d(f,g) < \epsilon$. ■

Proposition 6.7.5 *The metric topology on X/Θ is compact.*

Proof. With the usual topology, $[0,2]$ is compact, so $T = [0,2]^{[0,1]}$ is compact in the product topology. We first show X is a closed subspace of T. Suppose $f \notin \mathrm{X}$. Then there are $x < y$ with $f(x) < f(y)$, so $f(y) = f(x) + \epsilon$ for some $\epsilon > 0$. The set of all $g \in T$ lying within $\epsilon/2$ of f in both the x and y coordinates is an open cylinder in T that contains f but does not contain any decreasing function. So X is closed in T, hence is compact under the subspace topology.

We next show that the canonical quotient map $\kappa : \mathrm{X} \to \mathrm{X}/\Theta$ is continuous with respect to the subspace topology on X and the metric topology on X/Θ. For $f \in \mathrm{X}$ and $\epsilon > 0$, we seek an open neighborhood of f in X mapped by κ into the ball in X/Θ of radius ϵ centered at f/Θ. This is precisely what is provided by Lemma 6.7.4. Then because X is compact, and κ is onto and continuous, it follows that X/Θ is compact. ■

Proposition 6.7.5 shows that the metric space topology on this lattice is a compact Hausdorff topology, and Theorem 6.7.3 shows that this metric space topology makes X/Θ a topological lattice. Corollary 6.6.6 shows that X/Θ is a completely distributive lattice, and hence a continuous lattice. But Theorem 6.1.22 provides that on any completely distributive lattice L, the Lawson topology is equal to the interval topology, and this is the unique compact Hausdorff topology that makes L a topological lattice. Combining these results gives the following.

Proposition 6.7.6 *The metric space topology on* X/Θ *is equal to its Lawson topology, and this is equal to its interval topology.*

Summarizing our results, gives the following.

Corollary 6.7.7 *The isomorphic lattices* $L_u, \mathrm{X}/\Theta$*, and* L/Φ *are complete, and completely distributive. Each of these lattices has a natural metric that makes it a compact Hausdorff topological lattice, and this compact Hausdorff topology agrees with its Lawson topology and its interval topology.*

When considering the De Morgan negation * on X of Definition 6.6.14, it is easy to see that $d(f,g) = d(f^*,g^*)$. Therefore * gives a De Morgan negation on X/Θ that is an isometry of X/Θ. This gives the following.

Theorem 6.7.8 *The isomorphic structures* $\mathrm{L}_u, \mathrm{X}/\Theta$*, and* L/Φ *are complete, completely distributive, compact Hausdorff, metric De Morgan algebras.*

Convex normal functions seem a natural setting for applications of type-2 fuzzy theory, and agreement convexly almost everywhere fits very well with such applications. So L/Φ, and its isomorphic structures X/Θ and L_u are naturally motivated by applications. They are also De Morgan algebras with a large number of very attractive order theoretic and topological properties. It would be of interest to see if there is some abstract characterization of this lattice, perhaps in terms of some kind of universal property.

6.8 T-norms and t-conorms

In this section, we develop the properties of convolution t-norms and t-conorms on the isomorphic lattices L_u and X/Θ. These results were first obtained for L_u in [65, 81, 82] using a result known as Nguyen's theorem that characterizes α-cuts of convolutions [84]. Our approach is somewhat different, and makes use of "straightening." Throughout, we will state our results only for t-norms. The dual results hold for t-conorms, and in most cases the proofs are identical. We only describe modifications necessary to obtain the corresponding result for t-conorms when needed.

Proposition 6.8.1 *Let $\triangle$ be a continuous t-norm on I and $f, g \in \mathrm{L}_u$. Then for any $z \in \mathrm{I}$, for the value*

$$(f \blacktriangle g)(z) = \bigvee\{f(x) \wedge g(y) : x \triangle y = z\}$$

there are $x, y \in \mathrm{I}$ with $x \triangle y = z$ and $(f \blacktriangle g)(z) = f(x) \wedge g(y)$.

Proof. Let $\Sigma = \{(x,y) : x \triangle y = z\}$. Since $\Sigma = \triangle^{-1}(\{z\})$ and $\triangle$ is continuous, Σ is a closed subset of $\mathrm{I} \times \mathrm{I}$. Since f and g are USC and $\wedge$ is continuous, the map $h : \mathrm{I} \times \mathrm{I} \to \mathrm{I}$ given by $h(x,y) = f(x) \wedge g(y)$ is USC. Then $(f \blacktriangle g)(z)$ is the supremum λ of h on Σ. For each $\alpha < \lambda$, the set $h^{-1}([\alpha, 1])$ is a non-empty closed subset of Σ. Since Σ is closed and bounded, it is compact. So the intersection of the decreasing family of non-empty closed sets $h^{-1}([\alpha, 1])$ where $\alpha < \lambda$, contains a point (x, y). This point has the desired properties. ∎

When using this result we say that $(f \blacktriangle g)(z)$ **attains its value at** (x, y). We also use the following notation. For closed intervals $[a, b]$ and $[c, d]$ of I, set

$$[\,a,b\,] \triangle [\,c,d\,] = \{z : z = x \triangle y \text{ for some } a \leq x \leq b \text{ and } c \leq y \leq d\}$$

Lemma 6.8.2 *If $\triangle$ is a continuous t-norm, then*

$$[\,a,b\,] \triangle [\,c,d\,] = [\,a \triangle c, b \triangle d\,]$$

Proof. Properties of a t-norm yield that $a \triangle c$ is the least element of $[\,a,b\,] \triangle [\,c,d\,]$ and $b \triangle d$ is the largest. It remains to show that all of the elements of the interval $[\,a \triangle c, b \triangle d\,]$ are realized. This follows since $[\,a,b\,] \triangle [\,c,d\,]$ is by definition the image of the connected subset $[\,a,b\,] \times [\,c,d\,]$ under the continuous map $\triangle : \mathrm{I} \times \mathrm{I} \to \mathrm{I}$, so is a connected subset of I, and therefore an interval. ∎

We now have the following formulation of Nguyen's theorem [84] for our setting. Here, we recall that for any convex function, its α-cuts are intervals, and for USC functions, the α-cuts are closed.

Proposition 6.8.3 *Let* $\triangle$ *be a continuous t-norm on* I *and* $f, g \in \mathrm{L}_u$. *If* $\alpha \in$ I *and the* α*-cuts of* f *and* g *are the closed intervals* $f^{-1}[\alpha, 1] = [\,a, b\,]$ *and* $g^{-1}[\alpha, 1] = [\,c, d\,]$, *then the* α*-cut of* $f \blacktriangle g$ *is*

$$(f \blacktriangle g)^{-1}[\alpha, 1] = [\,a, b\,] \triangle [\,c, d\,]$$

Proof. By Proposition 6.8.1, $(f \blacktriangle g)(z) \geq \alpha$ if and only if there is a pair x, y with $x \triangle y = z$ and $f(x) \wedge g(y) \geq \alpha$. This is equivalent to having a pair x, y with x in the α-cut of f, y in the α-cut of g, and $x \triangle y = z$. So this is equivalent to having z in $[\,a, b\,] \triangle [\,c, d\,]$. ∎

Combining this result with Lemma 6.8.2 provides that the α-cuts of $f \blacktriangle g$ are closed. Thus $f \blacktriangle g$ is USC. This gives the following.

Theorem 6.8.4 *Let* $\triangle$ *be a continuous t-norm on* I *and* $f, g \in \mathrm{L}_u$. *Then* $f \blacktriangle g$ *belongs to* L_u. *Therefore* $(\mathrm{L}_u, \blacktriangle)$ *is a subalgebra of* $(\mathrm{L}, \blacktriangle)$. *A corresponding result holds for the convolution* $\blacktriangledown$ *of a continuous t-conorm* $\triangledown$.

In earlier sections, the algebra L_u has been isomorphically realized as a quotient of L by the congruence of agreement c.a.e. We consider convolution t-norms from this perspective. Recall that for any $f \in \mathrm{L}$, there is a unique member $\hat{f}$ of L_u that agrees with f c.a.e. This $\hat{f}$ can be obtained by straightening f, applying the operation ‡ of Proposition 6.6.9, then unstraightening the result. But $\hat{f}$ is simply the least USC function pointwise above f.

Theorem 6.8.5 *If* $\triangle$ *is a continuous t-norm on* I *and* $f, g \in \mathrm{L}$, *then* $f \blacktriangle g$ *agrees with* $\hat{f} \blacktriangle \hat{g}$ *c.a.e. So* $(\mathrm{L}_u, \blacktriangle)$ *is isomorphic to a quotient of* $(\mathrm{L}, \blacktriangle)$. *A corresponding result holds for the convolution* $\blacktriangledown$ *of a continuous t-conorm* $\triangledown$.

Proof. Note that if two convex normal functions take value 1 at the same place and agree a.e., then their straightened versions agree a.e., hence the functions agree c.a.e. In particular, if one convex normal function lies pointwise beneath another, and they agree a.e., then they agree c.a.e.

Since f lies pointwise under $\hat{f}$, and g lies pointwise under $\hat{g}$, then $f \blacktriangle g$ lies pointwise under $\hat{f} \blacktriangle \hat{g}$. For each $z \in \mathrm{I}$, by Proposition 6.8.1, $(\hat{f} \blacktriangle \hat{g})(z)$ attains its value at some pair x, y. Since f agrees with $\hat{f}$ a.e., there are only countably many points x where f and $\hat{f}$ differ, and also only countably many points y where g and $\hat{g}$ differ. So there are only countably many pairs x, y where $\hat{f} \blacktriangle \hat{g}$ attains its value and either f differs from $\hat{f}$ or g differs from $\hat{g}$. For all other pairs x, y, the functions $\hat{f} \blacktriangle \hat{g}$ and $f \blacktriangle g$ agree at $z = x \triangle y$. Thus, they agree a.e., and hence c.a.e.

Finally, suppose that $f_1, f_2 \in \mathrm{L}$ agree c.a.e., and $g_1, g_2 \in \mathrm{L}$ agree c.a.e. Since $\hat{f}_1$ is the unique member of L_u that agrees with f_1 c.a.e., then $\hat{f}_1 = \hat{f}_2$ and similarly $\hat{g}_1 = \hat{g}_2$. So $f_1 \blacktriangle g_1$ agrees with $\hat{f}_1 \blacktriangle \hat{g}_1$ c.a.e., and this equals $\hat{f}_2 \blacktriangle \hat{g}_2$, which agree with $f_2 \blacktriangle g_2$ c.a.e. So agreement c.a.e. is compatible with the operation $\blacktriangle$. It follows that $(\mathrm{L}_u, \blacktriangle)$ is isomorphic to the quotient of $(\mathrm{L}, \blacktriangle)$ by the congruence of agreement c.a.e. ∎

Having seen that convolutions of continuous t-norms give rise to operations on L_u, we explore the properties of these operations. Since $(\mathrm{L}_u, \blacktriangle)$ can be realized as either a subalgebra or quotient of $(\mathrm{L}, \blacktriangle)$, the operation $\blacktriangle$ on L_u is a lattice-ordered t-norm as well. But $\blacktriangle$ on L_u has the further property of preserving arbitrary joins in each coordinate. This was a result first established in [65]. Our approach is different. We begin with the following.

Proposition 6.8.6 *If p_j $(j \in J)$ is a family in L_u, then the following hold for $\bigsqcup$ the join in L_u:*

1. $(\bigsqcup_J p_j)^L = \bigsqcup_J p_j^L$.
2. $(\bigsqcup_J p_j)^R = \bigsqcup_J p_j^R$.

Proof. For any $f \in \mathrm{M}$, by Theorem 1.4.5 $f^L = f \sqcup \overline{1}$ and $f^R = f \sqcap \overline{1}$, where $\overline{1}$ is the constant function taking value 1. The first item below follows by properties of arbitrary joins in any complete lattice, and the second follows by meet continuity in L_u.

$$(\bigsqcup_J p_j)^L = (\bigsqcup_J p_j) \sqcup \overline{1} = \bigsqcup_J (p_j \sqcup \overline{1}) = \bigsqcup_J p_j^L$$
$$(\bigsqcup_J p_j)^R = (\bigsqcup_J p_j) \sqcap \overline{1} = \bigsqcup_J (p_j \sqcap \overline{1}) = \bigsqcup_J p_j^R$$

This establishes the result. ■

In the terminology of [27], this result shows that L and R are **complete operators** on L_u. Next, we establish our result in two special cases. Again, the proofs for t-norms work equally for t-conorms.

Lemma 6.8.7 *Let $\triangle$ be a continuous t-norm on I and $\blacktriangle$ be its convolution. If f, g_j $(j \in J)$ are decreasing functions in L_u with their join $g = \bigsqcup_J g_j$ in L_u, then*

$$f \blacktriangle g = \bigsqcup_J (f \blacktriangle g_j)$$

Proof. Since each g_j is decreasing, it follows from Proposition 6.8.6 that g is also decreasing. Then the straightened versions of all functions involved are themselves. So Lemma 6.6.5 gives that joins $\bigsqcup$ in L_u are given by pointwise join $\bigvee$ almost everywhere. Also, Theorem 6.8.5 shows that agreement c.a.e. is a congruence with respect to $\blacktriangle$. So it is enough to show that

$$f \blacktriangle \bigvee_J g_j = \bigvee_J (f \blacktriangle g_j)$$

For $z \in \mathrm{I}$, we have

$$\begin{aligned}(f \blacktriangle \bigvee_J g_j)(z) &= \bigvee\{f(x) \wedge \bigvee_J g_j(y) : x \vartriangle y = z\} \\ &= \bigvee\{\bigvee_J f(x) \wedge g_j(y) : x \vartriangle y = z\} \\ &= \bigvee_J \bigvee\{f(x) \wedge g_j(y) : x \vartriangle y = z\} \\ &= \bigvee_J (f \blacktriangle g_j)(z)\end{aligned}$$

This establishes the result. ■

Before the next proof, we describe joins in L_u of increasing functions. Since the straightened version of an increasing function p is $2-p$, the ordering $\sqsubseteq$ of increasing functions is $\geq$. The join $\bigsqcup$ in L_u is given by taking the pointwise join of their straightened versions, unstraightening this, and then finding the unique function in L_u that agrees with the result c.a.e. Since the pointwise meet of USC functions is USC, the join in L_u of a family of increasing USC functions is given by pointwise meet $\bigwedge$.

Lemma 6.8.8 *Let $\vartriangle$ be a continuous t-norm on* I *and $\blacktriangle$ be its convolution. If f, g_j $(j \in J)$ are increasing functions in L_u with their join $g = \bigsqcup_J g_j$ in L_u, then*

$$f \blacktriangle g = \bigsqcup_J (f \blacktriangle g_j)$$

Proof. By Proposition 6.8.6 and Theorem 6.8.4, all functions involved are increasing and USC. Since $\blacktriangle$ is increasing in each argument, we have that $\bigsqcup_J (f \blacktriangle g_j) \sqsubseteq f \blacktriangle g$. This implies that $f \blacktriangle g \leq \bigwedge_J (f \blacktriangle g_j)$. For the other inequality, we show that for each $z, \lambda \in \mathrm{I}$,

$$\bigwedge_J (f \blacktriangle g_j)(z) \geq \lambda \quad \text{implies that} \quad (f \blacktriangle g)(z) \geq \lambda$$

For such z, λ, for each $j \in J$ we have $(f \blacktriangle g_j)(z) \geq \lambda$. By Proposition 6.8.1, there are x_j and y_j with

$$x_j \vartriangle y_j = z \quad \text{and} \quad f(x_j) \wedge g_j(y_j) \geq \lambda$$

Set $x = \inf\{x_j : j \in J\}$ and $y = \sup\{y_j : j \in J\}$. Then $x \vartriangle y_j \leq x_j \vartriangle y_j = z$ for each $j \in J$, hence by the continuity of $\vartriangle$, we have $x \vartriangle y \leq z$. A dual argument shows that $x \vartriangle y \geq z$, hence $x \vartriangle y = z$. Since each g_j is increasing and $y_j \leq y$, we have $g_j(y) \geq g_j(y_j) \geq \lambda$. Hence $g(y) \geq \lambda$. Since f is increasing and USC and $x = \inf\{x_j : j \in J\}$, then $f(x) = \inf\{f(x_j) : j \in J\} \geq \lambda$. Thus $(f \blacktriangle g)(z) \geq \lambda$. ■

Theorem 6.8.9 *Let $\vartriangle$ be a continuous t-norm on* I *and $\blacktriangle$ be its convolution. If f, g_j $(j \in J)$ are functions in L_u with their join $g = \bigsqcup_J g_j$ in L_u, then*

$$f \blacktriangle g = \bigsqcup_J (f \blacktriangle g_j)$$

A corresponding result holds for the convolution $\blacktriangledown$ *of a continuous t-conorm, namely*

$$f \blacktriangledown g = \bigsqcup_J (f \blacktriangledown g_j)$$

Proof. For any convex normal function p, we have $p = p^L \wedge p^R$. So to show the desired expression, it is enough to show that

$$(f \blacktriangle g)^L = (\bigsqcup_J (f \blacktriangle g_j))^L \tag{6.1}$$

$$(f \blacktriangle g)^R = (\bigsqcup_J (f \blacktriangle g_j))^R \tag{6.2}$$

By Proposition 5.5.1, $(f \blacktriangle g)^L = f^L \blacktriangle g^L$. Using this and Proposition 6.8.6,

$$(\bigsqcup_J (f \blacktriangle g_j))^L = \bigsqcup_J (f^L \blacktriangle g_j^L)$$

Combining these, to show (6.1), it suffices to show that

$$f^L \blacktriangle g^L = \bigsqcup_J (f^L \blacktriangle g_j^L)$$

Since all functions involved in this expression are increasing, this is given by Lemma 6.8.8. The argument to show (6.2) is similar using Lemma 6.8.7. The argument for a t-conorm is identical since Lemmas 6.8.7 and 6.8.8 hold also for t-conorms. ■

So in the terminology of [27], if $\vartriangle$ is a continuous t-norm, and $\triangledown$ is a continuous t-conorm, then the convolutions $\blacktriangle$ and $\blacktriangledown$ restrict to complete operators on L_u. The following is a standard property of such complete operators. For the pertinent definitions, see Definitions 5.1.2, 5.1.3, and 5.1.4.

Corollary 6.8.10 *Let* $\vartriangle$ *be a continuous t-norm on* I *and* $\blacktriangle$ *be its convolution. Then* $(\mathrm{L}_u, \blacktriangle, 1_1)$ *is a residuated lattice-ordered monoid. Also if* $\triangledown$ *is a continuous t-conorm, then* $(\mathrm{L}_u, \blacktriangledown, 1_0)$ *is a residuated lattice-ordered monoid.*

Proof. That this structure is a lattice-ordered monoid follows from the facts that $\blacktriangle$ gives a lattice-ordered monoid on L with 1_1 as the unit, and that $(\mathrm{L}_u, \blacktriangle)$ is a subalgebra of $(\mathrm{L}, \blacktriangle)$. These are given in Proposition 5.5.1, Theorem 5.5.3, and Theorem 6.8.4. To see that $\blacktriangle$ is residuated, suppose $f, h \in \mathrm{L}_u$. We must show there is a largest g with $f \blacktriangle g \sqsubseteq h$. Let g_j $(j \in J)$ be an indexing of all functions with $f \blacktriangle g_j \sqsubseteq h$. Then for $g = \bigsqcup_J g_j$, Theorem 6.8.9 gives $f \blacktriangle g = \bigsqcup_J (f \blacktriangle g_j)$, hence $f \blacktriangle g \sqsubseteq h$. So g is the largest with $f \blacktriangle g \sqsubseteq h$. ■

Using properties of the De Morgan negation, we can obtain the following result that says the convolution of any continuous t-norm or continuous t-conorm is a **complete dual operator**.

Corollary 6.8.11 *Let $\triangle$ be a continuous t-norm on* I *and $\blacktriangle$ be its convolution. If f, g_j $(j \in J)$ are functions in* L_u *with their meet $g = \bigsqcap_J g_j$ in* L_u*, then*

$$f \blacktriangle g = \bigsqcap_J (f \blacktriangle g_j)$$

A corresponding result holds for the convolution $\blacktriangledown$ of a continuous t-conorm, namely

$$f \blacktriangledown g = \bigsqcap_J (f \blacktriangledown g_j)$$

Proof. By Proposition 5.4.1, $(f \blacktriangle g)^* = f^* \blacktriangledown g^*$ where $\blacktriangledown$ is the convolution of the dual t-conorm of $\triangle$. Since * is a De Morgan negation on L_u, we have $(\bigsqcap_J g_j)^* = \bigsqcup_J g_j^*$ (Exercise 20). Then using the result that the convolution of a continuous t-conorm is a complete operator,

$$(f \blacktriangle g)^* = f^* \blacktriangledown \bigsqcup_J g_j^* = \bigsqcup_J (f^* \blacktriangledown g_j^*) = \bigsqcup_J (f \blacktriangle g_j)^* = (\bigsqcap_J f \blacktriangle g_j)^*$$

This gives the result. ∎

We summarize these results as follows.

Theorem 6.8.12 *Let $\triangle$ be a continuous t-norm on* I *with convolution $\blacktriangle$ and $\triangledown$ be a continuous t-conorm on* I *with convolution $\blacktriangledown$. Then for f, g_j $(j \in J)$ functions in* L_u*, using $\bigsqcup$ and $\bigsqcap$ for join and meet in* L_u*,*

$$f \blacktriangle \bigsqcup_J g_j = \bigsqcup_J (f \blacktriangle g_j) \qquad f \blacktriangle \bigsqcap_J g_j = \bigsqcap_J (f \blacktriangle g_j)$$

$$f \blacktriangledown \bigsqcup_J g_j = \bigsqcup_J (f \blacktriangledown g_j) \qquad f \blacktriangledown \bigsqcap_J g_j = \bigsqcap_J (f \blacktriangledown g_j)$$

Thus $\blacktriangle$ and $\blacktriangledown$ are complete operators and complete dual operators on L_u

These results have topological consequences. We recall that a function in two variables $f(x, y)$ is **separately continuous** if holding one variable fixed always produces a continuous function of one variable; that is, the functions $\phi(x) = f(x, b)$ and $\psi(y) = f(a, y)$ are continuous functions for each choice of a, b. This is weaker than requiring that f is continuous under the product topology, a property sometimes called **joint continuity**.

The convolution of a continuous t-norm is a function of two variables $\blacktriangle : \mathrm{L}_u \times \mathrm{L}_u \to \mathrm{L}_u$, and on L_u the metric, Lawson, and interval topologies agree. The following is an immediate consequence of Theorem 6.1.21.

Corollary 6.8.13 *Let $\triangle$ be a continuous t-norm on* I *and $\blacktriangle$ be its convolution. Then $\blacktriangle$ is is separately continuous when considered as a function from $\mathrm{L}_u \times \mathrm{L}_u \to \mathrm{L}_u$ under the metric topology. The corresponding result holds for the convolution of a continuous t-conorm.*

We do not know whether the convolution of a continuous t-norm or t-conorm must be jointly continuous.

6.9 Summary

The ordering of the lattice L of convex normal functions is simplified through the technique of straightening. This is used to show that the lattice L is complete, but that it does not satisfy any infinite distributive laws. A new algebra L_u is created from L that is not only complete, but satisfies all infinite distributive laws. This algebra L_u is realized isomorphically both as the subalgebra of L consisting of functions that are upper semicontinuous, and as a quotient of L by the congruence of agreement convexly almost everywhere. This view of identifying functions that agree c.a.e. has natural motivation in applications.

The algebra L_u has many desirable features. It is complete, and completely distributive. It is a De Morgan algebra under *. It has a natural metric space topology where distance $d(f,g)$ is obtained via the integral $\int |f(x)-g(x)|\,dx$. Under this metric space topology, it is a compact Hausdorff topological De Morgan algebra. This metric topology is equal to its Lawson topology and to its interval topology.

The convolution $\blacktriangle$ of a continuous t-norm both restricts to L_u when it is considered as a subalgebra of L, and is compatible with the congruence of agreement c.a.e. when L_u is realized as a quotient of L. The operation $\blacktriangle$ preserves arbitrary meets and arbitrary joins in each coordinate. The same results hold for the convolution $\blacktriangledown$ of a continuous t-conorm. Thus both $\blacktriangle$ and $\blacktriangledown$ are residuated and coresiduated. They are also separately continuous functions with respect to the metric space topology on L_u.

6.10 Exercises

1. Give the definition of a continuous function between two metric spaces.

2. Prove that the distance $d((x_1,x_2),(y_1,y_2)) = \sqrt{(x_1-y_1)^2+(x_2-y_2)^2}$ between two points in the plane is a metric on $\mathbb{R}^2$.

3. The **taxicab metric** on the plane $\mathbb{R}^2$ is given by $d((x_1,x_2),(y_1,y_2)) = |x_1-y_1|+|x_2-y_2|$. Show that this is a metric. Why is this called the taxicab metric?

4. Prove that the usual distance and taxicab metrics on $\mathbb{R}^2$ have exactly the same open sets.

5. Prove that the intersection of two open subsets of $\mathbb{R}$ is open and that the union of arbitrarily many open subsets of $\mathbb{R}$ is open.

6. Prove that the collection of open sets of a topological space is a distributive lattice when partially ordered by set inclusion. It is additionally complete and satisfies the meet-distributive law, but these are more difficult to establish.

7. Show that the reals $\mathbb{R}$ with the usual lattice structure and topology form a topological lattice; that is, prove that $\wedge$ and $\vee$ are continuous.

8. Show that the subspaces of a vector space V form a lattice when partially ordered by set inclusion. Show that if V is finite-dimensional, then the dimension function on V is a valuation.

9. Show that if d is a quasi-metric on a set X, then the relation θ defined by $x\,\theta\,y$ if $d(x,y) = 0$ is an equivalence relation.

10. Show that $[0,1)\cup\{2\}$ is a sublattice of $[0,2]$ that is complete as a lattice in its own right, but is not a complete sublattice.

11. Prove that every complete chain is completely distributive and that the power set of any set is completely distributive.

12. Prove that the product of completely distributive lattices is completely distributive, and that a complete sublattice of a completely distributive lattice is completely distributive.

13. Prove that in I, $a \ll b$ if and only if $a < b$.

14. Prove Proposition 6.2.2.

15. Prove that the pointwise join and pointwise meet of two USC functions is USC, and similarly for two LSC functions.

16. Prove that the pointwise join of arbitrarily many LSC functions is LSC, and that the pointwise meet of arbitrarily many USC functions is USC.

17. Show that the intersection of two Scott open sets is Scott open, and that the union of arbitrarily many Scott open sets is Scott open.

18. Show that the relation Θ given in Definition 6.6.4 is a congruence on X.

19. Give the details of the proof of Corollary 6.6.6.

20. Show that in a complete De Morgan algebra, $(\bigwedge_J a_j)^* = \bigvee_J a_j^*$ and $(\bigvee_J a_j)^* = \bigwedge_J a_j^*$.

Chapter 7

Varieties Related to M

A variety of algebras is a class of algebras that can be defined by equations. In this chapter we investigate the variety generated by M, the truth value algebra of type-2 fuzzy sets. One important result is that this variety is generated by a finite algebra, enabling an algorithm for determining when an equation holds in this truth value algebra. This provides a method similar to that of truth tables for determining the validity of equations in the truth value algebra for type-2 fuzzy sets. The results of this chapter are found in [46].

7.1 Preliminaries

Much of this chapter deals with the area of mathematics known as universal algebra. The reader should review the definitions of an n-ary operation, a constant, a type, an algebra, a subalgebra, a homomorphism, an isomorphism, and a reduct from the preliminaries of Chapters 1, 3 and 4. The reader should also review the various examples of algebras, such as lattices, Boolean algebras, and groups from these preliminaries.

A key feature here will be the study of equations. We begin with a definition of the building blocks of equations, terms.

Definition 7.1.1 *A **term** for an algebra A is a well-formed expression built from a set of variables and the operations of the algebra. An **equation*** $s = t$

consists of a pair of terms. An algebra A ***satisfies*** *$s = t$ if the equation holds for all assignments of the variables in the equations to elements of A.*

For example, the equation $(x * y) * z = x * (y * z)$ of associativity holds in any group. Further, groups are defined by a set of equations that include associativity and such familiar equations as $x * e = x$, and $x * x^{-1} = e$.

Definition 7.1.2 *A* ***variety*** *$\mathcal{V}$ of algebras is a class of algebras of a given type that is defined by a set of equations.*

Many familiar classes of algebras are varieties. These include groups, abelian groups, rings, and lattices. The variety of lattices is defined by the equations saying that both operations $\wedge$ and $\vee$ are commutative, associative, idempotent, and the absorption laws $x \wedge (x \vee y) = x$ and $x \vee (x \wedge y) = y$.

Definition 7.1.3 *For an algebra A,* ***the variety*** *$\mathcal{V}(A)$* ***generated by*** *A is the class of all algebras of the same type as A that satisfy all equations that are valid in A.*

Birkhoff's Theorem below provides a link between the purely semantic notion of a variety, and the algebraic constructs of products, subalgebras, and homomorphisms. For a family of algebras A_i $(i \in I)$ of the same type, their **product** $\prod_I A_i$ is the product of their underlying sets with operations defined componentwise. A **subalgebra** of an algebra A is a subset $S \subseteq A$ that is closed under the operations of A. A **homomorphism** $\varphi : A \to B$ from an algebra A to an algebra B of the same type is a function between their underlying sets such that for each n-ary operation f we have $\varphi(f(a_1, \ldots, a_n)) = f(\varphi(a_1), \ldots, \varphi(a_n))$ for each $a_1, \ldots, a_n \in A$.

Theorem 7.1.4 *[Birkhoff] A class of algebras is a variety if and only if it is closed under taking homomorphic images, subalgebras, and products.*

We need another notion, that of a congruence on an algebra. Congruences are equivalence relations that are compatible with the operations of an algebra. They play the role for arbitrary algebras that normal subgroups play for groups. In particular, each gives a quotient that is an algebra defined on the set of equivalence classes of the congruence, and each homomorphic image of an algebra is isomorphic to such a quotient. The precise definition follows.

Definition 7.1.5 *A* ***congruence*** *on an algebra A is an equivalence relation θ on the underlying set of A such that for each n-ary operation f and each family $a_1\theta b_1, \ldots, a_n\theta b_n$ we have $f(a_1, \ldots, a_n)\, \theta f(b_1, \ldots, b_n)$.*

For further details regarding universal algebra, the reader should consult [10], available freely from the web. Exercises at the end of this chapter point to some basic ideas needed.

7.2 The variety $\mathcal{V}(\mathrm{M})$

As discussed in Chapter 2, the algebra $\mathrm{M} = (\mathrm{M}, \sqcap, \sqcup, ^*, 1_0, 1_1)$ satisfies the following set of equations for $f, g, h \in \mathrm{M}$.

1. $f \sqcup f = f$; $f \sqcap f = f$.
2. $f \sqcup g = g \sqcup f$; $f \sqcap g = g \sqcap f$.
3. $f \sqcup (g \sqcup h) = (f \sqcup g) \sqcup h$; $f \sqcap (g \sqcap h) = (f \sqcap g) \sqcap h$.
4. $f \sqcup (f \sqcap g) = f \sqcap (f \sqcup g)$.
5. $1_1 \sqcap f = f$; $1_0 \sqcup f = f$.
6. $f^{**} = f$.
7. $(f \sqcup g)^* = f^* \sqcap g^*$; $(f \sqcap g)^* = f^* \sqcup g^*$.

Also of interest is the reduct of M to $(\mathrm{M}, \sqcap, \sqcup)$, and the variety $\mathcal{V}(\mathrm{M}, \sqcap, \sqcup)$ generated by this reduct. An equation involving only the operations $\sqcap$ and $\sqcup$ is satisfied by M if and only if it is satisfied by this reduct. In particular, $(\mathrm{M}, \sqcap, \sqcup)$ satisfies equations (1)–(4) above.

Definition 7.2.1 *An algebra with two binary operations $\sqcap$ and $\sqcup$ is a* ***Birkhoff system*** *if each of these operations is commutative, associative, and idempotent, and it satisfies* ***Birkhoff's equation*** $x \sqcap (x \sqcup y) = x \sqcup (x \sqcap y)$.

Thus $(\mathrm{M}, \sqcap, \sqcup)$ is a Birkhoff system, and the variety $\mathcal{V}(\mathrm{M}, \sqcap, \sqcup)$ it generates is a subvariety of the variety of Birkhoff systems. We will see that this is a proper subvariety by demonstrating an equation that is valid in $(\mathrm{M}, \sqcap, \sqcup)$ but not valid in all Birkhoff systems, and we will see that the variety $\mathcal{V}(\mathrm{M})$ is a proper subvariety of the variety defined by equations (1)–(7) by giving an equation that is valid in M but is not valid in all algebras that satisfy (1)–(7). However, the following basic problem about these varieties remains open.

Problem 7.2.2 *Find a set of equations that defines the variety $\mathcal{V}(\mathrm{M})$ and one that defines $\mathcal{V}(\mathrm{M}, \sqcap, \sqcup)$.*

In regard to this problem, we remark that there is no reason to believe a finite set of equations can be found to describe either variety. However, we will show there is a decision procedure to determine when an equation holds in either variety. In classical logic, a set of equations defining the variety of Boolean algebras corresponds to an axiomatization of classical propositional logic. There are two well-known decision procedures to determine if an equation holds in propositional logic—one can put each expression in disjunctive normal form, or one can use the method of truth tables. Much of the remainder of this chapter will be devoted to providing such decision procedures for the truth value algebra M.

7.3 Local finiteness

In this section, we establish a basic property of the variety $\mathcal{V}(\mathrm{M})$, namely, that of local finiteness.

Definition 7.3.1 *An algebra A is **locally finite** if each finite subset of A generates a finite subalgebra of A, and a variety $\mathcal{V}$ is **locally finite** if each algebra in this variety is locally finite.*

A primary example of a locally finite variety is the variety of distributive lattices, a fact that will be of key importance here. To establish that M is locally finite, we make use of the auxiliary operations L and R defined on M given in Chapter 1, and the description of $\sqcap$ and $\sqcup$ in terms of the operations L, R and the pointwise meet and join operations $\wedge$ and $\vee$ given in Theorem 1.4.5. Then the local finiteness of M will ultimately be obtained from the local finiteness of the distributive lattice $(\mathrm{M}, \wedge, \vee)$.

Lemma 7.3.2 *For a subset $S \subseteq \mathrm{M}$, the subalgebra of* M *generated by S is contained in the sublattice of $(\mathrm{M}, \wedge, \vee)$ generated by S', where*

$$S' = \{f, f^* f^L, f^R, (f^*)^L, (f^*)^R, f^{LR} : f \in S\} \cup \{1_0, 1_1, 1_0^L\}$$

The proof of this lemma is found in [46], and is left here as a recommended exercise. This result has a number of interesting consequences.

Theorem 7.3.3 *The algebra $\mathrm{M} = (\mathrm{M}, \sqcap, \sqcup, ^*, 1_0, 1_1)$ is locally finite with a finite uniform upper bound on the size of a subalgebra in terms of the size of a generating set.*

Proof. Let S be a subset of M with n elements. By Lemma 7.3.2, the subalgebra of M generated by S is contained in the sublattice of $(\mathrm{M}, \wedge, \vee)$ generated by $S' = \{f, f^*, f^L, f^R, (f^*)^L, (f^*)^R, f^{LR} : f \in S\} \cup \{1_0, 1_1, 1_0^L\}$. Since S' has at most $7n + 3$ elements, the sublattice of the distributive lattice $(\mathrm{M}, \wedge, \vee)$ generated by S', has at most $2^{2^{7n+3}}$ elements. ■

Corollary 7.3.4 *The variety $\mathcal{V}(\mathrm{M})$ generated by* M *is locally finite.*

Proof. This is an immediate consequence of the previous theorem since M is locally finite with a uniform upper bound on the size of a subalgebra in terms of the size of its generating set [5]. ■

Corollary 7.3.5 *There are equations satisfied by* M *that are not consequences of equations (1)–(7) on page 129, and there are equations satisfied by $(\mathrm{M}, \sqcap, \sqcup)$ that are not consequences of equations (1)–(4) on page 129.*

Proof. If all equations satisfied by M were consequences of these equations, then $\mathcal{V}(\mathrm{M})$ would consist of all the algebras satisfying these equations. Since this variety is locally finite, it would follow that any algebra satisfying equations (1)–(7) would be locally finite. But ortholattices satisfy these equations, and not all ortholattices are locally finite [8]. A similar argument holds when we restrict to the operations $\sqcup, \sqcap$ since the variety $\mathcal{V}(\mathrm{M}, \sqcap, \sqcup)$ must also be locally finite, and not all the algebras satisfying equations (1)–(4) are locally finite, since all lattices satisfy (1)–(4), but not all lattices are locally finite [10]. ∎

We will soon produce an explicit equation that is valid in $\mathcal{V}(\mathrm{M})$ that is not a consequence of (1)–(7), and an equation valid in $\mathcal{V}(\mathrm{M}, \sqcap, \sqcup)$ that is not a consequence of (1)–(4). In fact, the same equation will do both jobs.

7.4 A syntactic decision procedure

We first recall one of the two primary decision procedures used in classical propositional logic. Propositional expressions are terms in the language of Boolean algebras. Each term is equivalent to one that is a join of elements, each of which is a meet of either variables or complements of variables

$$(a_{11} \wedge \cdots \wedge a_{1n_1}) \vee \cdots \vee (a_{k1} \wedge \cdots \wedge a_{kn_k})$$

Here each a_{ij} is a **literal**, meaning it is either a variable or the complement of a variable. This type of expression is called a **disjunctive normal form**. Its dual, an expression that is a meet of elements, each of which is a join of literals, is called a **conjunctive normal form**. The following is a well-known result of basic importance [22].

Theorem 7.4.1 *For each term t in the language of Boolean algebras, there is a term s in disjunctive normal form with the equation $s = t$ valid in all Boolean algebras. Further, up to obvious permutations, this term s is unique.*

This gives a decision procedure to determine when an equation $t_1 = t_2$ is valid in all Boolean algebras, meaning that the propositional expressions t_1 and t_2 are logically equivalent. One finds the terms s_1 and s_2 in disjunctive normal form equivalent to t_1 and t_2, and sees whether they agree. (See Exercise 8 and Exercise 9.) We now consider a syntactic decision procedure to determine when an equation holds in $\mathcal{V}(\mathrm{M}, \sqcap, \sqcup)$.

Definition 7.4.2 *Let V be a set of variables, and define a set V' whose elements are called **literals** by setting $V' = \{x, x^L, x^R, x^{LR} : x \in V\}$.*

Basic formulas from Chapter 1 expressing $\sqcap$ and $\sqcup$ in terms of pointwise meet $\wedge$ and join $\vee$ and the operations L and R allow any term in the operations $\sqcap$ and $\sqcup$ to be written as an equivalent term in the operations $\wedge, \vee, L, R$. Assuming the terms s and t have been rewritten in such a way, we seek a decision procedure to determine when $s \leq t$ under the pointwise order $\leq$ of M. The key step is the following lemma. Here we note that $x^L \vee x^R = x^{LR}$ holds in M, so any occurrence of $x^L \vee x^R$ may be replaced by x^{LR}.

Lemma 7.4.3 *Suppose the term m is a pointwise meet of literals and the term j is a pointwise join of literals. Then $m \leq j$ if and only if there is a variable $x \in V$ for which at least one of the following holds.*

1. *x occurs in m and one of x, x^L, x^R, x^{LR} occurs in j.*
2. *x^L occurs in m and one of x^L, x^{LR} occurs in j.*
3. *x^R occurs in m and one of x^R, x^{LR} occurs in j.*
4. *x^{LR} occurs in m and x^{LR} occurs in j.*

Proof. From the definitions of L and R, we see that $x \leq x^L$ and $x^R \leq x^{LR}$ hold in m. So any one of these conditions is sufficient to ensure $m \leq j$. For the converse, suppose that for each variable x, none of these conditions apply. For each variable x, we produce an element $f_x \in$ M so that when the variables are assigned so that x is interpreted as f_x, we have that m evaluates to a function taking value 1 at $\frac{1}{2}$ and j evaluates to a function taking value 0 at $\frac{1}{2}$.

(a) If x occurs in m, set $f_x = 1_{\{\frac{1}{2}\}}$, the characteristic function of the singleton $\{\frac{1}{2}\}$. (In this case none of x, x^L, x^R, x^{LR} occur in j.)

(b) If x does not occur in m and both x^L, x^R do occur in m, set $f_x = 1_{\{\frac{1}{4},\frac{3}{4}\}}$. (In this case, none of x^L, x^R, x^{LR} can occur in j, but perhaps x does occur in j.)

(c) If neither x, x^R occurs in m and x^L occurs in m, set $f_x = 1_{\{\frac{1}{4}\}}$. (In this case at most x, x^R occur in j.)

(d) If neither x, x^L occurs in m and x^R occurs in m, set $f_x = 1_{\{\frac{3}{4}\}}$. (In this case at most x, x^L occur in j.)

(e) If none of x, x^L, x^R occur in m, and x^{LR} does occur in m, and x^L does not occur in j, set $f_x = 1_{\{\frac{1}{4}\}}$. (In this case, x^{LR} cannot occur in j.)

(f) If none of x, x^L, x^R occur in m, and x^{LR} does occur in m, and x^R does not occur in j, set $f_x = 1_{\{\frac{3}{4}\}}$. (In this case, x^{LR} cannot occur in j. Note also that if x^{LR} occurs in m, at most one of x^L, x^R occurs in j because we agreed to replace $x^L \vee x^R$ with x^{LR} in j.)

(g) Finally, if none of x, x^L, x^R, x^{LR} occur in m, let f_x be the constant function 0.

It is now straightforward to check that when the variables are assigned so that x is interpreted as f_x, we have that m evaluates to a function taking value 1 at $\frac{1}{2}$ and that j evaluates to a function taking value 0 at $\frac{1}{2}$. Thus, if none of conditions (1)–(4) apply, we have $m \nleq j$. ∎

Theorem 7.4.4 *There is an algorithm to decide when an equation $s = t$ holds in $(m, \sqcap, \sqcup)$.*

Proof. First, transform s and t into terms s' and t' in the operations $\wedge, \vee, L, R$ with s equivalent to s' and t equivalent to t'. Then use the fact that M is a distributive lattice under pointwise meet and join to transform s' into a term s'' that is a join of meets of literals, and to transform t' into a term t'' that is a meet of joins of literals. Then s' is equivalent to s'' and t' is equivalent to t''. Also, $s'' \leq t''$ if and only if for each meet of literals M that comprises s'', and each join of literals j that comprises t'', we have $m \leq j$. Use Lemma 7.4.3 to determine this. We thus have an algorithm to determine if $s' \leq t'$, and we can use this also to determine if $t' \leq s'$. Having both $s' \leq t'$ and $t' \leq s'$ is equivalent to having s' equivalent to t', hence s equivalent to t. ∎

Remark 7.4.5 The above techniques can be adapted to give a syntactic algorithm to determine when an equation $s = t$ holds in M. Set the literals to be $V' = \{x, x^L, x^R, x^*, x^{*L}, x^{*R}, x^{LR} : x \in V\} \cup \{1_0, 1_1, 1_0^L\}$. Then s can be written as a join of meets of literals, and t can be written as a meet of joins of literals. So it suffices to determine when $m \leq j$ for m a meet of literals and j a join of literals. The supply of literals is now quite rich, so this leads to a sizable number of cases. This is left as an extended exercise for the reader.

Much of the remainder of this chapter is devoted to establishing a simple semantic decision procedure to determine the validity of equations in M and in $(\mathrm{M}, \sqcap, \sqcup)$. This is a method akin to the familiar truth tables of propositional logic. While the final result is quite tidy, it requires some effort.

7.5 The algebra E of sets in M

In this section we begin the process of constructing M from simpler pieces. Our first step is the following.

Definition 7.5.1 *Let* E *be the set of all elements of* M *taking values in the two-element set* $\{0, 1\}$*. In other words,* E *is the set of all characteristic functions of subsets of the interval* I.

Proposition 7.5.2 *The subset* E *of* M *is closed under the operations* $\sqcap, \sqcup, ^*, 1_0, 1_1, \wedge, \vee, L, R$. *Thus* $\mathrm{E} = (\mathrm{E}, \sqcap, \sqcup, ^*, 1_0, 1_1)$ *is a subalgebra of* M.

Proof. It is well known that E is closed under pointwise meet and join $\wedge, \vee$. It is closed under * since applying * to the characteristic function of a set A produces the characteristic function of $\{1 - x : x \in A\}$. It clearly is closed under L, R. Since $\sqcap, \sqcup$ can be expressed in terms of $\wedge, \vee, L, R$ it follows that it is closed under these operations also. ■

Definition 7.5.3 *For each a with $0 \le a < 1$, we define $\varphi_a : \mathrm{M} \to \mathrm{M}$ as follows:*

$$(\varphi_a(f))(x) = \begin{cases} 1 & \textit{if } a < f(x)) \\ 0 & \textit{otherwise} \end{cases}$$

Proposition 7.5.4 *For $0 \le a < 1$ the map φ_a is a homomorphism with respect to all of the operations $\sqcap, \sqcup, ^*, 1_0, 1_1, \wedge, \vee, L, R$.*

Proof. We abbreviate φ_a to φ. Clearly $\varphi(1_0) = 1_0$ and $\varphi(1_1) = 1_1$. Next, note that the following are equivalent:

$$\begin{aligned} &\varphi(f \wedge g)(x) = 1 \\ &f(x) \wedge g(x) > a \\ &f(x) > a \text{ and } g(x) > a \\ &\varphi(f)(x) = 1 \text{ and } \varphi(g)(x) = 1 \\ &(\varphi(f) \wedge \varphi(g))(x) = 1 \end{aligned}$$

It follows that $\varphi(f \wedge g) = \varphi(f) \wedge \varphi(g)$. The arguments for the operations $\vee, ^*, L, R$ are similar, and are left as an exercise. Finally, since $\sqcap$ and $\sqcup$ can be expressed in terms of $L, R, \wedge, \vee$, it follows that φ preserves these also. ■

A homomorphism from an algebra A to itself is called an **endomorphism**. An endomorphism $\varphi : A \to A$ is called a **retraction** if $\varphi \circ \varphi = \varphi$. We now have the following result, first observed in [112] for the case $a = 0$.

Corollary 7.5.5 *For each $0 \le a < 1$, the map φ_a is a homomorphism from* M *onto* E. *In fact, it is a retraction.*

Proof. The above result shows that φ_a is a homomorphism. By definition, for any $f \in \mathrm{M}$, we have $\varphi_a(f)$ takes only the values $0, 1$, so belongs to E. It is routine to show that if $f \in \mathrm{E}$, then $\varphi_a(f) = f$. So φ_a is a retraction, and hence is onto E. ■

Theorem 7.5.6 *The algebras* M *and* E *generate the same variety.*

Proof. As E is a subalgebra of M, we have $\mathcal{V}(\mathrm{E}) \subseteq \mathcal{V}(\mathrm{M})$. For the other containment, consider the product map,

$$\prod_{a \in [0,1)} \varphi_a : \mathrm{M} \longrightarrow \prod_{a \in [0,1)} \mathrm{E}$$

By general considerations, this map is a homomorphism. Suppose that $f, g \in \mathrm{M}$ with $f \neq g$. Let x be such that $f(x) \neq g(x)$, and pick a strictly between $f(x)$ and $g(x)$. Then $\varphi_a(f)(x) \neq \varphi_a(g)(x)$. It follows from Exercise 5 that the product map is an embedding. So M is isomorphic to a subalgebra of a power of E, showing $\mathcal{V}(\mathrm{M}) \subseteq \mathcal{V}(\mathrm{E})$. ■

7.6 Complex algebras of chains

In this section, we give an alternate way to view the algebra E. The idea is standard, and comes from the complex algebra 2^G of a group G. This consists of all subsets of G with the operations on these subsets given by: $A \circ B = \{a \circ b : a \in A, b \in B\}$, $A^{-1} = \{a^{-1} : a \in A\}$ and $e = \{e\}$, where e is the identity of the group.

Definition 7.6.1 *For a bounded chain $(C, \wedge, \vee, 0, 1)$, the* ***complex algebra*** *2^C consists of all subsets of C with constants $1_0 = \{0\}$, $1_1 = \{1\}$, and binary operations $\sqcap, \sqcup$ given by*

$$A \sqcap B = \{a \wedge b : a \in A, b \in B\}$$
$$A \sqcup B = \{a \vee b : a \in A, b \in B\}$$

An **involution** on a chain is an order inverting map $'$ from the chain to itself of period two.

Definition 7.6.2 *For a bounded chain C with involution, its complex algebra 2^C is as above with a unary operation * given by $A^* = \{a' : a \in A\}$.*

Just as it is fruitful to consider auxiliary operations on M, it is useful also to consider additional operations on these complex algebras of chains.

Definition 7.6.3 *If C is a bounded chain with or without involution, the binary operations $\wedge, \vee$ on its complex algebra 2^C are set intersection and set union, and the unary operations L, R on this complex algebra are upset and downset, respectively. Specifically,*

$$A^L = \{c : a \le c \text{ for some } a \in A\}$$
$$A^R = \{c : c \le a \text{ for some } a \in A\}$$

Lemma 7.6.4 *In the complex algebra 2^C of a chain C, the following hold:*

1. $A \sqcap B = (A \cup B) \cap A^R \cap B^R$.
2. $A \sqcup B = (A \cup B) \cap A^L \cap B^L$.

Proof. Let $c \in A \sqcap B$. Then $c = a \wedge b$ for some $a \in A$ and $b \in B$. Without loss of generality, assume that $a \leq b$, so $c = a$ and $c \leq b$. Then $c \in (A \cup B) \cap A^R \cap B^R$. Conversely, suppose $c \in (A \cup B) \cap A^R \cap B^R$. Without loss of generality, assume that $c \in A$. So $c = a$ for some $a \in A$, and since $c \in B^R$, then $c \leq b$ for some $b \in B$. Then $c = a \wedge b$, so $c \in A \sqcap B$. The argument for $A \sqcup B$ is similar. ■

The following result is a consequence of the fact that an equation $p = q$ that is satisfied by A and has each variable occur the same number of times on each side of the equation is also satisfied by the complex algebra of A. See Exercise 12.

Proposition 7.6.5 *The complex algebra of a bounded chain with involution satisfies equations (1)–(7) given at the beginning of Section 2.*

As a first step to link complex algebras with our earlier considerations, note the following.

Proposition 7.6.6 *The subalgebra* E *of* M *of all characteristic functions of sets is isomorphic to the complex algebra of the unit interval* I *with the standard negation* $x' = 1 - x$.

Proof. The map that sends a set A to its characteristic function 1_A is a bijection from 2^{I} to E that preserves $\wedge, \vee$. Since $1_{\{0\}}$ and $1_{\{1\}}$ are the constants of E, this bijection preserves the constants as well. Since $(1_A)^*(x) = 1_A(1-x)$, it follows that $(1_A)^* = 1_{A^*}$, so * is preserved. Since $(1_A)^L$ is the least increasing function above 1_A and A^L is the upset of A, it follows that $(1_A)^L = 1_{A^L}$. Similarly $(1_A)^R = 1_{A^R}$. Since this bijection preserves $L, R, \wedge, \vee$, and $\sqcap, \sqcup$ are expressed in terms of these operations in both the complex algebra and in E, it follows that these operations are preserved. ■

One more link is needed to connect more fully the algebra E to complex algebras of chains.

Proposition 7.6.7 *If C and D are bounded chains, or bounded chains with involution, and $\varphi : C \to D$ is a homomorphism, then $\varphi[\,\cdot\,] : 2^C \to 2^D$ is a homomorphism where $\varphi[A]$ is the image of the set A under the map φ.*

Proof. For $A, B \subseteq C$ we have $\varphi[A \sqcap B] = \{\varphi(a \wedge b) : a \in A, b \in B\}$. Since φ is a homomorphism, this equals $\{\varphi(a) \wedge \varphi(b) : a \in A, b \in B\}$, which is $\varphi[A] \sqcap \varphi[B]$. The arguments for the operations $\sqcup, ^*, 1_0, 1_1$ are similar. ■

7.7 Varieties and complex algebras of chains

Here we show that the variety generated by M is generated also by the complex algebra of a 5-element chain with involution, and that the variety generated by the reduct of M obtained by omitting the negation * is generated by the complex algebra of a 3-element chain.

Definition 7.7.1 *Let* 3 *be the three-element bounded chain* $0 < u < 1$.

Theorem 7.7.2 *For C a bounded chain with at least three elements, the variety $\mathcal{V}(2^C)$ generated by the complex algebra of C equals the variety $\mathcal{V}(2^3)$ generated by the complex algebra of* 3.

Proof. Suppose C is a bounded chain with at least three elements and bounds $0, 1$. Clearly there is an embedding $\varphi : 3 \to C$. By Proposition 7.6.7, $\varphi[\cdot] : 2^3 \to 2^C$ is a homomorphism, and since φ is an embedding, so is this map. This shows $\mathcal{V}(2^3) \subseteq \mathcal{V}(2^C)$.

For each $c \in C$, define a map $\varphi_c : C \to 3$. If $c \neq 0, 1$ set

$$\varphi_c(x) = \begin{cases} 1 & \text{if } x > c \\ u & \text{if } x = c \\ 0 & \text{if } x < c \end{cases}$$

If c is either of the bounds $0, 1$, set $\varphi_c(0) = 0, \varphi_c(1) = 1$, and $\varphi_c(x) = u$ for all $0 < x < 1$. Then each $\varphi_c : C \to 3$ is a homomorphism of bounded chains. By Proposition 7.6.7, it follows that each $\varphi_c[\cdot] : 2^C \to 2^3$ is a homomorphism, so the product map $2^C \to (2^3)^C$ is a homomorphism. We show this map is an embedding. Suppose A, B are subsets of C with $A \neq B$. We may assume some c belongs to A and not to B. If $c \neq 0, 1$, we have u belongs to $\varphi_c[A]$ and not to $\varphi_c[B]$. If $c = 0$, then 0 belongs to $\varphi_c[A]$ and not to $\varphi_c[B]$, and if $c = 1$, then 1 belongs to $\varphi_c[A]$ and not to $\varphi_c[B]$. So this product map is an embedding, and this shows $\mathcal{V}(2^C) \subseteq \mathcal{V}(2^3)$. ■

Theorem 7.7.3 *The varieties $\mathcal{V}((\mathrm{M}, \sqcap, \sqcup, 1_0, 1_1))$, $\mathcal{V}((\mathrm{E}, \sqcap, \sqcup, 1_0, 1_1))$, and $\mathcal{V}(2^3)$ are equal.*

Proof. Note that $(\mathrm{E}, \sqcap, \sqcup, 1_0, 1_1)$ is the algebra E without the operation *, and $(\mathrm{M}, \sqcap, \sqcup, 1_0, 1_1)$ is the algebra M without *. Theorem 7.5.6 gives $\mathcal{V}(\mathrm{M}) = \mathcal{V}(\mathrm{E})$. This means that M and E satisfy the same equations in the operations $\sqcap, \sqcup, ^*, 1_0, 1_1$, so they satisfy the same equations in the operations $\sqcap, \sqcup, 1_0, 1_1$. This means that $\mathcal{V}((\mathrm{E}, \sqcap, \sqcup, 1_0, 1_1)) = \mathcal{V}((\mathrm{M}, \sqcap, \sqcup, 1_0, 1_1))$. In Proposition 7.6.6, we showed E is isomorphic to 2^{I} where I is the unit interval considered as a bounded chain with the standard negation. So $(\mathrm{E}, \sqcap, \sqcup, 1_0, 1_1)$ is isomorphic to 2^{I} where I is considered as just a bounded chain. The remainder of the theorem then follows immediately from Theorem 7.7.2. ■

We turn our attention to the matter of chains with involutions. Here the proofs are similar to the ones just given, but a bit lengthier. We refer the reader to [46] for details.

Definition 7.7.4 *Let* 5 *be the five-element bounded chain* $0 < p < q < r < 1$ *with the only possible involution* $'$.

Theorem 7.7.5 *For C any bounded chain with involution having at least five elements, the variety $\mathcal{V}(2^C)$ equals the variety $\mathcal{V}(2^5)$.*

Theorem 7.7.6 *The varieties $\mathcal{V}(\mathrm{M})$, $\mathcal{V}(\mathrm{E})$, and $\mathcal{V}(2^5)$ are equal, and they are equal to the variety $\mathcal{V}(2^C)$ where C is any bounded chain with involution having at least five elements.*

Corollary 7.7.7 *The variety $\mathcal{V}(\mathrm{M})$ is generated by a single finite algebra and therefore is locally finite.*

7.8 The algebras 2^3 and 2^5 revisited

Before putting the results of the previous section to use, we shall refine them further. This requires a more nuts and bolts consideration of the complex algebras 2^3 and 2^5. We begin with 2^3. This is an 8-element algebra whose structure is given below.

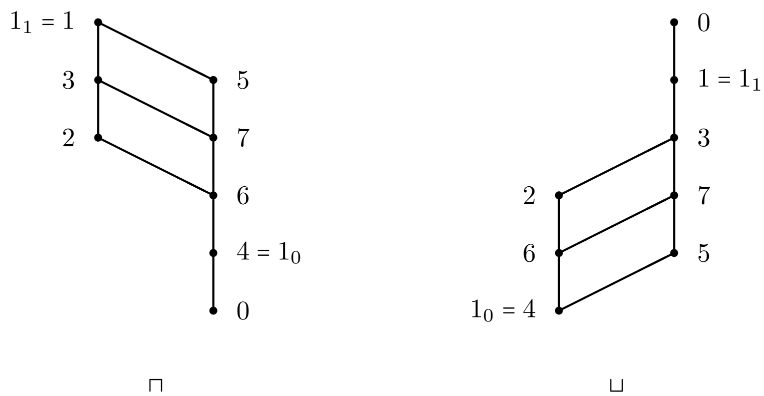


FIGURE 7.1: The algebra 2^3

To understand this figure, note that the algebra 2^3 has operations $\sqcap, \sqcup, 1_0, 1_1$. Each of the operations $\sqcap, \sqcup$ is a semilattice operation, so can be drawn as a poset. The operation $\sqcap$ is the meet operation in the poset at left, and $\sqcup$ is the join operation of the poset at right. The constants $1_0, 1_1$ are shown in the figure. The elements of 2^3 are the subsets of the set $\{0, u, 1\}$. We first represent a subset such as $\{0, u\}$ by a triple giving its characteristic function 110. Here a 1 on the spot farthest left means 0 is in the set, a 1 in the middle spot means u is in the set, and a 1 in the rightmost spot means 1 is in the set. This represents the eight elements of 2^3 as the binary representations of the numbers $0, \ldots, 7$. We convert these binary numbers to their decimal counterparts to label the figure. So the subset $\{0, u\}$ is converted to 110, and finally to 6.

Lemma 7.8.1 *Consider the equivalence relations $\theta_1, \theta_2, \theta_3$ on 2^3 that have the associated partitions:*

$$\begin{aligned}
\theta_1 &:: \{0\}, \{1, 2, 3\}, \{4, 5, 6, 7\} \\
\theta_2 &:: \{0\}, \{1, 3, 5, 7\}, \{2, 4, 6\} \\
\theta_3 &:: \{0\}, \{1\}, \{2, 3, 6, 7\}, \{4\}, \{5\}
\end{aligned}$$

Then $\theta_1, \theta_2, \theta_3$ are congruences and intersect to the identical relation.

This lemma was obtained using the Universal Algebra Calculator, a software package to work with general algebras. It allows us to sharpen the results of the previous section as follows.

Theorem 7.8.2 *The variety $\mathcal{V}((\mathrm{M}, \sqcap, \sqcup, 1_0, 1_1))$ is generated by the five-element algebra in Figure 7.2.*

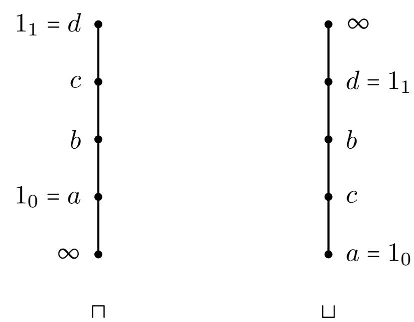


FIGURE 7.2: A five-element “chain”

Proof. Since $\theta_1 \cap \theta_2 \cap \theta_3$ intersect to the identical relation, we have that 2^3 is isomorphic to a subalgebra of the product $2^3/\theta_1 \times 2^3/\theta_2 \times 2^3/\theta_3$. Upon computing these quotients, the first two are isomorphic to subalgebras of the third. It follows that the variety generated by 2^3 equals the variety generated by $2^3/\theta_3$, and it is this quotient that is shown in Figure 7.2. ■

We next consider the situation when the constants $1_0, 1_1$ are no longer considered as basic operations of the algebra. Our aim is to find a generator for the variety $\mathcal{V}((\mathrm{M}, \sqcap, \sqcup))$. We consider the algebra in Figure 7.3. It is obtained by removing the element ∞ from the algebra in Figure 7.2, and by no longer considering the constants $1_0, 1_1$ to be basic operations.

Theorem 7.8.3 *The variety $\mathcal{V}((\mathrm{M}, \sqcap, \sqcup))$ is generated by the 4-element algebra in Figure 7.3.*

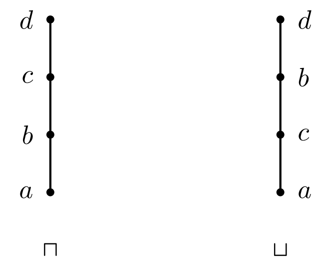


FIGURE 7.3: A four-element "chain"

Proof. Let A be the algebra in Figure 7.2, A' be the algebra obtained from A by not considering $1_0, 1_1$ to be basic operations, and let B be the algebra of Figure 7.3. Theorem 7.8.2 gives that $\mathcal{V}((\mathrm{M}, \sqcap, \sqcup, 1_0, 1_1)) = \mathcal{V}(A)$, so by the same argument as in Theorem 7.7.3, $\mathcal{V}((\mathrm{M}, \sqcap, \sqcup)) = \mathcal{V}(A')$. We show $\mathcal{V}(A') = \mathcal{V}(B)$. Since B is a subalgebra of A', $\mathcal{V}(B) \subseteq \mathcal{V}(A')$. For the other containment, let C be the subalgebra of B consisting of the elements $\{b, c\}$. Consider the algebra $B \times C$, and note that there is a congruence θ on this algebra with $\{(a,b), (b,b), (c,b), (d,b)\}$ as its only non-trivial equivalence class. Then the quotient $(B \times C)/\theta$ is isomorphic to A', giving $\mathcal{V}(A') \subseteq \mathcal{V}(B)$. ∎

We turn to the case of the variety generated by M. We have seen that this variety is generated by the complex algebra 2^5 of a 5-element chain with involution. However, this complex algebra is still quite complicated. It has 32 elements and a rather intricate structure. Using techniques similar to those above, we can simplify matters [46]. Consider the algebra in Figure 7.4.

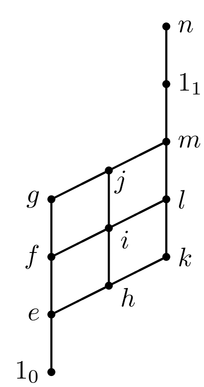


FIGURE 7.4: A 12-element algebra

This figure shows the $\sqcup$ operation as the join operation of this semilattice.

The operation * is determined by having $f^* = f$, $j^* = j$, $k^* = k$, $n^* = n$, and having the period two * interchange the elements $1_0^* = 1_1$, $e^* = l$, $g^* = i$, and $h^* = m$. Then $\sqcap$ is given by the equation $x \sqcap y = (x^* \sqcup y^*)^*$.

Theorem 7.8.4 *The variety $\mathcal{V}(\mathrm{M})$ is generated by the 12-element algebra in Figure 7.4.*

These results, showing that the variety $\mathcal{V}(\mathrm{M})$ is generated by the twelve-element algebra above, that $\mathcal{V}((\mathrm{M}, \sqcap, \sqcup, 1_0, 1_1))$ is generated by the five-element "chain," and that $\mathcal{V}((\mathrm{M}, \sqcap, \sqcup))$ is generated by the four-element "chain" give improved methods to determine if certain kinds of equations hold. For an equation $s \approx t$ involving only the operations $\sqcap, \sqcup$, it is enough to determine if it holds in the four-element "chain." This requires testing every possible combination of these four elements for the variables. For an equation using the operations $\sqcap, \sqcup$ and the constants $1_0, 1_1$, we must test it in the five-element "chain," and for one using * as well, we must test it in the twelve-element algebra. This is an extension of the familiar method of truth tables from classical propositional logic. We conclude with an application of this technique.

Definition 7.8.5 *The* ***doubled distributive law*** *is the equation*

$$[x \sqcap (y \sqcup z)] \sqcap [(x \sqcap y) \sqcup (x \sqcap z)] = [x \sqcap (y \sqcup z)] \sqcup [(x \sqcap y) \sqcup (x \sqcap z)]$$

The doubled distributive law is obtained in an obvious syntactic manner from the usual distributive law; it is $p \sqcap q = p \sqcup q$ where $p = q$ is the usual distributive law. It is easily seen that a lattice satisfies the doubled distributive law if and only if it satisfies the usual distributive law.

Proposition 7.8.6 *The doubled distributive law is an equation that is valid in* M *that is not a consequence of the equations (1)–(7) listed at the start of Section 7.2.*

Proof. To see that the doubled distributive law is valid in M one must only check that it is valid in the 4-element algebra of Figure 7.3. That this equation is not a consequence of equations (1)–(7) follows since there is an orthocomplemented lattice that is not distributive. This will satisfy (1)–(7), but not the generalized distributive law. ∎

Definition 7.8.7 *The* ***symmetric distributive law*** *is the equation*

$$(x \sqcup y) \sqcap (x \sqcup z) \sqcap (y \sqcup z) = (x \sqcap y) \sqcup (x \sqcap z) \sqcup (y \sqcap z)$$

The symmetric distributive law is equivalent to each of the join distributive law and the meet distributive law in every lattice, but not in every Birkhoff system. Using the method of truth tables, one can verify the following.

Proposition 7.8.8 *The symmetric distributive law is an equation that is valid in* M *that is not a consequence of the equations (1)–(7) listed at the start of Section 7.2.*

In Chapter 3, we gave a fairly wide array of subalgebras of M that were lattices. Each turned out to be a De Morgan algebra. This was not an accident.

Corollary 7.8.9 *Every subalgebra of* $(\mathrm{M}, \sqcap, \sqcup)$ *that is a lattice is a distributive lattice. Therefore every subalgebra of* M *whose* $\sqcap, \sqcup$ *reduct is a lattice is a De Morgan algebra.*

7.9 Summary

We have shown that the variety generated by $(\mathrm{M}, \sqcap, \sqcup, {}^*, 1_0, 1_1)$ is equal to the variety generated by its subalgebra E, where E is the algebra that consists of characteristic functions of sets. This variety is also generated by the complex algebra of any bounded chain with involution that has at least 5 elements, as well as by an algebra with 12 elements.

The varieties generated by $(\mathrm{M}, \sqcap, \sqcup, 1_0, 1_1)$ and $(\mathrm{M}, \sqcap, \sqcup)$ are generated by the corresponding reducts of E, as well as by the complex algebra of any bounded chain, or chain, with at least 3 elements. The variety generated by $(\mathrm{M}, \sqcap, \sqcup, 1_0, 1_1)$ is generated by the complex algebra of a 5-element algebra that is a bichain with constants, and the variety generated by $(\mathrm{M}, \sqcap, \sqcup)$ is generated by a 4-element bichain.

These results allow a decision procedure to determine when an equation holds in M—one tests to see whether it holds in the 12-element algebra that generates the same variety as M. If the equation does not use negation or the constants, one can test whether it holds in the 4-element bichain that generates the same variety as the reduct $(\mathrm{M}, \sqcap, \sqcup)$. This is analogous to the method of truth tables to determine whether an equation is a classical tautology. A syntactic decision procedure to determine whether an equation holds is also given.

7.10 Exercises

1. Give an example of an algebra $(G, *)$ with a single binary operation $*$ that is a group in the usual sense of the term, but that has a subalgebra that is not a group in the usual sense of the term.

2. Define the variety of rings with unit; that is, give the type of these algebras, and equations defining the variety.

3. Prove that if $(G, *, ^{-1}, e)$ is a group and N is a normal subgroup of G, then $\theta_N = \{(x, y) : xyx^{-1}y^{-1} \in N\}$ is a congruence on G.

4. Give all congruences on the Klein-4 group $(G, *, ^{-1}, e)$. Hint: There are five of them.

5. Suppose A is an algebra, and that for each $i \in I$ that $\varphi_i : A \to A_i$ is a homomorphism. Let $\varphi : A \to \prod_I A_i$ be the homomorphism given by $\varphi(a)(i) = \varphi_i(a)$. We say this family of homomorphisms separates points if for each $a, b \in A$ with $a \neq b$ there is $i \in I$ with $\varphi_i(a) \neq \varphi_i(b)$. Show that if this family of homomorphisms separates points, then φ is an embedding.

6. Prove that the equations $1_1^* = 1_0$, $1_0^* = 1_1$, $(f \sqcup g) \sqcap (f \sqcap g) = f \sqcap g$, and $(f \sqcup g) \sqcup (f \sqcap g) = f \sqcup g$ follow from the equations (1)–(7) on page 129.

7. Prove Lemma 7.3.2 by showing that the closure of S under the operations $\sqcap, \sqcup, ^*, 1_0, 1_1$ and L, R is contained in the sublattice of $(\mathrm{M}, \wedge, \vee)$ generated by S'.

8. Express the expression $x \wedge (y' \vee (x \wedge z))$ in disjunctive normal form and in conjunctive normal form.

9. Use conjunctive normal form to prove that a subalgebra of a Boolean algebra that is generated by a set with n elements has at most 2^{2^n} elements.

10. Give the decision procedure to determine when $m \leq j$ where m is a meet of literals and j is a join of literals as in Remark 7.4.5.

11. Complete the argument in Proposition 7.5.4 to show φ_a is a homomorphism.

12. Suppose A is an algebra. Show that if an equation $p = q$ has each variable occurring the same number of times on each side of the equation, then if $p = q$ is valid in A, it is valid in the complex algebra of A. Provide an example from group theory that not all equations are preserved this way.

13. Show that in a lattice the doubled distributive law and the distributive law are equivalent.

14. Use the method of truth tables to show that the doubled distributive law holds in M.

15. Use the method of truth tables to show that the symmetric distributive law holds in M.

16. Use the syntactic method of disjunctive normal form to show that the doubled distributive law holds in M.

17. Use the method of truth tables to determine whether or not the doubled modular law holds in M.

18. Use the method of truth tables to determine whether or not either distributive law holds in M.

Chapter 8

Type-2 Fuzzy Sets and Bichains

In the previous chapter, it was shown that the variety $\mathcal{V}(\mathrm{M})$ generated by M is generated by a 12-element De Morgan bisemilattice, and that the variety generated by the reduct $(\mathrm{M}, \sqcap, \sqcup)$ is generated by a 4-element bichain. This gives an algorithm similar to the method of truth tables to determine whether an equation holds in M. In this chapter, we continue the study of the variety generated by $(\mathrm{M}, \sqcap, \sqcup)$. We place this in the context of other known varieties, and try to find an equational basis for it. Such an equational basis would give an axiomatization for the $\sqcap$, $\sqcup$ fragment of the type-2 truth value algebra much as the equations defining Boolean algebras give an axiomatization of the truth value algebra of classical logic. This remains an open problem. It has been shown in the previous chapter that additional equations past the basic ones of Chapter 2 are required to define this variety, and possibly infinitely many are required. We give a conjecture involving the notion of splittings for such an equational basis, but further work on this problem is required.

8.1 Preliminaries

Congruences were introduced in Definition 7.1.5 as equivalence relations on an algebra that are compatible with its operations. It is easily seen that the intersection of a family of congruences of an algebra A is a congruence (Exercise 1). It follows that the congruences of an algebra form a lattice.

Definition 8.1.1 *For an algebra A,* $\mathrm{Con}(A)$ *is its lattice of congruences.*

An algebra is **congruence distributive** if its congruence lattice is distributive, and a variety is **congruence distributive** if each algebra in the variety is congruence distributive. An **axiomatization**, or **equational basis**, of a variety $\mathcal{V}$ is a set of equations that is satisfied by exactly those algebras of the appropriate type that belong to $\mathcal{V}$. For example, the equations given in Definition 1.1.2 axiomatize the variety of lattices. There is a deep result regarding equational axiomatizations of congruence distributive varieties due to Baker [10].

Theorem 8.1.2 *A congruence distributive variety that is generated by a finite algebra can be axiomatized by a finite set of equations.*

Every algebra A has a smallest congruence relation $\Delta = \{(a,a) : a \in A\}$ and a largest congruence relation $\nabla = \{(a,b) : a,b \in A\}$. An algebra A is **subdirectly irreducible** if there is a smallest congruence on A among those that are not equal to Δ. Subdirectly irreducible algebras are basic building blocks of varieties as is shown by the following result of Birkhoff [10].

Theorem 8.1.3 *If A belongs to a variety $\mathcal{V}$, then A is a subalgebra of a product of subdirectly irreducible algebras that belong to $\mathcal{V}$. So every variety is generated by its subdirectly irreducibles.*

In the case of congruence distributive varieties, much more is true. The following is a consequence of a result of Jónsson [10].

Theorem 8.1.4 *If A is a finite algebra and the variety $\mathcal{V}(A)$ generated by A is congruence distributive, then the subdirectly irreducible algebras in $\mathcal{V}(A)$ are homomorphic images of subalgebras of A.*

Funayama and Nakayama showed that the variety of lattices is congruence distributive [10]. Moreover, if A is any algebra that has a lattice L as a reduct, then the congruence lattice of A is a sublattice of the congruence lattice of L (Exercise 2). It follows that if each member of a variety $\mathcal{V}$ has a reduct that is a lattice, then $\mathcal{V}$ is congruence distributive. Thus De Morgan algebras and Boolean algebras are also congruence distributive varieties.

Not all varieties are congruence distributive. For example, the varieties of sets (Exercise 4) and groups are not congruence distributive. Another example is most pertinent here. The proof of the following is given as Exercises 9 and 10.

Theorem 8.1.5 *The variety of semilattices is generated by the 2-element semilattice and is not congruence distributive.*

There is another context in which lattices arise in the study of general algebras. Suppose $\mathcal{V}$ is a variety and $\mathcal{V}_j$ $(j \in J)$ is a family of varieties, each contained in $\mathcal{V}$. Then $\bigcap_J \mathcal{V}_j$ is also a subvariety of $\mathcal{V}$. This can be seen using either of the two characterizations of a variety—as a collection of algebras of the same type that is closed under homomorphic images, subalgebras, and products (Exercise 11), or as the collection of all algebras of a given type that satisfy some set of equations (Exercise 12). This yields the following.

Proposition 8.1.6 *For any variety $\mathcal{V}$, the collection of subvarieties of $\mathcal{V}$ is a lattice* $\mathrm{Sub}(\mathcal{V})$ *called the* ***lattice of subvarieties*** *of $\mathcal{V}$.*

Our primary tool to study the lattice of subvarieties will be the notion of splittings. While this is defined for any lattice, it is the application of this notion to the lattice $\mathrm{Sub}(\mathcal{V})$ of subvarieties of $\mathcal{V}$ that will be of interest here.

Definition 8.1.7 *An ordered pair (u, w) of elements of a lattice L is a* ***splitting pair*** *if for each $x \in L$, either $x \leq u$ or $w \leq x$, but not both. Thus, (u, w) is a splitting pair if u is the largest element of L that does not lie above w.*

Our interest is in splitting pairs $(\mathcal{U}, \mathcal{W})$ of the lattice $\mathrm{Sub}(\mathcal{V})$ of subvarieties of a variety $\mathcal{V}$. These are called **splitting pairs of subvarieties**. The following can be found in [57].

Proposition 8.1.8 *Let $(\mathcal{U}, \mathcal{W})$ be a splitting pair of subvarieties of $\mathcal{V}$. Then there is a subdirectly irreducible algebra S that generates $\mathcal{W}$, the variety $\mathcal{U}$ is the largest subvariety of $\mathcal{V}$ that does not contain S, and $\mathcal{U}$ is defined by the equations defining $\mathcal{V}$ and one additional equation.*

The subdirectly irreducible algebra S in Proposition 8.1.8 is a **splitting algebra** in $\mathcal{V}$, the variety $\mathcal{U}$ is the **splitting variety** of S in $\mathcal{V}$, and the additional equation defining the splitting variety is the **splitting equation** of S in $\mathcal{V}$. Our source of splitting algebras is tied to the following.

Definition 8.1.9 *An algebra P in a variety $\mathcal{V}$ is* ***weakly projective*** *in $\mathcal{V}$ if for any algebra $A \in \mathcal{V}$ and any homomorphism $f : A \to P$ onto P, there is a subalgebra B of A so that the restriction $f|_B : B \to P$ is an isomorphism.*

We note that the term "projective" without the modifier "weakly" is reserved for the notion where the homomorphism P is not necessarily onto, but just an epimorphism in the categorical sense [73].

Definition 8.1.10 *For a variety $\mathcal{V}$ and an algebra $S \in \mathcal{V}$, define $\mathcal{V}_S$ to be the class of all algebras in $\mathcal{V}$ that do not contain a subalgebra that is isomorphic to S.*

$$\mathcal{V}_S = \{A \in \mathcal{V} : S \nleq A\}$$

The following is our key result [57, Lemma 2.10, Theorem 2.11].

Theorem 8.1.11 *Let S be an algebra that is subdirectly irreducible and weakly projective in $\mathcal{V}$. Then S is a splitting algebra in $\mathcal{V}$, and $(\mathcal{V}_S, \mathcal{V}(S))$ is a splitting pair of subvarieties of $\mathcal{V}$.*

When applying this theorem, the splitting equation of S is also obtained via expressions for the elements that generate the least non-trivial congruence of the algebra S. This will be illustrated in Section 8.4.

8.2 Birkhoff systems and bichains

Bisemilattices were described in Definition 2.1.2 as algebras that have two semilattice operations. To make longer expressions more readable, we alter notation and use $\cdot$ and $+$ for these semilattice operations, and use the familiar arithmetic conventions that juxtaposition denotes $\cdot$, and that $\cdot$ binds more strongly than $+$. With these conventions, we have the following version of Definition 2.1.10.

Definition 8.2.1 *A **Birkhoff system** is an algebra $(B, \cdot, +)$ with two semilattice operations $\cdot$ and $+$ that satisfies **Birkhoff's equation***

$$x(x + y) = x + xy \tag{8.1}$$

In the preliminaries of Chapter 2, we have seen that every bisemilattice $(B, \cdot, +)$ gives rise to two partial ordering $\leq_{\cdot}$ and $\leq_{+}$ on B where

$$\begin{aligned} x \leq_{\cdot} y \quad &\text{if and only if} \quad x \cdot y = x \\ x \leq_{+} y \quad &\text{if and only if} \quad x + y = y \end{aligned}$$

Further, these two partial orderings completely describe $(B, \cdot, +)$. The order $\leq_{\cdot}$ is called the **meet order**, and the order $\leq_{+}$ is called the **join order**. When drawing diagrams of finite bisemilattices, the meet order will be drawn on the left, and the join order on the right. We note that these are partial orderings on the same underlying set B.

Definition 8.2.2 *A **bichain** is a bisemilattice where both its meet order and its join order are chains.*

Figure 8.1 shows a 4-element bichain B. Its meet order is at left, so $2 \cdot 4 = 2$, and its join order is at right, so $2 + 4 = 4$. Note that $2 \cdot 3 = 2$ and $2 + 3 = 2$. This bichain is isomorphic to the algebra in Figure 7.3 and therefore generates the same variety as $(\mathrm{M}, \sqcap, \sqcup)$. The bichain B has a particularly simple form that is described in the following proposition.

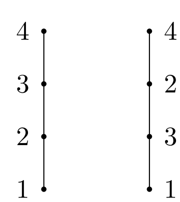


FIGURE 8.1: The 4-element bichain B

Proposition 8.2.3 *A finite bichain with n elements is isomorphic to a unique bichain whose underlying set is $\{1, \ldots, n\}$ for some n, and whose meet order is $1 <. \cdots <. n$. This is called the* ***standard form*** *of a finite bichain.*

Corollary 8.2.4 *Up to isomorphism, there are $n!$ n-element bichains.*

When drawing a diagram of a bichain in standard form, it is not necessary to show its meet order since that is determined. Figure 8.2 shows the join orders of the two standard form 2-element bichains and six standard form 3-element bichains. Note that 2_l is the 2-element lattice and in 2_s, $1 \cdot 2 = 1$ and $1 + 2 = 1$, so its two semilattice operations are equal.

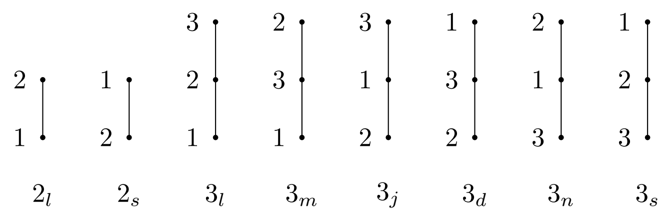


FIGURE 8.2: The 2- and 3-element bichains

Given two elements x, y of a bichain, we have $x \cdot y$ is equal to either x or y, and that $x + y$ is equal to either x or y. This immediately gives the following.

Proposition 8.2.5 *Every subset of a bichain is a subalgebra of it.*

Given a set S, any partial ordering on S where any two elements have a greatest lower bound gives a semilattice operation $\cdot$ on S, and any partial ordering where any two elements have a least upper bound gives a semilattice operation $+$ on S. Thus any two such partial orderings will give a bisemilattice structure $(S, \cdot, +)$ that has the two given orders as its meet and join orders. However, it is not the case that any two such orders will produce a bisemilattice that satisfies Birkhoff's equation. The following basic observation is therefore perhaps quite unexpected.

Proposition 8.2.6 *Every bichain is a Birkhoff system.*

Proof. Suppose that x, y are elements of some bichain. By Proposition 8.2.5, $\{x, y\}$ is a subalgebra S of this bichain. Since S has 2 elements, it is isomorphic to either 2_l or 2_s of Figure 8.2. But 2_l is a 2-element lattice, so in it $x(x + y)$ and $x + xy$ are both equal to x. In 2_s the operations $\cdot$ and $+$ are equal, so in it $x(x + y) = x + (x + y) = x + y$ and $x + xy = x + (x + y) = x + y$. In either case, Birkhoff's equation is satisfied. ■

8.3 Varieties of Birkhoff systems

One purpose of this chapter is to place the variety generated by the reduct $(\mathrm{M}, \sqcap, \sqcup)$ in the context of other known varieties. Since Birkhoff systems are defined by equations, they form a variety. There is a large literature on the variety of Birkhoff systems, and its subvarieties [76, 88, 91, 92]. For a current account, see [42, 43]. We begin by listing several equations of interest.

Definition 8.3.1 *Consider the following equations:*

(BS) $x(x+y) = x + xy$.

(SL) $xy = x + y$.

(mD) $x(y+z) = xy + xz$.

(jD) $x + yz = (x+y)(x+z)$.

The equation (BS) is Birkhoff's equation. The equation (SL) says that the two semilattice operations are equal. The equations (mD) and (jD) are known as the **meet distributive law** and the **join distributive law**.

Definition 8.3.2 *Let* BS *be the variety of Birkhoff systems and* DL *be the variety of distributive lattices. Let* SL*,* mDB*, and* jDB *be the varieties of all Birkhoff systems that satisfy* (SL), (mD), *and* (jD), *respectively, and let* DB *be the variety of all Birkhoff systems that satisfy* (mD) *and* (jD).

Each of 2_l and 3_l (Figure 8.2) is a distributive lattice, and each generates the variety DL. Similarly, each of 2_s and 3_s has its two semilattice operations agree, and each generates SL. It was shown in [76] that 3_d generates DB, that 3_m generates mDB, and that 3_j generates jDB.

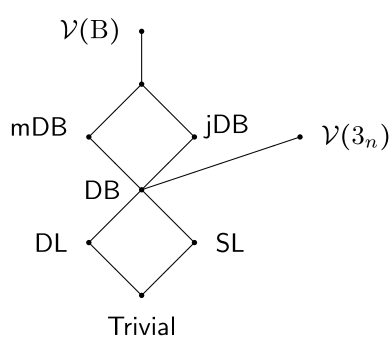


FIGURE 8.3: Containments between varieties

Figure 8.3 shows the relationships among these varieties. In [42, 43], it was shown that the portion of Figure 8.3 that does not involve $\mathcal{V}(\mathrm{B})$, correctly indicates all joins and meets, and that there are no varieties contained in any of these other than the ones indicated. These results are nontrivial since the variety of Birkhoff systems fails to be congruence distributive, and we do not address them here. We do consider our primary concern, the placement of the variety $\mathcal{V}(\mathrm{B})$ in this figure.

Proposition 8.3.3 *The variety* $\mathcal{V}(\mathrm{B})$ *properly contains* $\mathsf{mDB} \vee \mathsf{jDB}$.

Proof. It is easily seen that B has subalgebras isomorphic to 3_m and 3_j. Thus $\mathcal{V}(\mathrm{B}) \supseteq \mathsf{mDB} \vee \mathsf{jDB}$. To see that the containment is strict, consider the equation

$$(y+u)(xy+x+z) = (y+u)(yz+x+z)$$

This equation can be seen to hold in both 3_m and 3_j, and therefore holds in the variety $\mathsf{mDB} \vee \mathsf{jDB}$. However, it does not hold in B, as can be seen by substituting 1 for x, 2 for y, 3 for z and 4 for u. ■

Proposition 8.3.4 *The variety* $\mathcal{V}(\mathrm{B})$ *does not contain* $\mathcal{V}(3_n)$.

Proof. It was noted in Proposition 7.8.6 that B satisfies the following equation, called the **doubled distributive law**.

$$[x(y+z)] \cdot [xy+xz] = [x(y+z)] + [xy+xz] \tag{S}$$

This equation fails in 3_n, substituting 2 for x, 1 for y, and 3 for z. ■

There is a further variety of interest, one that lies between $\mathcal{V}(\mathrm{B})$ and the variety of all Birkhoff systems. This is BCh, the **variety generated by all bichains**. Using results of [5], Proposition 8.2.5 has the following consequence.

Proposition 8.3.5 *The variety* BCh *is locally finite.*

Since there are Birkhoff systems that are not locally finite, such as certain non-distributive lattices, it follows that BCh is a proper subvariety of BS. Thus, there must be equations that are valid in BCh that are not valid in all Birkhoff systems. The following result presents several such equations.

Proposition 8.3.6 *The following equations are valid in every bichain, but fail in some Birkhoff system.*

1. $x(xy+xz) = xy+xz$.
2. $x(x+y)(xz+y) = x(x+y)(xz+y+z)$.

Proof. Since these equations involve 3 variables, it is enough to show they are valid in every 3-generated subalgebra of a bichain. But by Proposition 8.2.5, these are exactly the 3-element bichains of Figure 8.2. With basic reasoning, such as the first equation holds if $y = 1$, this is not difficult. A Birkhoff system where the equation fails is given by the software Prover9/Mace4 [75]. ■

In [42, 43, 49], there are further equations given that are valid in BCh but not in BS. It remains an open problem to find an equational basis of BCh. It is not even known whether this variety has a finite basis. As we will see, these questions may impact the matter of finding an equational basis for $\mathcal{V}(\mathrm{B})$.

Figure 8.4 illustrates the situation of the variety $\mathcal{V}(\mathrm{B})$ in the lattice of subvarieties of BS. A more complete picture is given in [42, 43, 49]. This figure also includes the variety L of lattices, and a well-known variety Q called the variety of **quasilattices**.

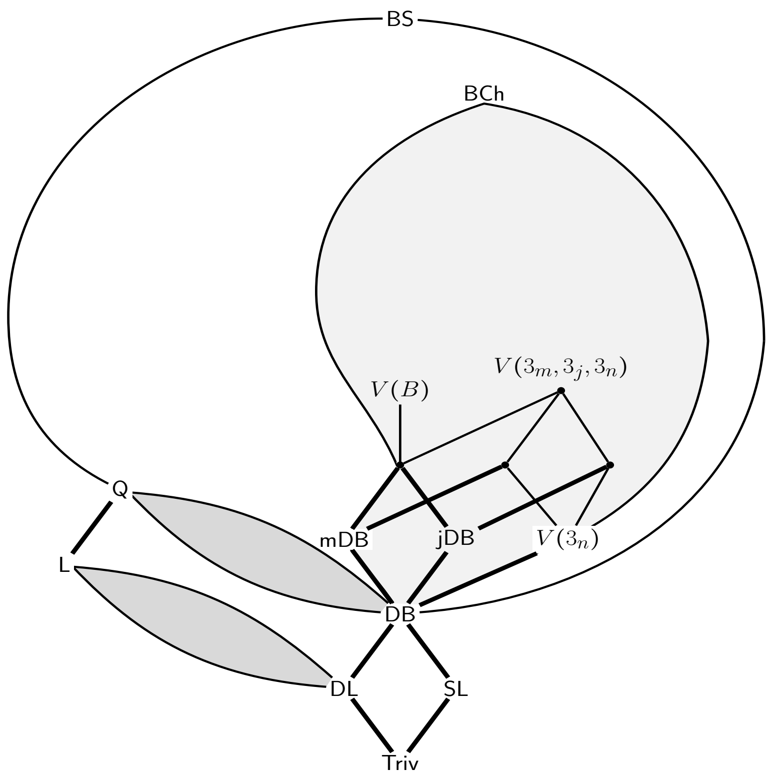


FIGURE 8.4: Large-scale view of the lattice of subvarieties of BS

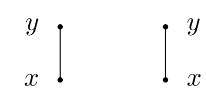
In Figure 8.4, L is the variety of lattices, and Q is the variety of quasilattices. Thick lines represent covers, and thin lines indicate proper containment. Not all joins and meets are as indicated.

8.4 Splitting bichains

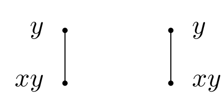
In this section, we show that 3_n is splitting in the variety BS, and hence also in BCh. So there is a largest subvariety of BCh that does not contain 3_n. This variety is defined by the equations that define BCh and the splitting equation of 3_n in BCh, which we show is the doubled distributive law. Proposition 8.3.4 shows that $\mathcal{V}(\mathrm{B})$ is a subvariety of BCh that does not contain 3_n, and we conjecture that $\mathcal{V}(\mathrm{B})$ is the splitting variety. We begin with a simpler result that illustrates the methods involved.

Proposition 8.4.1 *The 2-element distributive lattice 2_l is weakly projective in the variety* BS.

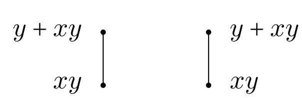
Proof. Suppose that $A \in \mathsf{BS}$ and that $f : A \to 2_l$ is a homomorphism onto 2_l. We must show that there is a subalgebra of A that is mapped isomorphically by f onto 2_l. Since f is onto, there are $x, y \in A$ with $f(x) = 1$ and $f(y) = 2$. The diagram below depicts how we would like for x, y to sit in A, but neither the meet order nor the join order is correct.



We repair the meet order by replacing x with xy. Note that $y \cdot (xy) = xy$ so the meet order is now correct. Also $f(xy) = 1$ and $f(y) = 2$. However, the situation for the join order in the following diagram is not correct.



We repair the join order by replacing y with $y + xy$. The join order is now correct since $xy + (y + xy) = y + xy$. Also, $f(xy) = 1$ and $f(y + xy) = 2$.



It is possible that the changes made to the join order have made new troubles with the meet order, but this is not the case. Birkhoff's equation gives that $xy(y + xy) = xy + xyy = xy$. Therefore $\{xy, y + xy\}$ is a subalgebra of A that is mapped isomorphically by f onto 2_l. ■

As with every 2-element algebra, 2_l is subdirectly irreducible. So by Theorem 8.1.11, 2_l is splitting in BS. The general method to find its splitting equation is as follows. Take two distinct elements of 2_l that belong to the least non-trivial congruence of 2_l, in this case 1 and 2. In the process to show that 2_l is weakly projective, formulas were found that give elements mapped to these elements, in this case xy and $y+xy$. The splitting equation is $xy = y+xy$.

Proposition 8.4.2 *The 2-element distributive lattice* 2_l *is splitting in* BS *and its splitting variety is* SL.

Proof. In any Birkhoff system $x + y + xy = x + y$ holds (Exercise 15). So the splitting equation $xy = y + xy$ implies that $x + y + xy = x + xy = xy$. Conversely, $x+y = xy$ implies that $xy = y+xy$. So the splitting equation for 2_l is equivalent in BS to the equation $x + y = xy$ that defines the variety SL. ∎

We now turn to the matter that is of primary interest to us, the splitting of 3_n. The process follows exactly that outlined above, but is a bit more complicated.

Proposition 8.4.3 *The bichain* 3_n *is weakly projective in* BS.

Proof. Suppose that $A \in \mathsf{BS}$ and that $f : A \to 3_n$ is a homomorphism onto 3_n. Since f is onto, there are $x, y, z \in A$ with $f(x) = 1$, $f(y) = 2$ and $f(z) = 3$. The diagram below depicts how we would like for x, y, z to sit in A, but neither the meet order nor the join order is correct.

$$\begin{array}{rcccl} z & \bullet & \qquad & \bullet & y \\ & | & & | & \\ y & \bullet & & \bullet & x \\ & | & & | & \\ x & \bullet & & \bullet & z \end{array}$$

We repair the meet order by replacing y with yz and x with xyz. Note that $f(xyz) = 1$, $f(yz) = 2$, and $f(z) = 3$.

$$\begin{array}{rcccl} z & \bullet & \qquad & \bullet & yz \\ & | & & | & \\ yz & \bullet & & \bullet & xyz \\ & | & & | & \\ xyz & \bullet & & \bullet & z \end{array}$$

Next, the join order is repaired. We note that f works as it should.

$$\begin{array}{rcccl} z & \bullet & \qquad & \bullet & z + yz + xyz \\ & | & & | & \\ z + yz + xyz & \bullet & & \bullet & z + xyz \\ & | & & | & \\ z + xyz & \bullet & & \bullet & z \end{array}$$

Consider again the meet order. Using Birkhoff's equation repeatedly, we obtain

$$\begin{aligned} z(z + yz + xyz) &= z(z + yz(x + yz)) \\ &= z + yz(x + yz) \\ &= z + yz + xyz \end{aligned}$$

Therefore, the meet order is repaired as follows.

$$\begin{array}{rc} z & \bullet \\ & | \\ z + yz + xyz & \bullet \\ & | \\ (z + xyz)(z + yz + xyz) & \bullet \end{array} \qquad\qquad \begin{array}{cl} \bullet & z + yz + xyz \\ | & \\ \bullet & (z + xyz)(z + yz + xyz) \\ | & \\ \bullet & z \end{array}$$

We now see that the join order is repaired. Note first that Birkhoff's equation gives $(z + xyz)(z + yz + xyz) = z + xyz + yz(z + xyz)$. So the join of the bottom and middle element of the join order is correct. For the join of the middle and top element of the join order, Birkhoff's equation yields

$$\begin{aligned} (z + yz + xyz) + (z + xyz)(z + yz + xyz) &= (z + yz + xyz)(z + yz + xyz) \\ &= z + yz + xyz \end{aligned}$$

So the join order is also correct. Thus, the elements in the final diagram form a subalgebra of A that is mapped by f isomorphically onto 3_n. So 3_n is weakly projective. ■

The algebra 3_n is subdirectly irreducible with its least nontrivial congruence collapsing the pair $(1, 2)$. So by Theorem 8.1.11, 3_n is splitting. The formulas for elements in the above proof that are mapped to 1 and 2 are $z + yz + xyz$ and $(z + xyz)(x + yz + xyz)$. So the splitting equation of 3_n in the variety BS is

$$z + yz + xyz = (z + xyz)(x + yz + xyz) \tag{8.2}$$

While this equation is not compelling, when we restrict attention to BCh matters become more interesting.

Theorem 8.4.4 *The bichain 3_n is splitting in BCh and its splitting equation in BCh is the doubled distributive law*

$$[x(y + z)] \cdot [xy + xz] = [x(y + z)] + [xy + xz]$$

Proof. Since 3_n is weakly projective in BS and it belongs to BCh, it is also weakly projective in the smaller variety BCh. It is subdirectly irreducible, and therefore is splitting in BCh, and its splitting equation in BCh is the same as its splitting equation in BS, namely (8.2). Equation (8.2) is not equivalent to the doubled distributive law in BS. But in the presence of the equations of

Proposition 8.3.6 that are valid in BCh, the software package Prover9 [75] gives proofs that (8.2) and the doubled distributive law are equivalent equations in BCh. ∎

In [47], a much more general result was established. Every finite bichain that does not contain a subalgebra isomorphic to 3_d is weakly projective in the variety BS. This has many implications in describing the lattice of subvarieties of BS [42, 43].

8.5 Toward an equational basis of $\mathcal{V}(\mathrm{M}, \sqcap, \sqcup)$

The variety $\mathcal{V}(\mathrm{B})$ is the variety generated by $(\mathrm{M}, \sqcap, \sqcup)$. While an equational basis for this variety remains an open problem, we conjecture that it is the splitting variety of 3_n in BCh. If so, Theorem 8.4.4 would provide that this variety has an equational basis, namely, an equational basis of BCh (still unknown) and the doubled distributive law. Here we give results that lend credence to this conjecture. We begin with the following technical result.

Lemma 8.5.1 *For a finite bichain C, let $C \cup \{\infty\}$ be the bichain formed from C by adding a new element to the bottom of the $\sqcap$-order and to the top of the $\sqcup$-order. Let $C \cup \{b\}$ be formed from C by adding a new element to the bottom of both orders; and let $C \cup \{t\}$ be formed from C by adding a new element to the top of both orders. Then if $C \in \mathcal{V}(\mathrm{B})$, so are $C \cup \{\infty\}$, $C \cup \{b\}$, and $C \cup \{t\}$.*

Proof. We first show that $\mathrm{B} \cup \{\infty\}$, $\mathrm{B} \cup \{b\}$, and $\mathrm{B} \cup \{t\}$ belong to $\mathcal{V}(\mathrm{B})$. Note that $\mathrm{B} \cup \{\infty\}$ is the quotient of $\mathrm{B} \times 2_s$ by the congruence θ that has one nontrivial block consisting of $\mathrm{B} \times \{1\}$, $\mathrm{B} \cup \{b\}$ is the subalgebra of $\mathrm{B} \times 2_l$ consisting of $\mathrm{B} \times \{2\}$ and $(1, 1)$, and $\mathrm{B} \cup \{t\}$ is the subalgebra of $\mathrm{B} \times 2_s$ consisting of $\mathrm{B} \times \{1\}$ and $(4, 2)$. Since 2_l and 2_s belong to $\mathcal{V}(\mathrm{B})$, so do these algebras.

Assume C belongs to $\mathcal{V}(\mathrm{B})$. Then there is a set J, a subalgebra $S \leq \mathrm{B}^J$, and an onto homomorphism $\varphi : S \to C$. Consider the constant function $\overline{\infty}$ in $(\mathrm{B} \cup \{\infty\})^J$ whose constant value is the new element ∞ added to B. In $\mathrm{B} \cup \{\infty\}$, $x \sqcap \infty = \infty$ and $x \sqcup \infty = \infty$. It follows that $S \cup \{\infty\}$ is a subalgebra of this power, and φ extends to a homomorphism from $S \cup \{\infty\}$ onto $C \cup \{\infty\}$. The arguments for $C \cup \{b\}$ and $C \cup \{t\}$ are similar, using powers of $\mathrm{B} \cup \{b\}$ and $\mathrm{B} \cup \{t\}$. ∎

To show that $\mathcal{V}(\mathrm{B})$ is the splitting variety of 3_n in BCh, by Theorem 8.1.11, we must show that if A is any algebra in BCh, then A belongs to $\mathcal{V}(\mathrm{B})$ if and only if 3_n is not a subalgebra of A. We cannot establish this for all algebras in BCh, but we can establish it for those algebras that are themselves bichains. If we knew that every subvariety of BCh was generated by the bichains that it contains, this would establish that $\mathcal{V}(\mathrm{B})$ is the splitting variety of 3_n in BCh. If

BCh were congruence distributive, this result would be simple. Unfortunately, there is no progress on this basic problem.

Theorem 8.5.2 *For a bichain C, the following are equivalent:*

1. $C \in \mathcal{V}(\mathrm{B})$.
2. 3_n *is not a subalgebra of* C.
3. C *satisfies the doubled distributive law.*

Proof. That the first condition implies the third follows from the fact that B satisfies the doubled distributive law. That the third condition implies the second follows from the fact that 3_n does not satisfy the doubled distributive law. This can be seen by taking $x = 2$, $y = 1$, and $z = 3$. The task is to show that the second condition implies the first. Assume that C is a bichain and that 3_n is not a subalgebra of C.

To show that $C \in \mathcal{V}(\mathrm{B})$, it is sufficient to show that every finite sub-bichain of C belongs to $\mathcal{V}(\mathrm{B})$. Indeed, if $C \notin \mathcal{V}(\mathrm{B})$, there is some equation valid in B that fails in C. This equation involves only finitely many variables, so there is some finitely generated subalgebra of C that does not belong to $\mathcal{V}(\mathrm{B})$. By Proposition 8.2.5, every subset of C is a subalgebra of C. So it is enough to show that every finite bichain C that does not contain a subalgebra that is isomorphic to 3_n belongs to $\mathcal{V}(\mathrm{B})$.

We show by induction on $n = |C|$ that if C does not contain a subalgebra isomorphic to 3_n then $C \in \mathcal{V}(\mathrm{B})$. For $n \leq 3$, all n-element bichains are given in the figure in the previous section, and all but 3_n have been shown to belong to $\mathcal{V}(\mathrm{B})$. Suppose C has $n \geq 4$ elements. Assume the $\sqcap$-order of C is $1 < 2 < \cdots < n$. If the bottom element of the $\sqcup$-order of C is 1, then C is isomorphic to $C' \cup \{b\}$ where C' is the sub-bichain $\{2, \ldots, n\}$ of C. Then by the inductive hypothesis and Lemma 8.5.1, $C \in \mathcal{V}(\mathrm{B})$. A similar argument handles the cases where either 1 or n is the top element of the $\sqcup$-order of C. Set

$$U = \{k : 2 \leq k \leq n \text{ and } k \text{ precedes 1 in the } \sqcup\text{-order}\}$$

$$V = \{k : 2 \leq k \leq n \text{ and 1 precedes } k \text{ in the} \sqcup\text{-order}\}$$

Since 1 is not the bottom or top of the $\sqcup$-order, U and V are non-empty. Also, since 3_n is not a subalgebra of C, if $u \in U$ and $v \in V$, then $u < v$. Also, since n is not the top element of the $\sqcup$-order, V must have at least two elements. So there is some $2 \leq k \leq n - 2$ with $U = \{2, \ldots, k\}$ and $V = \{k + 1, \ldots, n\}$.

There are congruences θ, ϕ on C with θ collapsing $\{1, \ldots, k\}$ and nothing else, and ϕ collapsing V and nothing else. Note that C/θ is isomorphic to the sub-bichain $\{1, k + 1, \ldots, n\}$ of C, and that C/ϕ is isomorphic to the sub-bichain $\{1, \ldots, k, k + 1\}$ of C. It follows from the inductive hypothesis that C/θ and C/ϕ belong to $\mathcal{V}(\mathrm{B})$. Since θ and ϕ intersect to the diagonal, C is a subalgebra of their product, so belongs to $\mathcal{V}(\mathrm{B})$. ∎

There is another path that leads to information about the equations that are valid in $\mathcal{V}(\mathrm{B})$. First a definition.

Definition 8.5.3 *For an equation $s = t$ using the operations $\cdot$ and $+$, let its* ***doubled version*** *be the equation $s \cdot t = s + t$.*

For example, what we have called the doubled distributive law

$$[x(y+z)] \cdot [xy + xz] = [x(y+z)] + [xy + xz]$$

is in fact the doubled version of the meet distributive law $x(y+z) = xy + xz$. There are other forms of distributivity that are valid in distributive lattices, such as the join distributive law. However, the meet distributive law is not equivalent to the join distributive law in BS. Indeed, one defines the variety mDB and the other jDB. Therefore, the following result is somewhat surprising.

Proposition 8.5.4 *Let $s = t$ be an equation that is valid in the 2-element distributive lattice 2_l. Then its doubled version $s \cdot t = s + t$ is valid in $\mathcal{V}(\mathrm{B})$.*

Proof. Note there is a congruence on B that collapses only the two middle elements $\{2, 3\}$, and the resulting quotient is a distributive lattice. Take any equation $s = t$ that holds in all distributive lattices. If this equation is to fail in B for some choice of elements, it must be that s and t evaluate to 2 and 3. Since $\{2, 3\}$ is a subalgebra of B that is isomorphic to the 2-element semilattice, it then follows that $s \cdot t = s + t$ holds in B. ■

8.6 Summary

In the previous chapter, we have shown that the variety generated by the truth value algebra of type-2 fuzzy sets with only its two semilattice operations in its type is generated by a 4-element algebra B that is a bichain.

In this chapter, we have investigated properties of this variety $\mathcal{V}(\mathrm{B})$. We have placed it in the context of known varieties and located it in a picture of the lattice of subvarieties of Birkhoff systems. This has a side benefit of providing information about the lattice of subvarieties of $\mathcal{V}(\mathrm{B})$.

We have shown that a certain 3-element bichain 3_n is splitting in Birkhoff systems, and conjecture that $\mathcal{V}(\mathrm{B})$ is the splitting variety of 3_n in the variety BCh generated by all bichains. If this is the case, it follows that $\mathcal{V}(\mathrm{B})$ is defined by the equations that define BCh (still unknown) and the doubled distributive law. Results are given that lend credence to this conjecture. In addition to the doubled distributive law, it is shown that $\mathcal{V}(\mathrm{B})$ satisfies the doubled version of each equation valid in distributive lattices.

It is somewhat remarkable how intractable is the variety $\mathcal{V}(\mathrm{B})$. It has good properties, such as being locally finite and a fairly simple type theory in the sense of tame congruence theory [53]. However, it is not congruence distributive, has subdirectly irreducible algebras of arbitrarily large infinite cardinality [76], and is immune to the many results describing when varieties admit a finite equational basis. Many equational properties related to this variety seem to push the limits of automated theorem provers such as Prover9 [75]. In sum, this remains an interesting area of study for both its relation to fuzzy theory, and for purely algebraic reasons.

8.7 Exercises

1. Show that if θ_i $(j \in J)$ is a family of congruences on an algebra A, then $\theta = \bigcap_J \theta_j$ is a congruence on A.
2. Suppose that $(A, (f_j)_J)$ is an algebra and θ is a congruence of this algebra. If $J' \subseteq J$, show that θ is a congruence of the reduct $(A, (f_j)_{J'})$.
3. Let X be a set. Then X can be considered as an algebra with no operations, so the congruences on X are the equivalence relations on X. Draw the congruence lattice of X in the following situations:
   (a) X has 1 element.
   (b) X has 2 elements.
   (c) X has 3 elements.

   (Hint: There are 1, 2, and 5 equivalences relations).
4. Show that the variety of sets is not congruence distributive.

   (Hint: consider the previous exercise).
5. Show that the variety of sets is generated by any 2-element set.

   (Hint: Show that any set is isomorphic to a subset of 2^X for some sufficiently large set X. This requires some knowledge of set theory).
6. Show that there is an algebra A that is congruence distributive, but the variety $\mathcal{V}(A)$ that it generates is not congruence distributive.

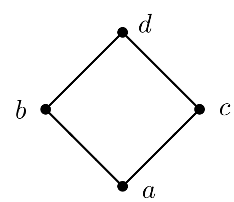
   (Hint: Consider the previous exercise).
7. Suppose that $(S, \wedge)$ is a semilattice and that $F \subseteq S$ has the following properties:
   (a) If $x, y \in F$, then $x \wedge y \in F$.
   (b) If $x \in F$ and $x \leq y$, then $y \in F$.

Show that $\theta = \{(x,y) : x,y \in F \text{ or } x,y \notin F\}$ is a congruence on S.

8. Use the previous exercise to show that if S is a meet semilattice with more than 2 elements, then for any $x,y \in S$ with $x \neq y$, there is a congruence $\theta_{x,y}$ of S that is not equal to Δ and $(x,y) \notin \theta_{x,y}$. Use this to conclude that the only subdirectly irreducible semilattice has 2 elements.

9. Use the previous exercise to show that the variety of semilattices is generated by the 2-element semilattice. (Hint: Use Theorem 8.1.3.)

10. Draw the congruence lattice of the 4-element meet semilattice shown below. Conclude that semilattices are not congruence distributive. Why is this congruence lattice distributive when the join operation is also taken as basic?



11. Show that if $\mathcal{V}_j$ $(j \in J)$ is a family of subvarieties of a variety $\mathcal{V}$, then $\bigcap_J \mathcal{V}_j$ is closed under homomorphic images, subalgebras, and products, thus is a variety.

12. Suppose that $\mathcal{V}_j$ $(j \in J)$ is a family of subvarieties of a variety $\mathcal{V}$ and that Σ_j is a set of equations that axiomatizes $\mathcal{V}_j$. Show that $\bigcup_J \Sigma_j$ axiomatizes $\bigcap_J \mathcal{V}_j$.

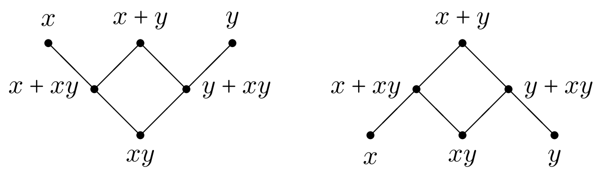
13. Let X be a set and $\mathcal{P}(X)$ be its power set. For $x \in X$, let $U = \{y \in X : x \neq y\}$ and $W = \{x\}$. Show that (U,W) is a splitting pair of $\mathcal{P}(X)$. Are there any other splitting pairs in $\mathcal{P}(X)$? Describe them all.

14. Are there any splitting pairs in the lattice $\mathrm{I} = [0,1]$?

15. Show that in any Birkhoff system, $x + y + xy = x + y$ and $(x + y)xy = xy$.

16. Show that 2_s is weakly projective in BS.

17. The following algebra is the free Birkhoff system on 2 generators x, y. In it, the only relationships that hold are ones forced by the equations defining BS. For example, $x(x+y) = x+xy$, but this is forced by Birkhoff's equation. Show that all other relationships that hold in this algebra, such as $(x + xy)y = xy$, are consequences of the equations defining BS.



18. The free Birkhoff system on 2 generators has the property that any mapping of $\{x, y\}$ into a Birkhoff system A extends to a homomorphism from the free Birkhoff system into A. Show how the mapping of $\{x, y\}$ into 2_l that maps x to 1 and y to 2 extends to a homomorphism. Show the same for 2_s.

Chapter 9

Categories of Fuzzy Relations

Goguen [34] introduced a categorical treatment of fuzzy sets that was later extended by Winter [116] and given the name Goguen categories. This approach deals not only with theoretical aspects of fuzzy sets, but also with applications such as fuzzy control. At heart, it takes the familiar category of matrices of real numbers, and modifies this to a category of matrices whose entries are elements of I, where matrix addition and multiplication are computed using the operations $\wedge$ and $\vee$ of I. This is called the category of fuzzy relations. This approach is flexible enough to allow any complete completely distributive lattice to be used in place of I. In particular, the algebra L_u of convex normal functions modulo agreement c.a.e. can be used in this setting. Our aim in this chapter is to outline the basics of various categories of fuzzy relations and their role in the study of fuzzy sets.

9.1 Preliminaries

We begin with a review of a topic that is likely familiar to nearly all, that of matrices, and matrix multiplication. However, we do this review with an eye toward employing these ideas in a different setting.

Definition 9.1.1 *For natural numbers m and n, an $m \times n$ **real matrix** is an indexed family $A = (a_{ij})$ where $1 \leq i \leq m$ and $1 \leq j \leq n$.*

An $m \times n$ matrix is written as a rectangular array of numbers with m rows and n columns, with the entry a_{11} in the top left and the entry a_{mn} in the bottom right. For example, a 2×3 matrix is shown below.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$$

Already, there is a subtlety to discuss. Suppose that we are given sets $X = \{\text{Homework}, \text{Exam}\}$ and $Y = \{\text{Bob}, \text{Ted}, \text{Alice}\}$, and that for each $x \in X$ and $y \in Y$ we have the score obtained on item x by student y. To represent this data as a matrix according to the above definition, we first have to enumerate the elements of X as x_1, x_2 and the elements of Y as y_1, y_2, y_3. For the most part, this is of importance only for a visual description of the data. An alternate way to provide a visual description is to label the rows and columns.

	Bob	Ted	Alice
Homework	63	85	74
Exam	70	82	61

Definition 9.1.2 *For sets X and Y, an $X \times Y$* ***real matrix*** *is an indexed family of real numbers $A = (a_{xy})$ where $x \in X$ and $y \in Y$.*

More concisely, an $X \times Y$ real matrix is simply a function $\Phi : X \times Y \to \mathbb{R}$. However, we find it useful to use the familiar matrix terminology, especially since we will eventually be taking products of such matrices.

$$\begin{pmatrix} 4 & 2 & 1 \\ \boxed{3 \;\; 4 \;\; 5} \end{pmatrix} \begin{pmatrix} 1 & 3 & 2 & 0 \\ 2 & 1 & 5 & 3 \\ 4 & 8 & 1 & 5 \end{pmatrix} = \begin{pmatrix} 12 & 22 & 19 & 11 \\ 31 & 53 & \boxed{31} & 37 \end{pmatrix}$$

If A is an $m \times k$ matrix and B is a $k' \times n$ matrix B, the product AB exists if and only if $k = k'$. In the case that $k = k'$, the product is an $m \times n$ matrix C where the entry c_{ij} is given as follows:

$$c_{ij} = \sum_{p=1}^{k} a_{ip} b_{pj}$$

In the example above, $c_{23} = a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33}$. So $3 \cdot 2 + 4 \cdot 5 + 5 \cdot 1 = 31$. This product has an obvious counterpart for matrices whose entries are indexed by sets.

Definition 9.1.3 *Suppose that X, Y, Z are finite sets. If A is an $X \times Y$ matrix, and B is a $Y \times Z$ matrix, then their* ***matrix product*** *C is the $X \times Z$ matrix where the entry c_{xz} is given as follows.*

$$c_{xz} = \sum_{y \in Y} a_{xy} b_{yz}$$

It is well known that matrix multiplication is associative. This means that if A is an $m \times k$ matrix, B is a $k \times l$ matrix, and C is an $l \times n$ matrix, then $A(BC) = (AB)C$. We later extend this result to a different setting, and understanding the ingredients that make this hold will be of importance. For this reason, we review the proof, but do so in the setting of matrices indexed by sets.

Proposition 9.1.4 *Suppose that W, X, Y, Z are finite sets, that A is an $W \times X$ matrix, that B is an $X \times Y$ matrix, and that C is a $Y \times Z$ matrix. Then $A(BC) = (AB)C$.*

Proof. Let $D = A(BC)$, $E = (AB)C$, $P = AB$, and $Q = BC$. Note that D and E are $W \times Z$ matrices, that P is a $W \times Y$ matrix, and that Q is a $X \times Z$ matrix. Then for $w \in W$ and $z \in Z$, we have

$$\begin{aligned} d_{wz} &= \sum_X [a_{wx} q_{xz}] \\ &= \sum_X \left[a_{wx} \sum_Y b_{xy} c_{yz} \right] \\ &= \sum_X \sum_Y a_{wx} b_{xy} c_{yz} \end{aligned}$$

On the other hand,

$$\begin{aligned} e_{w,z} &= \sum_Y p_{wy} c_{yz} \\ &= \sum_Y \left[\sum_X a_{wx} b_{xy} \right] c_{yz} \\ &= \sum_Y \sum_X a_{wx} b_{xy} c_{yz} \end{aligned}$$

So $D = E$, providing the result. ■

For each natural number $n \geq 1$, the $n \times n$ identity matrix Id_n is the one that has 1's along the main diagonal, and is 0 elsewhere. For example, the 3×3 identity matrix is shown below. Again, with an eye toward generalizations in a later section, we give a detailed account of identity matrices and their basic properties, but we do so in the setting of matrices indexed by sets.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Definition 9.1.5 *For a set X, the **identity matrix for** X is the $X \times X$ matrix Id_X whose (x,y) entry is given by the Kronecker delta function δ_{xy} where*

$$\delta_{xy} = \begin{cases} 1 & \textit{if } x = y \\ 0 & \textit{otherwise} \end{cases}$$

Proposition 9.1.6 *Let X, Y be finite sets, and let A be an $X \times Y$ matrix. Then $\mathrm{Id}_X A = A = A \,\mathrm{Id}_Y$.*

Proof. The (x,y) entry of $\mathrm{Id}_X A$ is $c_{xy} = \sum_{p \in X} \delta_{xp} a_{py} = a_{xy}$, and the (x,y) entry of $A\,\mathrm{Id}_Y$ is $d_{xy} = \sum_{q \in Y} a_{xq} \delta_{qy} = a_{xy}$. ■

We consider next a related topic, that of relations. From the preliminaries of Chapter 1, a relation R on a set X is a subset of $X \times X$; that is, a set of ordered pairs (x_1, x_2) of elements of X. We write $x_1 R x_2$ to mean that $(x_1, x_2) \in R$. This notion can be extended.

Definition 9.1.7 *For sets X and Y, a **relation** R from X to Y is a subset $R \subseteq X \times Y$. So a relation from X to Y is a set of ordered pairs (x,y) where $x \in X$ and $y \in Y$. We write $x R y$ to mean $(x,y) \in R$.*

For example, suppose that $X = \{a,b\}$ and $Y = \{p,q,r\}$. Consider the relation R from X to Y with a related to p and q, and b related to q and r. Then $R = \{(a,p),(a,q),(b,q),(b,r)\}$. We have $a R p$, but not $a R r$. We can associate to R an $X \times Y$ matrix whose (x,y) entry is 1 if $x R y$ and 0 otherwise. This is shown below. The rows of this matrix should be labeled a and b, and the columns p, q, and r.

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

We turn now to a method to "multiply" relations.

Definition 9.1.8 *Suppose that X, Y, Z are sets, that R is a relation from X to Y, and that S is a relation from Y to Z. Their **relational product** $R \circ S$ is the relation from X to Z given by*

$$R \circ S = \{(x,z) : \textit{there is a } y \in Y \textit{ with } x R y \textit{ and } y R z\}$$

Continuing the example above, suppose that $Z = \{u,v,w\}$ and that S is the relation from Y to Z given by $S = \{(p,u),(p,v),(q,v),(r,w)\}$. Then a computation gives $R \circ S = \{(a,u),(a,v),(b,v),(b,w)\}$.

To consider the connection between relational product and matrix product, consider the following matrix product. The matrices at left are those for R and S. For the product to have a nonzero entry in its (x,z) entry, there must be a $y \in Y$ with the matrix for R having a 1 in its (x,y) entry and the matrix for S having a 1 in its (y,z) entry. In fact, the matrix at right below contains the number of such y that occur.

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

Thus the matrix for S is not the matrix at right in the above equation, but is obtained from it by changing each nonzero entry in this matrix to a 1. In effect, the relation product is obtained as a matrix product using the ordinary multiplication of numbers, but replacing + with max.

Definition 9.1.9 *For a set X, let Id_X be the relation $R = \{(x,x) : x \in X\}$. This is called the* ***identity relation*** *on X.*

It is well known, and will follow from results of the following section, that a relational product is associative when defined, and that for a relation R from a set X to a set Y, that $\mathrm{Id}_X \circ R = R = R \circ \mathrm{Id}_Y$. In essence, a relational product has the same properties as for matrix multiplication. We next place these properties in a broader context, that of categories.

Definition 9.1.10 *A* ***category*** *$\mathcal{C}$ consists of the following items.*

1. *A collection $\mathcal{O}$ of* ***objects****.*
2. *For objects X, Y, a set $\mathcal{C}(X,Y)$ called the* ***morphisms*** *from X to Y.*
3. *An operation $\circ$ of* ***composition*** *that provides for each morphism A from X to Y, and B from Y to Z, a morphism $A \circ B$ from X to Z.*
4. *For each object X, an* ***identity morphism*** *Id_X from X to itself.*

It is further required that $\circ$ is associative when defined, and that for each morphism A from an object X to an object Y, that $\mathrm{Id}_X \circ A = A = A \circ \mathrm{Id}_Y$.

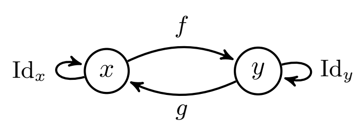
Categories are a very general notion that occur in a very wide range of forms [73]. We present a number of examples.

Example 9.1.11 The most obvious example of a category is the category Set of sets. Its objects are all sets. For sets X and Y, the morphisms $\mathrm{Set}(X,Y)$ are all functions from X to Y. The composition is usual function composition, and the identity morphism for a set X is the identity function on X. We note that for a morphism A from X to Y, and a morphism B from Y to Z, that the composite morphism is written here as $A \circ B$ rather than in the usual way of $B \circ A$ for function composition. For our current purposes, this is more convenient.

Example 9.1.12 The category Group has as its objects all groups, and as its morphisms all homomorphisms between groups. Composition is again usual function composition, and the identity morphisms on a group G is the identity function on G.

Example 9.1.13 The category Vect has as its objects all real vector spaces, and as its morphisms all homomorphisms (linear transformations) between them. Composition is again usual function composition, and the identity morphism on a vector space V is the identity function on V. This can be modified so that the objects are only the finite-dimensional vector spaces, giving the category FDVect.

Example 9.1.14 Before giving the impression that the objects of a category must be sets, and the morphisms between them functions, we point out the similarity between categories and directed graphs. For instance, the category $\mathcal{C}$ depicted below has 2 objects, x and y, and 4 morphisms, $\mathrm{Id}_x, \mathrm{Id}_y, f, g$. The composition of morphisms behaves as it must with respect to the identity morphisms, and has $f \circ g = \mathrm{Id}_x$ and $g \circ f = \mathrm{Id}_y$.



Our earlier discussion shows that the following is a valid definition of a category.

Definition 9.1.15 *The **category of matrices*** Mat *has as objects all finite sets. For objects X and Y, the morphisms from X to Y are all $X \times Y$ matrices. The composition of suitable morphisms is given by matrix multiplication, and the morphism* Id_X *is the identity matrix on X.*

The names given for the categories in the above definitions and examples are relatively standard. There is, however, something unfortunate about the choice of these names. The category Set of sets is clearly named after its objects, while the category Mat of matrices is named after its morphisms. In fact, both categories have the same objects! It might be better to call the category of Example 9.1.11 the category Fun of functions, and that of Example 9.1.13 the category Lin of linear transformations, but this has never become entrenched.

Definition 9.1.16 *The **category of relations*** Rel *has as objects all sets. For objects X and Y, the morphisms from X to Y are all relations R from X to Y. The composition of suitable morphisms is given by relation composition, and the morphism* Id_X *is the identity relation on X.*

There is an obvious similarity between the categories Mat and Rel. In fact, there is a more general setting to which both belong. Suppose that $(S, +, \cdot, 0, 1)$ is an algebra with two binary operations $+$ and $\cdot$ and constants 0 and 1. We can attempt to form a category of matrices over S whose objects are finite sets and whose morphisms are matrices with entries in S. However, to define matrix multiplication and have it satisfy needed conditions such as associativity, properties are required of S. A sufficient condition is that S be a semiring [36].

Definition 9.1.17 *An algebra* $(S, +, \cdot, 0, 1)$ *is a* ***semiring*** *if it satisfies the following conditions:*

1. $+$ *is commutative and associative.*
2. $\cdot$ *is associative.*
3. *For each* $a \in S$, $0 + a = a$, $1 \cdot a = a = a \cdot 1$, *and* $0 \cdot a = 0 = a \cdot 0$.
4. *For each* $a, b, c \in S$, $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ *and* $(a + b) \cdot c = (a \cdot c) + (b \cdot c)$.

There is also a link between the category FDVect of finite-dimensional real vector spaces and Mat. We don't wish to delve into the technical meaning of the "equivalence of categories." The reader can take the essential meaning from the proof of the following result. See [73] for details.

Theorem 9.1.18 *The category of finite-dimensional vector spaces is equivalent to* Mat*.*

Proof. For each finite set X there is a vector space F_X that has X as a basis, and every finite-dimensional real vector space is isomorphic to some F_X. If X and Y are finite sets and A is an $X \times Y$ matrix, then there is a unique linear transformation $T_A : F_X \to F_Y$ so that for a basis element $x \in X$ we have $T_A(x)$ is the element of F_Y given by

$$T_A(x) = \sum_Y a_{xy} y$$

Essentially, T_A is given by the matrix transpose A^T considered as a mapping $A^T : F_X \to F_Y$. Then T_{AB} corresponds to $(AB)^T = B^T A^T$, which corresponds to the composite of the maps T_B and T_A, which we have agreed to write $T_A \circ T_B$. (See Example 9.1.13.) So T provides a bijection from $\mathrm{Mat}(X, Y)$ to $\mathrm{Vect}(F_X, F_Y)$ with $T_{AB} = T_A \circ T_B$. This gives an equivalence. ■

In the following sections, we will work with categories similar to Mat and Rel. In gaining intuition for these categories, it is best to keep in mind Theorem 9.1.18 and view them as cousins of the category of vector spaces. So to an extent, we will be working with a variant of linear algebra.

9.2 Fuzzy relations

In this section we introduce categories related to Mat and Rel that are of use in studying fuzzy sets. There will be two categories of primary instance, one that applies to fuzzy sets, and one for type-2 fuzzy sets. Both are instances of a more general pattern that is no more difficult to describe, and we work in the more general setting.

Definition 9.2.1 *Let D be a distributive lattice. For sets X and Y, an $X \times Y$ D-**matrix**, also called an $X \times Y$ D-**relation**, is an indexed family $A = (a_{xy})$ where $a_{xy} \in D$ for each $x \in X$ and $y \in Y$.*

Ordinary matrix multiplication computes entries as a sum of products. This can easily adapted to the setting of D-matrices replacing the product with meet, and sum with join. A key point here is that the distributive law $a \wedge \bigvee_Y b_y = \bigvee_Y (a \wedge b_y)$ holds in any distributive lattice for joins involving only finite indexing sets.

Definition 9.2.2 *Let D be a distributive lattice and X, Y, Z be finite sets. If A is an $X \times Y$ D-matrix, and B is a $Y \times Z$ D-matrix, then the product AB is the $X \times Z$ D-matrix C, where*

$$c_{xz} = \bigvee_Y (a_{xy} \wedge b_{yz})$$

The proof of the associativity of ordinary matrix multiplication given in Proposition 9.1.4 can easily be adapted to yield the following result. The proof is left as an exercise (Exercise 3).

Proposition 9.2.3 *For a distributive lattice D, multiplication of D-matrices over appropriately matching finite sets is associative.*

We recall that a bounded distributive lattice is one with a largest element 1 and a least element 0. We leave as an exercise (Exercise 4) the fact that every bounded distributive lattice is a semiring in the sense of Definition 9.1.17.

Definition 9.2.4 *Let D be a bounded distributive lattice and X be a set. The $X \times X$ D-identity matrix Id_X is the one whose (x, y) entry is given by the Kronecker delta function δ_{xy}, where*

$$\delta_{xy} = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}$$

We leave as an exercise (Exercise 5) the following result.

Proposition 9.2.5 *Let D be a bounded distributive lattice, X, Y be finite sets, and A be an $X \times Y$ D-matrix. Then $\mathrm{Id}_X A = A = A \,\mathrm{Id}_Y$.*

Propositions 9.2.3 and 9.2.5 provide the following.

Proposition 9.2.6 *Let D be a bounded distributive lattice. Then there is a category whose objects are all finite sets and whose morphisms are the D-matrices indexed by finite sets composed via D-matrix multiplication.*

We next aim to extend matters to the setting of arbitrary sets. There are applications in analysis that extend ordinary matrix multiplication to (usually) countably infinite sets, but this involves taking infinite sums and can be somewhat intricate.

Fortunately, the situation for infinite D-matrices and their products is straightforward. We require completeness of D so that infinite joins exist. We also require the appropriate fragment of infinite distributivity, that $a \wedge \bigvee_Y b_y = \bigvee_Y (a \wedge b_y)$ for arbitrary sets Y. We recall from Definition 1.4.4 that this is known as meet continuity.

Definition 9.2.7 *Let D be a complete meet continuous distributive lattice, and let X, Y, Z be arbitrary sets. For an $X \times Y$ D-matrix A and a $Y \times Z$ D-matrix B, their product is the $X \times Z$ D-matrix C whose (x, z) entry is*

$$c_{xz} = \bigvee_Y (a_{xy} \wedge b_{yz})$$

Note that this join involved is over the set Y and may be infinite.

The proof of the following result is similar to ones in the finite case. It is left as an exercise (Exercise 6).

Theorem 9.2.8 *Let D be a complete meet continuous distributive lattice. Suppose that W, X, Y, Z are sets, that A is an $W \times X$ D-matrix, that B is an $X \times Y$ D-matrix, and that C is a $Y \times Z$ D-matrix. The following hold:*

1. $A(BC) = (AB)C$.
2. $\mathrm{Id}_W A = A = A\,\mathrm{Id}_X$.

So for a complete meet continuous distributive lattice D, this allows us to view the collection of all sets with the D-matrices over them as a category. Such categories will be the central theme of this chapter. We will give them two names to emphasize both their connection to linear algebra, as well as to match with more commonly used names in fuzzy theory.

Definition 9.2.9 *Let D be a complete meet continuous distributive lattice. The category whose objects are all sets and whose morphisms are D-matrices over sets will be called the* ***category of D-matrices*** *and written* Mat_D*. It will also be called the* ***category of D-relations*** *and written* Rel_D*.*

There are a number of complete meet-continuous distributive lattices that arise in fuzzy theory. The 2-element distributive lattice 2 is connected with classical sets. As mentioned, the category of 2-relations is simply Rel. (See Exercise 9.) The lattice I is used in connection with fuzzy sets, and $\mathrm{I}^{[2]}$ with interval-valued fuzzy sets. Both are complete meet-continuous distributive lattices. The lattice L_u connected with type-2 fuzzy sets is also a complete meet continuous distributive lattice.

Definition 9.2.10 *The* ***category of fuzzy relations*** *is* Rel_{I}*, the category of D-relations for D the lattice* I*. We denote this* FRel.

Definition 9.2.11 *The* ***category of type-2 fuzzy relations*** *is the category* Rel_{L_u} *of D-relations for D the lattice* L_u*. We denote this* 2 FRel.

The categories FRel and 2 FRel have an interesting modification. For any continuous t-norm $\vartriangle$ on I, its convolution $\blacktriangle$ gives a t-norm on L_u. A type of matrix multiplication can be defined using this t-norm in place of the t-norm of meet. This is described below.

Definition 9.2.12 *Let $*$ be a binary operation on a complete distributive lattice D. If X, Y, Z are sets, A is an $X \times Y$ D-matrix, and B is a $Y \times Z$ D-matrix, define the $*$-product AB to be the $X \times Z$ D-matrix C whose (x, y) entry is*

$$c_{xz} = \bigvee_Y (a_{xy} * b_{yz})$$

Proposition 9.2.13 *Suppose D is a complete lattice and that $*$ is a binary operation on D that satisfies the following:*

1. *$*$ is commutative and associative.*
2. *$a * \bigvee_Y b_y = \bigvee_Y (a * b_y)$ for every a and family b_y $(y \in Y)$ in D.*
3. *$a * 1 = a$ and $a * 0 = 0$ for every $a \in D$.*

Then there is a category whose objects are all sets, whose morphisms are all D-matrices, and whose rule of composition is the $*$-product of D-matrices.

The proof is identical to the previous cases. We note that the distributivity of D is not required, although we will employ this only for certain distributive lattices. Also, the conditions above are related to the notion of a **quantale** [93]. Finally, we have assumed commutativity of $*$ since we will have it in the cases of interest, and it makes the results easier to state. Without it, we would need the infinite distributive law of item 2 on both sides.

Theorem 9.2.14 *Let $\vartriangle$ be a continuous t-norm on* I*. There is a category* $\text{FRel}_{\vartriangle}$ *that has the same objects, morphisms, and identity morphisms as* FRel*, but whose rule of composition is $\vartriangle$-product rather than the usual product of matrices over* I.

This result is a direct consequence of Proposition 9.2.13 since a continuous t-norm $\vartriangle$ on I satisfies the three conditions of the proposition. Similarly, the following result is a direct consequence of Proposition 9.2.13 using properties of $\blacktriangle$ on L_u given in Theorems 5.6.2 and 6.8.9.

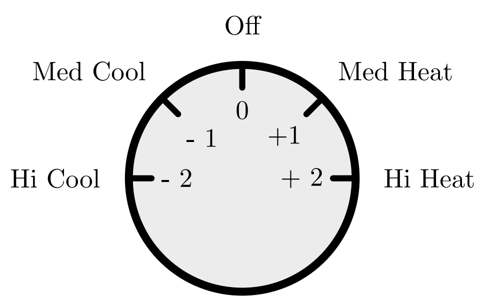
Theorem 9.2.15 *Let $\triangle$ be a continuous t-norm on* I *and* $\blacktriangle$ *the restriction of its convolution to* L_u. *There is a category* $2\,\mathrm{FRel}_{\blacktriangle}$ *that has the same objects, morphisms, and identity morphisms as* $2\,\mathrm{FRel}$, *but whose rule of composition is* $\blacktriangle$*-product rather than the usual product of matrices over* L_u.

It is also possible to modify matters so that multiplication is constructed taking an infinite meet of binary joins, or using an infinite meet in conjunction with a continuous t-conorm or its convolution. The reader can formulate these results on their own.

9.3 Rule bases and fuzzy control

In this section we provide motivation for the use of categories of matrices, or relations, in the study of (type-1) fuzzy sets by considering an example of their use in fuzzy control.

A description of the setup A room has a device that can be used to heat or cool it. The device has a knob that has 5 settings, −2, −1, 0, +1, +2. A setting of −2 rapidly puts cold air into the room, and a setting of +2 rapidly puts hot air into the room, with the other values performing in between. We have a sensor that will measure the temperature of the room as either 50, 60, 70, 80, or 90 degrees. We want to build a controller that will make an adjustment to the setting of the knob based on the temperature given by the sensor.



We define the following sets:

$$X = \{50, 60, 70, 80, 90\} \quad \text{the possible values of temperature}$$
$$Y = \{-2, -1, 0, +1, +2\} \quad \text{the possible knob settings}$$

The aim is to find a function $\Phi : X \to Y$ that describes for a given measurement of the temperature T of the room, the appropriate adjustment $\Phi(T)$ to be applied to the knob.

We begin by assigning **linguistic variables** COLD, NICE, and HOT to room temperature. Experts would be asked to associate fuzzy subsets of X to each of these. We also associate linguistic variables for knob settings. To keep matters simple, we use only two. To avoid confusion, we will call these AIR and FURNACE. Experts would be asked to associate fuzzy subsets of Y to these. These are indicated below.

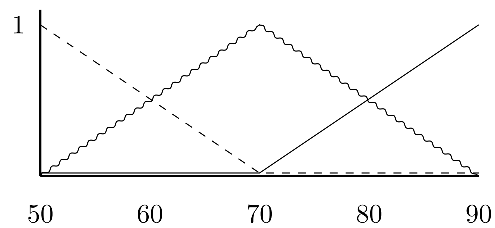


FIGURE 9.1: Fuzzy subsets for COLD (dashed), NICE (wavy), HOT (solid)

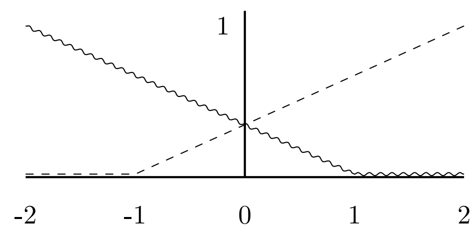


FIGURE 9.2: Fuzzy subsets for AIR (wavy), FURNACE (dashed)

We can represent these fuzzy sets using tables, or as the corresponding matrices, which we call P and Q. We use $.\overline{3} = .3333\ldots$ and so forth.

$$P = \begin{pmatrix} 1 & .5 & 0 & 0 & 0 \\ 0 & .5 & 1 & .5 & 0 \\ 0 & 0 & 0 & .5 & 1 \end{pmatrix}$$

	50	60	70	80	90
COLD	1	.5	0	0	0
NICE	0	.5	1	.5	0
HOT	0	0	0	.5	1

$$Q = \begin{pmatrix} 1 & .\overline{6} & .\overline{3} & 0 & 0 \\ 0 & 0 & .\overline{3} & .\overline{6} & 1 \end{pmatrix}$$

	-2	-1	0	1	2
AIR	1	$.\overline{6}$	$.\overline{3}$	0	0
FURNACE	0	0	$.\overline{3}$	$.\overline{6}$	1

The next step is to give a **rule base** saying what actions in terms of the adjustments AIR and FURNACE are to be taken in terms of certain outcomes COLD, NICE, or HOT of a measurement of temperature. The rules chosen are given below as both a table and a matrix R. This choice of rules may not be the best to actually build a controller, but are fine for giving an example.

$$R = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

	COLD	NICE	HOT
AIR	0	1	1
FURNACE	1	0	0

FIGURE 9.3: A rule base described as a matrix and as a table

These rules say the following. If the temperature is COLD, apply FURNACE, if temperature is NICE, apply AIR, and if temperature is HOT, apply AIR. The rules for COLD and HOT seem reasonable, but what about NICE? Could we instead do nothing, so have a column of 0's for NICE? Could we do both and have a column of 1's for NICE? Could we do both to some degree, and choose whatever values in I we like for this column or the others? All of these choices are compatible with the following mathematics. What is best depends on what winds up building the controller that works best.

Constructing our control function Having assembled the pieces that go into making our control function $\Phi : X \to Y$, we now consider how to combine them. As one would expect, the basic tool is matrix multiplication. But from Theorem 9.2.14, we have many choices for how matrix multiplication is calculated. For this example, it will be calculated with the continuous t-norm of ordinary multiplication for product and join in I for sum.

An actual measurement is made of the temperature T. This will give us one of the values in X. Suppose that the value $T = 80$ is obtained. We represent this as a column vector $\hat{T}$ that has a 1 in its fourth spot and 0's elsewhere. This process is sometimes called **fuzzifying** the value T.

$$\begin{pmatrix} 1 & .5 & 0 & 0 & 0 \\ 0 & .5 & 1 & .5 & 0 \\ 0 & 0 & 0 & .5 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ .5 \\ .5 \end{pmatrix}$$

FIGURE 9.4: Computation of $P(\hat{T})$

Since $\hat{T}$ is a column vector with a 1 in its fourth spot and 0s elsewhere, $P(\hat{T})$ is the fourth column of P as a column vector. It represents the degrees to which the temperature 80 is COLD, NICE, and HOT, so 0, .5, and .5.

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ .5 \\ .5 \end{pmatrix} = \begin{pmatrix} .5 \\ 0 \end{pmatrix}$$

FIGURE 9.5: Computation of $RP(\hat{T})$

We next apply the rule base and compute $RP(\hat{T})$. The top entry is obtained as $(0 \vartriangle 0) \vee (1 \vartriangle .5) \vee (1 \vartriangle .5) = .5$, where the t-norm $\vartriangle$ is ordinary multiplication. The value of $RP(\hat{T})$ gives the degree to which a reading of 80 for temperature calls for an adjustment of AIR and FURNACE. It calls for an adjustment of AIR with degree .5 and FURNACE with a degree of 0.

$$\begin{pmatrix} 1 & 0 \\ .\overline{6} & 0 \\ .\overline{3} & .\overline{3} \\ 0 & .\overline{6} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} .5 \\ 0 \end{pmatrix} = \begin{pmatrix} .5 \\ .\overline{3} \\ .1\overline{6} \\ 0 \\ 0 \end{pmatrix}$$

FIGURE 9.6: Computation of $Q^T RP(\hat{T})$

Next we compute $Q^T RP(\hat{T})$ where Q^T is the transpose of Q. This gives the fuzzy subset of Y obtained by **superposing** the fuzzy subset for AIR with degree .5 and the fuzzy subset for FURNACE with degree 0. The other values of temperature can be handled similarly, but there is a more efficient way. The column vector just obtained is simply the fourth column of the matrix $F = Q^T RP$. The others will be the other columns of this matrix.

$$\begin{pmatrix} 0 & .5 & 1 & .5 & 1 \\ 0 & .\overline{3} & .\overline{6} & .\overline{3} & .\overline{6} \\ .\overline{3} & .1\overline{6} & .\overline{3} & .1\overline{6} & .\overline{3} \\ .\overline{6} & .\overline{3} & 0 & 0 & 0 \\ 1 & .5 & 0 & 0 & 0 \end{pmatrix}$$

	50	60	70	80	90
-2	0	.5	1	.5	1
-1	0	$.\overline{3}$	$.\overline{6}$	$.\overline{3}$	$.\overline{6}$
0	$.\overline{3}$	$.1\overline{6}$	$.\overline{3}$	$.1\overline{6}$	$.\overline{3}$
+1	$.\overline{6}$	$.\overline{3}$	0	0	0
+2	1	.5	0	0	0

FIGURE 9.7: $F = Q^T RP$ as a matrix and table

We now have a process, called **fuzzifying**, to produce from a measurement of T a fuzzy subset of Y (a column of the matrix above). Thus we have a map

$\Phi' : X \to \mathrm{Map}(Y, \mathrm{I})$ that takes an element T of X such as 80, fuzzifies it to form $\hat{T}$, then multiplies $F(\hat{T})$ where F is the matrix $Q^T RP$. It remains to take a fuzzy subset ψ of Y and to produce from it an element of Y, a process called **defuzzifying**. A common method to do this is by taking the **center of mass** $\overline{\psi}$,

$$\overline{\psi} = \frac{\sum_Y \psi(y)\, y}{\sum_Y \psi(y)} \tag{9.1}$$

In many ways, this is the most problematic part of the procedure. A critical view shows that this choice of defuzzification relies on a structure of the set Y that may not always be present. It also makes use of operations of sum and product from the reals. Other choices are possible, such as choosing an element $y \in Y$ with $\psi(y)$ taking a maximum value. Difficulties aside, in our current example, this defuzzification yields the following result.

T	50	60	70	80	90
$\overline{\psi}$	$1.\overline{3}$	0	$-1.\overline{3}$	$-1.\overline{3}$	$-1.\overline{3}$

The results do not lie in the set Y, and we would somehow have to round these off to obtain our final knob adjustments. It seems likely that we did not produce a very effective controller from our choices, but hopefully the example illustrates the process.

Two generalizations of the process outlined here seem natural. First, as we have already remarked, arbitrary matrices with entries in I could be used as rule bases. This would in no way complicate the resulting analysis. A second place for generalization is in the treatment of temperature. The mathematics allows for any column vector with entries in I in place of $\hat{T}$. One can easily imagine circumstances where our sensor provides other than crisp values where this would be applicable.

9.4 Additional variables

In this section, we continue the example of the previous section as a means of motivating techniques related to having additional variables.

In addition to a measure of temperature T, we consider also an independent variable humidity H. Suppose humidity takes values in $X' = \{20, 40, 60, 80\}$ measured in percentage of moisture. We choose possible linguistic variables Dry and Wet for humidity, and associate to these the fuzzy subsets of X' below. These are again represented both as a table and as a matrix P'.

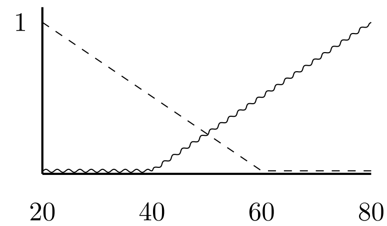


FIGURE 9.8: Fuzzy subsets for DRY (dashed), WET (wavy)

$$P' = \begin{pmatrix} 1 & .5 & 0 & 0 \\ 0 & 0 & .5 & 1 \end{pmatrix}$$

	20	40	60	80
DRY	1	.5	0	0
WET	0	0	.5	1

Next is a rule base that describes for each combination of temperature (COLD, NICE, HOT) and humidity (DRY, WET), the type of adjustment (AIR, FURNACE). This is given both as a table and as a matrix R_1. Again, our aim here is to build an example, not to build a good-quality controller.

	COLD DRY	COLD WET	NICE DRY	NICE WET	HOT DRY	HOT WET
AIR	0	0	0	1	1	1
FURNACE	1	1	1	0	0	0

FIGURE 9.9: The 2-variable rule base as a table

The matrix corresponding to this table is R_1 as follows.

$$R_1 = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

We would like to use a similar approach to that used before when we formed $Q^T RP$, but we need a matrix formed from P and P' of appropriate size to be multiplied on the left by R_1.

Definition 9.4.1 *Suppose that A is an $X \times Y$ matrix and that A' is an $X' \times Y'$ matrix. Their **Kronecker product** is the $(X \times X') \times (Y \times Y')$ matrix written $A \otimes A'$ whose $((x, x'), (y, y'))$ entry is $a_{xy} \cdot a'_{x'y'}$*

For a t-norm $\triangle$ on I, the notion of a **$\triangle$-Kronecker product** $A \otimes_\triangle A'$ of matrices over I uses $\triangle$ to multiply entries rather than ordinary product. Taking the Kronecker product of the matrices P and P' using ordinary multiplication, gives the following, expressed in table form.

	50 20	50 40	50 60	50 80	⋯ ⋯	90 20	90 40	90 60	90 80
Cold, Dry	1	.5	0	0	⋯	0	0	0	0
Cold, Wet	0	0	.5	1	⋯	0	0	0	0
⋮									
Hot, Dry	0	0	0	0	⋯	1	.5	0	0
Hot, Wet	0	0	0	0	⋯	0	0	.5	1

FIGURE 9.10: The Kronecker product $P \otimes P'$ in table form

This table has 6 rows and 20 columns. It is built by replacing each entry of P with a copy of the matrix P' multiplied by the entry of P that it replaces. The pattern above is especially simple because the entries in the corners of P are 1 in the top left and bottom right and 0 in the other corners.

Our fuzzy controller works as follows. One measures T and H. The column vectors $\hat{T}$ and $\hat{H}$ each have a 1 in exactly one spot and are 0 otherwise. So the Kronecker product $\hat{T} \otimes \hat{H}$ is also a column vector with a 1 in one spot and is 0 otherwise. This column vector is multiplied on the left by the matrix

$$Q^T R_1 (P \otimes P')$$

The result is then defuzzified using the center of mass to yield a value for adjustment as before.

More general situations The above example extends to a situation where we have a family of independent variables whose matrices of fuzzy subsets are given by $P_1, \ldots, P_n$. One makes a matrix for our rules R and applies

$$Q^T R (P_1 \otimes \cdots \otimes P_n)$$

to the Kronecker product of the measurements.

Alternately, we may have a situation with one independent variable whose matrix of fuzzy subsets is given by P and a family of dependent variables whose matrices of fuzzy subsets are given by $Q_1, \ldots, Q_m$. These may be adjustments to a cooler and to a humidifier based on a measure of temperature. Here the controller is given by

$$(Q_1 \otimes \cdots \otimes Q_m)^T RP$$

The general situation has a family of independent variables, as well as a family of adjustments, connected by a rule base R. Its controller is given by

$$(Q_1 \otimes \cdots \otimes Q_m)^T R\,(P_1 \otimes \cdots \otimes P_n)$$

9.5 The type-2 setting

We continue the example of the previous two sections in the context of convex normal type-2 fuzzy sets $\mathbf{L}_u$.

The basic setup is the same with a device to measure temperature, and the aim to make an adjustment to a knob that controls a device that heats and cools a room. Again, values of temperature lie in X and the possible adjustments of the knob lie in Y where

$$\begin{aligned} X &= \{50, 60, 70, 80, 90\} && \text{the possible values of temperature} \\ Y &= \{-2, -1, 0, +1, +2\} && \text{the possible knob settings} \end{aligned}$$

We use the linguistic variables Cold, Nice, and Hot for temperature; the linguistic variables Air and Furnace for the adjustment of the knob; and the same rule base as before. However, instead of associating fuzzy subsets of X and Y to these linguistic variables, we will assign to them certain type-2 fuzzy subsets.

We associate to Cold a type-2 fuzzy subset of X. More specifically, we associate to Cold a function $c : X \to \mathbf{L}_u$ from X into the set of convex normal upper semicontinuous functions from $\mathbb{I}$ to itself. This function c is shown below. It is comprised of five functions from $\mathbb{I}$ to itself, namely $c(50), \ldots, c(90)$. For convenience we denote these $c_{50}, \ldots, c_{90}$.

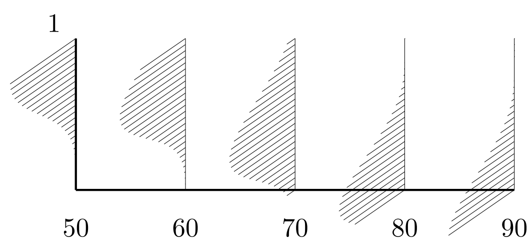


FIGURE 9.11: A type-2 fuzzy subset of X for Cold

In Figure 9.11, the temperature 50 has its associated function c_{50} being strongly centered around the value 1. So 50 being considered COLD with degree 1 is a view strongly held by experts. The function c_{70} for a temperature of 70 is strongly centered around the value .5. One can say that 70 being considered COLD with degree .5 is a strongly held view among the experts, while 70 being viewed as COLD with degree 0 or degree 1 is something rejected by experts.

Similar type-2 fuzzy subsets n and h of X are given for NICE and HOT, but we will not draw these. We then form a matrix P much as we did before. The essential difference is that the entries now are the functions comprising these fuzzy subsets, hence elements of L_u, rather than elements of I.

	50	60	70	80	90
COLD	c_{50}	c_{60}	c_{70}	c_{80}	c_{90}
NICE	n_{50}	n_{60}	n_{70}	n_{80}	n_{90}
HOT	h_{50}	h_{60}	h_{70}	h_{80}	h_{90}

FIGURE 9.12: A table corresponding to the matrix P

For AIR and FURNACE we have type-2 fuzzy subsets a and f of Y. The functions comprising these are the entries of a matrix Q.

	−2	−1	0	1	2
AIR	a_{-2}	a_{-1}	a_0	a_1	a_2
FURNACE	f_{-2}	f_{-1}	f_0	f_1	f_2

FIGURE 9.13: A table corresponding to the matrix Q

We use the same rule base as before, but need to treat it as a matrix with coefficients in L_u. Replace the entries 0 and 1 in the matrix of Figure 9.3 with 1_0 and 1_1 of L_u. This is using the embedding of I into L_u. Tables corresponding to the matrices Q and R are given in Figures 9.13 and 9.14.

	COLD	NICE	HOT
AIR	1_0	1_1	1_1
FURNACE	1_1	1_0	1_0

FIGURE 9.14: A table for the rule base matrix R

In Section 9.3, we multiplied matrices with entries in I using the t-norm $\triangle$ of ordinary multiplication as product and join in I as sum. To multiply matrices with entries in L_u, we will choose to use the convolution $\blacktriangle$ of $\triangle$ as product, and join in the lattice L_u as sum. By Theorem 9.2.15, this gives an associative multiplication. We use this to form the matrix

$$Q^T RP$$

The process of fuzzifying, is nearly the same as before. Given a measured value T for temperature, create a column vector $\hat{T}$ that has an entry for each value of X. Before we placed a 1 in the entry for the measured value of T and 0's elsewhere. Now, instead of using the bounds 0 and 1 of I, we use the bounds 1_0 and 1_1 of L_u. Place a 1_1 in the entry of $\hat{T}$ corresponding to the measured value of T, and 1_0 in the other entries.

We then form the following product. It yields a type-2 fuzzy subset γ of Y in the form of a column vector with one entry for each element of Y.

$$Q^T RP(\hat{T}) = \begin{pmatrix} \gamma_{-2} \\ \gamma_{-1} \\ \gamma_0 \\ \gamma_{+1} \\ \gamma_{+2} \end{pmatrix}$$

In Section 9.3, the corresponding column vector had entries in I. We defuzzified it using the center of mass formula (9.1). Now the entries $\gamma_{-2}, \dots, \gamma_{+2}$ are elements of L_u, hence functions from I to I. We describe one possibility to defuzzify this vector.

Since each function γ_i is convex, it is integrable [94]. So we can form its center of mass $\overline{\gamma_i}$ where

$$\overline{\gamma_i} = \frac{\int_0^1 t \cdot \gamma_i(t)\, dt}{\int_0^1 \gamma_i(t)\, dt}$$

Then setting $y_i = \overline{\gamma_i}$ for $i = -2, \dots, +2$ yields an ordinary fuzzy subset of Y that we can defuzzify using the center of mass formula (9.1) as before.

With the exception of a small modification in the defuzzification procedure, the use of type-2 fuzzy sets in the controller is identical to that of ordinary fuzzy sets. The further analysis of Section 9.4 is also seen to carry over exactly to the type-2 setting. Finally, we mention that the theory allows general matrices over L_u to be used as rule bases, and general type-2 fuzzy subsets of X to be used as inputs to the controller.

9.6 Symmetric monoidal categories

In this section, we put the use of matrices in fuzzy controllers in a broader context. Our basic ingredient is **symmetric monoidal categories**. The precise definition of this notion is given in [73]. As with many categorical concepts, it embodies a simple idea, but requires a number of technicalities (coherence conditions [73]). We give an abridged version below.

Definition 9.6.1 *A* ***symmetric monoidal category*** *is a category $\mathcal{C}$ with a distinguished object U and a binary operation $\otimes$ that does the following. For X and Y objects, $X \otimes Y$ is an object, and for $A : X \to Y$ and $A' : X' \to Y'$ morphisms, $A \otimes A' : X \otimes X' \to Y \otimes Y'$ is a morphism. It is required that the following conditions hold.*

1. *$X \otimes Y$ is isomorphic to $Y \otimes X$.*
2. *$X \otimes (Y \otimes Z)$ is isomorphic to $(X \otimes Y) \otimes Z$.*
3. *$X \otimes U$ is isomorphic to X.*
4. *When defined, $(A \otimes A') \circ (B \otimes B') = (A \circ B) \otimes (A' \circ B')$.*

In a symmetric monoidal category, $\otimes$ is often called a **tensor product**, and the object U the **tensor unit**. This terminology comes from the primary example of such categories, that of real vector spaces with $\otimes$ the usual tensor product of vector spaces. In this case, the tensor unit U is the one-dimensional vector space over the reals.

A related example is the category Mat of Definition 9.1.15 whose objects are finite sets and whose morphisms are matrices indexed by finite sets with entries in the reals. In this case, the tensor product $X \otimes Y$ of two sets is their ordinary set product $X \times Y$, and the tensor product of morphisms (matrices) $A : X \to Y$ and $A' : X' \to Y'$ is the Kronecker product $A \otimes A' : X \times X' \to Y \times Y'$. The tensor unit in Mat can be chosen to be any one-element set. We use $\{*\}$ for a standard one-element set.

Before the next results, we recall several facts. For any continuous t-norm $\triangle$ on I, Theorem 9.2.14 gives a category $\mathrm{FRel}_\triangle$. Objects of $\mathrm{FRel}_\triangle$ are sets. Morphisms are matrices indexed over sets with entries in I. Morphisms are composed via a variant of matrix multiplication that uses $\triangle$ as product and join in I as sum. We recall also the $\triangle$-Kronecker product of matrices described after Definition 9.4.1. The proofs of the following results are similar to those for Rel [41].

Theorem 9.6.2 *Let $\triangle$ be a continuous t-norm on* I. *Then* $\mathrm{FRel}_\triangle$ *is a symmetric monoidal category with the tensor product $\otimes_\triangle$ that is set product $X \times Y$*

on objects and $\triangle$-Kronecker product of matrices on morphisms. The tensor unit of this category is a one-element set $\{*\}$.

In this result, it is important to have the same t-norm $\triangle$ used for both the matrix multiplication and the formation of Kronecker products. Using different t-norms would cause problems with item 4 in Definition 9.6.1. There is also a version for the type-2 setting.

Theorem 9.6.3 *Let $\triangle$ be a continuous t-norm on* I *and $\blacktriangle$ be its convolution to L_u. Then $2\,\mathrm{FRel}_{\blacktriangle}$ is a symmetric monoidal category with the tensor product $\otimes_{\blacktriangle}$ that is the set product $X \times Y$ on objects and the $\blacktriangle$-Kronecker product of matrices on morphisms. The tensor unit of this category is a one-element set $\{*\}$.*

These categories are also known as **dagger categories**. They have an operation $\dagger$ where $X^{\dagger} = X$ for each object X, and $A^{\dagger} = A^{T}$ is the transpose of the matrix A. We note for $A : X \to Y$, that $A^{\dagger} : Y \to X$. So $\dagger$ reverses the flow of arrows. Further, they have a **biproduct** structure where the biproduct $X \oplus Y$ of objects is their disjoint union. See [41] for more details.

Definition 9.6.4 *In a symmetric monoidal category with tensor unit U, the morphisms $s : U \to U$ are called **scalars**.*

In any symmetric monoidal category, the scalars form a commutative monoid under composition, and in the presence of a biproduct structure, they are also equipped with a sum [73]. For the specific categories at hand, the scalars are matrices (a) with one entry. This leads to the following.

Proposition 9.6.5 *For a continuous t-norm $\triangle$ on* I*, the scalars of* $\mathrm{FRel}_{\triangle}$ *form a structure that is isomorphic to $(\mathrm{I}, \triangle, \vee)$, and the scalars of $2\,\mathrm{FRel}_{\blacktriangle}$ form a structure that is isomorphic to $(\mathrm{L}_u, \blacktriangle, \sqcup)$.*

We consider how the category FRel relates to fuzzy sets and controllers. For a set X, a fuzzy subset of X associates to each $x \in X$ a value in I, hence a vector with an entry for each element of X. Viewed as a row vector, this is a morphism from $\{*\}$ to X, and as a column vector it is the transpose, a morphism from X to $\{*\}$.

The linguistic variables Cold, Nice, and Hot are fuzzy subsets of X. We view them as morphisms from X to $\{*\}$. Taken together, they form a morphism P from X into the disjoint union $\{*\} \oplus \{*\} \oplus \{*\}$. Similarly, we view Air and Furnace as a morphism Q from Y into $\{*\} \oplus \{*\}$. The rule base R of Figure 9.3 can be viewed as a morphism from $\{*\} \oplus \{*\} \oplus \{*\}$ to $\{*\} \oplus \{*\}$. Then, the composite morphism from X to Y is given by

$$Q^{T} R P$$

Recall that we originally sought a function $\Phi : X \to Y$ for our controller. Of course, the morphism we have is not an ordinary function, and to turn it into one requires the processes of fuzzification and defuzzification. We have described how to do so, but currently there seems to be no natural categorical way to view this part of the process.

The modifications described in Section 9.4 obviously employ the monoidal structure of the category to incorporate additional dependent or independent variables. Also, a sequential process of taking the output from one controller (before defuzzification) and inputting it into another controller (without fuzzification) is given simply by composition. Lifting matters to the type-2 setting involves only passage to the category $2\,\mathrm{FRel}_{\blacktriangle}$ and is otherwise unchanged.

Finally, we remark that symmetric monoidal categories have been very usefully employed in other areas related to information flow, including logic, computation, and most recently quantum computation [1, 2]. Such programs become particularly applicable when the topic moves past small-scale implementations and more systematic study is required. Then, general tools for working with symmetric monoidal categories, such as the graphical calculus of [61, 100] become valuable. Also of interest is placing different, yet related studies, in broadly similar context.

9.7 Summary

Generalizations of matrices and matrix multiplication were given. These involved matrices indexed by sets and having entries in I or L_u. Multiplication of such matrices used meet, or another continuous t-norm, in place of product and join in place of sum.

For any continuous t-norm $\triangle$ on I, the product of matrices with entries in I taken using $\triangle$ is associative and has an identity. This allows us to define a category of fuzzy relations $\mathrm{FRel}_{\triangle}$ whose objects are sets and whose morphisms are such matrices under this product. For $\blacktriangle$ the convolution of $\triangle$ to L_u, results of Chapter 7 were used to show that the multiplication of matrices with entries in L_u taken using $\blacktriangle$ and join is also associative and has an identity. This allows us to define a category of type-2 fuzzy relations $2\,\mathrm{FRel}_{\blacktriangle}$.

An extended example was given illustrating the role played by matrices over I in working with fuzzy controllers. An assignment of linguistic variables and their fuzzy subsets to a quantity was shown to be given by such a matrix. A rule base for working with linguistic variables was shown to be given by such a matrix, and the computations for combining the linguistic variables and the rule base was shown to amount to the multiplication of the matrices or their transposes. The addition of additional independent, or dependent, variables was accomplished using the Kronecker product of matrices involved.

The fuzzy controller was extended to the type-2 setting. This replaced fuzzy subsets with members of L_u, certain convex normal functions. In practical terms, rather than an expert giving a specific degree such as .6 to a statement such as "70 degrees is COLD," the expert gives a function that peaks at .6 to express their confidence that varying degrees of truth apply to this statement. The subsequent analysis of the fuzzy controller was exactly as before, replacing matrices over I with matrices over L_u.

The categories $\mathrm{FRel}_\vartriangle$ and $2\,\mathrm{FRel}_\blacktriangle$ have a symmetric monoidal structure $\otimes$ given by products of sets and Kronecker products of matrices, and they also have a biproduct structure $\oplus$ given by disjoint unions of sets. The treatment of fuzzy controllers using matrices was formulated in a categorical context using such structure. It was also noted how other situations involving information flow in the sciences is treated by such symmetric monoidal categories.

9.8 Exercises

1. Let Metric be the category whose objects are all metric spaces and whose morphisms are all continuous maps between metric spaces. Prove that Metric is indeed a category.

2. Let G be a group, and define a category $\mathcal{C}_G$ as follows. Let there be a single object $*$. For each $g \in G$, consider g as a morphism from $*$ to itself. For morphisms g, h from $*$ to itself, define their composite to be their product gh taken in the group G. Prove that $\mathcal{C}_G$ is indeed a category.

3. Let D be a distributive lattice. Prove that when defined, multiplication of D-matrices is associative.

4. Prove that every bounded distributive lattice is a semiring in the sense of Definition 9.1.17.

5. Prove Proposition 9.2.5.

6. Prove Theorem 9.2.8.

7. Let S be a semiring in the sense of Definition 9.1.17. Give a proof that the category Mat_S whose objects are finite sets and whose morphisms are matrices over S is a category.

8. Referring to Exercise 7, prove that $\mathbb{R}$ is a semiring, and that the category Mat is the category $\mathrm{Mat}_{\mathbb{R}}$.

9. Let S be the 2-element distributive lattice 2. Referring to Exercise 7, prove that the category Mat_S is the category FRel whose objects are all finite sets, whose morphisms are the relations between them, and whose rule of composition is composition of relations.

10. Give an example of a distributive lattice D and a square D-matrix A whose determinant is non-zero, but with A not having an inverse.

11. Repeat the example (the whole of it) from Section 9.3 using matrix multiplication with $\wedge$ and $\vee$ from I rather than ordinary multiplication and $\vee$ as was done in the text.

12. Repeat the example (the whole of it) from Section 9.3 using a different rule base of your choice.

13. Consider the category Set whose objects are sets and whose morphisms are ordinary functions between sets under function composition. Show that the product $X \times Y$ of sets can be considered as a tensor product for this category.

Chapter 10

The Finite Case

This chapter addresses the situation when the unit interval that is used to construct M is replaced by two finite chains, possibly of different sizes. Again, the functions from one chain to the other form an algebra with operations defined via convolutions. These algebras are of interest on two counts. Many applications of fuzzy theory would involve only a finite subdivision of the chain I, so would take place in such algebras rather than the full algebra M. Also, these algebras are interesting mathematical entities per se. Much of the basic theory developed in previous chapters applies also in the finite setting. We consider topics such as subalgebras, automorphisms, and convex normal functions. There are also additional topics of interest in this setting, especially concerning the structure of these algebras and their irreducibles, and related orders.

10.1 Preliminaries

Here we consider more closely the matter of join irreducible elements in finite distributive lattices. We have seen such elements before in the study of automorphisms of M. But there are many aspects of such join irreducibles in the setting of finite distributive lattices that go beyond what we have so far considered. This is known as **Birkhoff duality**. (See Theorem 10.1.8.)

Definition 10.1.1 *Let L be a lattice and $x \in L$ be non-zero.*

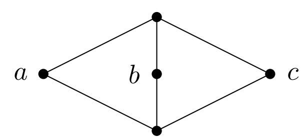
1. *x is **join irreducible** if $x = y \vee z$ implies that $x = y$ or $x = z$.*
2. *x is **join prime** if $x \leq y \vee z$ implies that $x \leq y$ or $x \leq z$.*

The definitions of **meet irreducible** and **meet prime** are defined dually, and as before, an element is **doubly irreducible** if it is both join irreducible and meet irreducible.

Proposition 10.1.2 *In any lattice, join prime implies join irreducible. In general, the two notions do not coincide, but they do coincide in any distributive lattice.*

Proof. Suppose that x is join prime and that $x = y \vee z$. Since x is the least upper bound of y and z, then $y \leq x$ and $z \leq x$. Since x is join prime and $x = y \vee z$, then $x \leq y \vee z$, so $x \leq y$ or $x \leq z$. Thus $x = y$ or $x = z$. So x is join irreducible.

The element a below is join irreducible, but not join prime since $a \leq b \vee c$. This lattice is not distributive since $a \wedge (b \vee c) \neq (a \wedge b) \vee (a \wedge c)$.



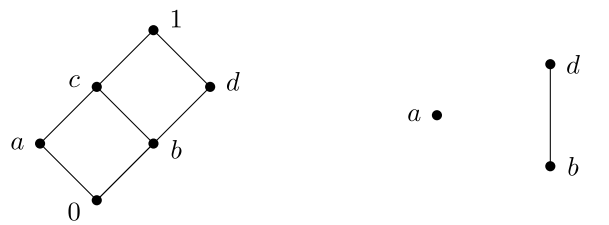
Finally, suppose that x is a join irreducible element in a distributive lattice. If $x \leq y \vee z$, then $x = x \wedge (y \vee z)$, hence by distributivity $x = (x \wedge y) \vee (x \wedge z)$. Since x is join irreducible, then $x = x \wedge y$ or $x = x \wedge z$. Thus either $x \leq y$ or $x \leq z$. So x is join prime. ■

Proposition 10.1.3 *In any finite lattice L, each element is the join of join irreducible elements.*

Proof. Note first that 0 is the join of the empty set, hence is a join of a set of irreducible elements. Suppose the statement is not true. Then among the elements of L that are not the join of join irreducible ones, there is one that is minimal, say x. If x is join irreducible, then x is the join of $\{x\}$, hence is the join of a set of one join irreducible element. If x is not join irreducible, then $x = y \vee z$ where $y < x$ and $z < x$. The minimality of x gives that both y and z are the join of join irreducibles, hence x is also the join of join irreducibles. ■

Definition 10.1.4 *For a lattice L, let $J(L)$ be the set of join irreducible elements of L. We consider $J(L)$ as a partially ordered set where the partial ordering is that inherited from L.*

This will be primarily employed for distributive lattices. An example is given below.



L $\quad$ $J(L)$

Definition 10.1.5 *A* ***downset*** *of a poset P is a subset $S \subseteq P$ such that $x \in S$ and $y \leq x$ implies $y \in S$. We let $D(P)$ be the collection of all downsets of P partially ordered by set inclusion.*

We note that the empty set is always a downset of a poset.

Proposition 10.1.6 *For any poset P, its downsets $D(P)$ form a distributive lattice.*

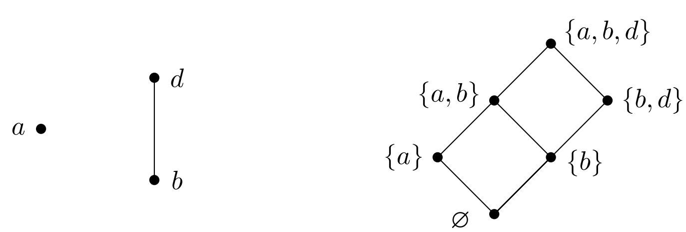
Proof. It is easy to see that the intersection of two downsets is a downset, and that the union of two downsets is a downset. So $D(P)$ is a sublattice of the power set of P, and therefore is a distributive lattice. ■

For an element p in a poset P, the **principle downset generated by** p is $\{q \in P : q \leq p\}$. This is often written as $p\downarrow$.

Lemma 10.1.7 *If P is a poset, then the join irreducible elements of $D(P)$ are exactly the principal downsets $p\downarrow$ where $p \in P$.*

Proof. If $p\downarrow$ is the union of two downsets, then at least one of the downsets must have p in it, and would then be equal to $p\downarrow$. So each $p\downarrow$ is join irreducible. Conversely, any downset S is the union of all $p\downarrow$ where p is a maximal element of S. If S is join irreducible, then it must equal some $p\downarrow$. ■

If we consider the poset $P = J(L)$ in the diagram above, its downsets are $\varnothing, \{a\}, \{b\}, \{a,b\}, \{b,d\}, \{a,b,d\}$. A diagram of $P = J(L)$ and its lattice of downsets is shown below.



$J(L)$ $\quad$ $D(J(L))$

One notices that in this example, the lattice of downsets of $J(L)$ is isomorphic to L, and the poset of join irreducibles of $D(P)$ is isomorphic to P. This is always the case.

Theorem 10.1.8 *Suppose that L is a finite distributive lattice and that P is a finite poset. Consider the maps*

$$\begin{array}{lll} \Phi : L \to D(J(L)) & \textit{given by} & \Phi(a) = \{j \in J(L) : j \le a\} \\ \Psi : P \to J(D(P)) & \textit{given by} & \Psi(p) = \{q \in P : q \le p\} \end{array}$$

Then Φ is a lattice isomorphism and Ψ is a poset isomorphism.

Proof. Let D be a downset of $J(L)$ and set $a = \bigvee D$. Then $D \subseteq \Phi(a)$. If j is a join irreducible in a distributive lattice, then by Proposition 10.1.2 it is join prime. So if $j \le a = \bigvee D$, then $j \le d$ for some $d \in D$, hence $j \in D$. So $\Phi(a) = D$. Thus Φ is onto. If $a \in L$, then a is the join of the join irreducibles beneath it, so $a = \bigvee \Phi(a)$. It follows that Φ is one-one. It is obvious that $\Phi(a \wedge b) = \Phi(a) \cap \Phi(b)$, so Φ preserves finite meets. Surely $\Phi(a) \cup \Phi(b) \subseteq \Phi(a \vee b)$. Suppose j is join irreducible and $j \le a \vee b$. Then as j is join prime, $j \le a$ or $j \le b$, hence $j \in \Phi(a) \cup \Phi(b)$. So Φ preserves finite joins as well. So Φ is a lattice isomorphism.

Lemma 10.1.7 shows that the join irreducibles of $D(P)$ are exactly the $p\downarrow$ where $p \in P$. So Ψ is a bijection, and it is easily seen that $p \le q$ if and only if $p\downarrow \subseteq q\downarrow$. So Ψ is an isomorphism of posets. ■

So there is a complete correspondence between finite posets and finite distributive lattices. Each finite distributive lattice is isomorphic to the downsets of some finite poset, and this finite poset is unique up to isomorphism. It is the poset of join irreducibles of the lattice. Conversely, each finite poset arises as the join irreducibles of a finite distributive lattice, and this lattice is unique up to isomorphism. More is true.

Proposition 10.1.9 *Let $f : L \to M$ be a bound preserving homomorphism between finite distributive lattices, and let $g : P \to Q$ be an order preserving map between finite posets. Consider*

$$\begin{array}{lll} J(f) : J(M) \to J(L) & \textit{given by} & J(f)(j) = \bigwedge\{x \in L : j \le f(x)\} \\ D(g) : D(Q) \to D(P) & \textit{given by} & D(g)(S) = \{p \in P : g(p) \in S\} \end{array}$$

Then $J(f)$ is an order-preserving map between posets and $D(g)$ is a bound-preserving lattice homomorphism.

The proof is left as an exercise (Exercise 6).

The collection of finite distributive lattices and the bound-preserving homomorphisms between them forms a category FDist, and the collection of finite posets and the order-preserving maps between them forms a category

FPos. Since J takes objects of FDist to objects of FPos and morphisms of FDist to morphisms of FPos, it can be considered as a map from FDist to FPos. Since J preserves identity maps and composition, it is known as a **contravariant functor**. The term "contravariant" indicates that it reverses the direction of morphisms.

In a similar way, D is a contravariant functor from FPos to FDist. The composites $J \circ D$ and $D \circ J$ provide objects isomorphic to the originals. Along with a some small technicalities [73], this amounts to the categories FDist and FPos being **dually equivalent**. In effect, rather than working with finite distributive lattices, one can work with finite posets, and conversely!

10.2 Finite type-2 algebras

In this section we introduce the basic algebras of this chapter, analogs of M in the finite case, and develop their basic properties.

Definition 10.2.1 *For a natural number n, let* n *be the algebra on the set* $\{1, 2, \dots, n\}$*, equipped with usual linear order* $\leq$*, with operations* $\vee$ *and* $\wedge$ *given by* $\max$ *and* $\min$*, constants* 1 *and* n*, and negation* $k^* = n - k + 1$*.*

We recall that the unit interval I with its constants and negation is a Kleene algebra. A similar situation holds for the finite case. The proof of the following is left as an exercise (Exercise 8).

Proposition 10.2.2 *For a natural number n, the algebra* n *is a Kleene algebra whose underlying lattice is a chain.*

We denote by m^{n} the set $\mathrm{Map}(\mathrm{n}, \mathrm{m}) = \{f : \mathrm{n} \to \mathrm{m}\}$ of all mappings from the set n into the set m. The algebra $\mathrm{m}^{\mathrm{n}} = (\mathrm{m}^{\mathrm{n}}, \sqcap, \sqcup, ^*, 0, 1)$ consists of the set m^{n} with operations given in the following definition. We remark that the notation used for the constants has changed from before since, in this new setting, the old notation no longer conveys an accurate meaning.

Definition 10.2.3 *The basic operations on* m^{n} *are the following.*

1. $(f \sqcap g)(i) = \bigvee_{j \wedge k = i} (f(j) \wedge g(k))$.
2. $(f \sqcup g)(i) = \bigvee_{j \vee k = i} (f(j) \wedge g(k))$.
3. $f^*(i) = f(n - i + 1)$.
4. $0(i) = \begin{cases} m & \text{if } i = 1 \\ 1 & \text{if } i \neq 1 \end{cases}$

5. $1(i) = \begin{cases} m & \text{if } i = n \\ 1 & \text{if } i \neq n \end{cases}$

These operations are completely analogous to the ones given for M in Chapter 1. There are two other useful operations on the functions in m^{n}, namely pointwise max and min. We also denote these by $\vee$ and $\wedge$, respectively. Just as in the case I^{I}, these operations help in determining the properties of the algebra m^{n} via the following auxiliary operations.

Definition 10.2.4 *For $f \in \mathrm{m}^{\mathrm{n}}$, let f^L and f^R be the elements defined by*

$$f^L(i) = \bigvee_{j \leq i} f(j) \quad \text{and} \quad f^R(i) = \bigvee_{i \leq j} f(j)$$

The operations $\sqcup$ and $\sqcap$ in m^{n} can be expressed in terms of the pointwise max and min of functions in two different ways, as follows. These are completely analogous to the ones for M given in Theorem 1.4.5.

Theorem 10.2.5 *The following hold for all $f, g \in \mathrm{m}^{\mathrm{n}}$.*

$$f \sqcup g = (f \wedge g^L) \vee (f^L \wedge g) = (f \vee g) \wedge (f^L \wedge g^L)$$
$$f \sqcap g = (f \wedge g^R) \vee (f^R \wedge g) = (f \vee g) \wedge (f^R \wedge g^R)$$

Just as in the case of M, it is fairly routine to verify the following properties of the algebra m^{n} using these auxiliary operations. The details of the proofs are almost exactly the same as for the algebra M. See Theorem 2.3.3.

Corollary 10.2.6 *Let f, g, $h \in \mathrm{m}^{\mathrm{n}}$. Some basic equations follow.*

1. $f \sqcup f = f$; $f \sqcap f = f$.
2. $f \sqcup g = g \sqcup f$; $f \sqcap g = g \sqcap f$.
3. $f \sqcup (g \sqcup h) = (f \sqcup g) \sqcup h$; $f \sqcap (g \sqcap h) = (f \sqcap g) \sqcap h$.
4. $f \sqcup (f \sqcap g) = f \sqcap (f \sqcup g)$.
5. $1 \sqcap f = f$; $0 \sqcup f = f$.
6. $f^{**} = f$.
7. $(f \sqcup g)^* = f^* \sqcap g^*$; $(f \sqcap g)^* = f^* \sqcup g^*$.

The following is then immediate.

Proposition 10.2.7 *Each algebra m^{n} is a De Morgan Birkhoff system.*

Except in trivial cases, m^{n} is not a lattice since the absorption law fails, and m^{n} fails to satisfy the distributive laws.

In the finite situation, we can draw diagrams of the bisemilattices m^{n} using their meet and join orders of Definition 2.1.7. In this situation, there is more that can be said about these orders, and that will be the subject of Section 10.4.

10.3 Subalgebras

The algebra m^{n} has subalgebras analogous to the subalgebras of M that were considered in Chapter 3. In this section, we study a number of these.

Definition 10.3.1 *For any $a \in \mathrm{n}$, define the **singleton function** s_a in m^{n} by*

$$s_a(k) = \begin{cases} 1 & \text{if } k \neq a \\ m & \text{if } k = a \end{cases}$$

The notation for singletons of M in Definition 3.2.1 has changed since it would be confusing in the current context. However, the ideas remain identical. Before, singletons were used in Theorem 3.2.3 to create a subalgebra of M that is isomorphic to I. The proof carries over exactly to give the following.

Proposition 10.3.2 *For any m, n, the singletons form a subalgebra of m^{n} that is isomorphic to n.*

There is an obvious analog of the functions in M that are characteristic functions of intervals. These were used to form a subalgebra of M that is isomorphic to $\mathrm{I}^{[2]}$. We will not develop this here, but move to analogs of the normal functions. This will be of particular importance, and we refine the notion somewhat.

Definition 10.3.3 *The **height** of a function $f \in \mathrm{m}^{\mathrm{n}}$ is the largest value that is attained by the function. The **normal** functions are those of height m. Let N_k be the set of functions of height k, and let $\mathrm{N} = \mathrm{N}_m$.*

Note that due to the finiteness of m^{n}, there is no distinction between the notions of normal and strictly normal, in contrast to the infinite case. The proof of the following proposition is immediate from definitions.

Proposition 10.3.4 *The following four conditions are equivalent for $f \in \mathrm{m}^{\mathrm{n}}$:*

1. *f has height k.*
2. *$f^{RL} = k$.*
3. *$f^{L}(n) = k$.*
4. *$f^{R}(1) = k$.*

The following proposition follows immediately, using Theorem 10.2.5.

Proposition 10.3.5 *For any $k \in \mathrm{m}$, the set N_k of functions of height k is a subalgebra of $(\mathrm{m}^{\mathrm{n}}, \sqcap, \sqcup, ^{*})$. The set $\mathrm{N} = \mathrm{N}_m$ of normal functions also contains the constants 0 and 1, hence is a subalgebra of $(\mathrm{m}^{\mathrm{n}}, \sqcap, \sqcup, ^{*}, 0, 1)$.*

Each function in m^{n} has a unique value k for its height. So m^{n} is the disjoint union of its subalgebras N_k. Also, the set N_k of functions of m^{n} of height k is literally equal to the algebra of normal functions of k^{n} once its constants are defined appropriately. This gives the following.

Theorem 10.3.6 *Each algebra* m^{n} *is the disjoint union of its subalgebras* N_k *of functions of height* k*,*

$$\mathrm{m}^{\mathrm{n}} = \bigoplus_{k=1}^{m} \mathrm{N}_k$$

Further, each N_k *is equal to the algebra of normal functions of* k^{n}*.*

With the definition for the height of a function in M as the constant value of f^{LR}, a corresponding theorem holds also for M. For each $\alpha \in \mathrm{I}$, the set N_α of functions in M of height α forms a subalgebra of $(\mathrm{M}, \sqcap, \sqcup, ^*)$, and M is the disjoint union of these subalgebras

$$\mathrm{M} = \bigoplus_{\alpha \in \mathrm{I}} \mathrm{N}_\alpha$$

Further, for each $\alpha \neq 0$, the algebra N_α is isomorphic to M once its constants are appropriately defined. This result is of greater use in the finite case, so it was not mentioned before.

Definition 10.3.7 *An element* $f \in \mathrm{m}^{\mathrm{n}}$ *is* ***convex*** *if whenever* $i \leq j \leq k$*, then* $f(j) \leq f(i) \wedge f(k)$*. Let* C *be the set of all convex functions in* m^{n}*.*

The convex functions $f \in \mathrm{m}^{\mathrm{n}}$ are exactly those that satisfy $f = f^L \wedge f^R$, or equivalently, those that are the pointwise meet of an increasing function and a decreasing function.

Proposition 10.3.8 *The set* C *of convex functions is a subalgebra of* m^{n}*.*

The proof is the same as that for Proposition 3.5.4. We also have the following, whose proof is the same as that for Theorem 3.5.7.

Theorem 10.3.9 *Given* $f \in \mathrm{M}$*, the distributive laws*

$$f \sqcup (g \sqcap h) = (f \sqcup g) \sqcap (f \sqcup h)$$
$$f \sqcap (g \sqcup h) = (f \sqcap g) \sqcup (f \sqcap h)$$

hold for all $g, h \in \mathrm{M}$ *if and only if* f *is convex. One of these distributive laws holds for a given* f *and for all* g, h *if and only if the other holds.*

The collection of functions in m^{n} that are both convex and normal will be a primary focus of a substantial part of this chapter. The types of questions we ask about them will be different than with the convex normal function of M, but still there will be much to say. Noting that the proof of Proposition 3.6.2 applies in the finite setting, we have the following.

Proposition 10.3.10 *The set* L *of convex normal functions is a subalgebra of* $\mathrm{m^n}$ *that is a De Morgan algebra.*

As a final comment, we have defined such subalgebras of $\mathrm{m^n}$ as L, C, and N. There will be situations where we consider an algebra $\mathrm{m^n}$, and another $\mathrm{p^q}$. In this case we use $\mathrm{L(m^n)}$ to mean the subalgebra L of convex normal functions of $\mathrm{m^n}$, and so forth.

10.4 The partial orders determined by ⊔ and ⊓

We consider the partial orders associated with ⊓ and ⊔. We use these to draw diagrams of $\mathrm{m^n}$. We also consider various properties of these orderings. In particular, although $\mathrm{m^n}$ is not a lattice, each partial order gives a lattice ordering. A description of joins and meets in each order is given.

Definition 10.4.1 *Define relations* $\sqsubseteq_\sqcap$ *and* $\sqsubseteq_\sqcup$ *on* $\mathrm{m^n}$ *as follows:*

$$\begin{array}{lll} f \sqsubseteq_\sqcap g & \text{if} & f \sqcap g = f \\ f \sqsubseteq_\sqcup g & \text{if} & f \sqcup g = g \end{array}$$

We call $\sqsubseteq_\sqcap$ *the* ***meet order*** *and* $\sqsubseteq_\sqcup$ *the* ***join order****.*

These relations can be expressed in terms of the pointwise order, using the auxiliary operations L and R. The proof of the following result is the same as that of Proposition 2.4.3.

Proposition 10.4.2 *For* f, g *in* $\mathrm{m^n}$ *we have the following:*

1. $f \sqsubseteq_\sqcup g$ *if and only if* $f \wedge g^L \le g \le f^L$.
2. $f \sqsubseteq_\sqcap g$ *if and only if* $f^R \wedge g \le f \le g^R$.

Since $\mathrm{m^n}$ is a bisemilattice, the relation $\sqsubseteq_\sqcap$ is a meet semilattice order with meets given by ⊓, and $\sqsubseteq_\sqcup$ is a join semilattice order with joins given by ⊔. So the operations ⊓ and ⊔ can be described by giving diagrams to describe these orders. Before giving an example, we make some general comments about describing elements of $\mathrm{m^n}$.

The elements of $\mathrm{m^n}$ may be viewed as n-tuples $(a_1, \ldots, a_n)$ of elements of m. When no confusion arises, we treat these as strings. For example, we write the element $(2, 1, 3, 2)$ of 3^4 as 2132. With this notation, the constant 0 is $(m, 1, \ldots, 1)$ and the constant 1 is $(1, \ldots, 1, m)$. Finally, $(a_1, a_2, \ldots, a_n)^* = (a_n, a_{n-1}, \ldots, a_1)$.

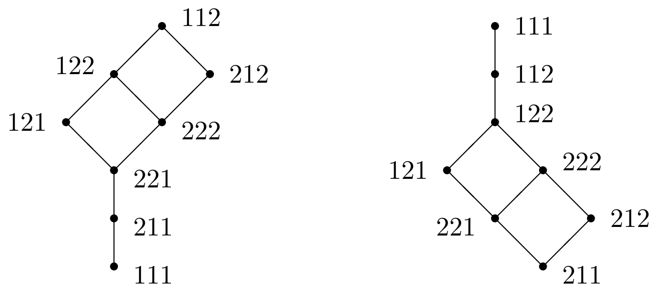


FIGURE 10.1: 2^3 under the meet order (left) and join order (right)

There are things to notice about Figure 10.1. If one flips the meet order at left upside down, it looks like the join order at right, but the labeling is changed. Each label is reversed. This is because $f \sqsubseteq_{\sqcap} g$ if and only if $g^* \sqsubseteq_{\sqcup} f^*$. The constant 0 is the bottom of the join order and the constant 1 is the top of the meet order.

Definition 10.4.3 *An* ***absorbing element*** *of a bisemilattice B is an element e such that $e \sqcap f = e = e \sqcup f$ for all $f \in B$.*

In Figure 10.1, the element 111 that occurs at the bottom of the meet order and the top of the join order is an absorbing element. This is a general situation.

Proposition 10.4.4 *In m^{n}, the element e that takes constant value 1 is an absorbing element, and in M the function that takes constant value 0 is an absorbing element.*

There is a further property that is apparent from looking at Figure 10.1, namely, both the meet and join orders are not only semilattice orders, but are lattice orders! This is a general situation, and will comprise the remainder of this section. We begin with the following restatement of the situation for any bisemilattice.

Proposition 10.4.5 *In m^{n}, any two elements f and g have a least upper bound in the join order $\sqsubseteq_{\sqcup}$ and this is given by $f \sqcup g$. Similarly, f and g have a greatest lower bound in the meet order $\sqsubseteq_{\sqcap}$ and this is given by $f \sqcap g$.*

Of course, this result is true in any bisemilattice, and in particular in M, as we have used before. However, the following result is special to the finite case, and examples of [51] show that it does not hold in M.

Theorem 10.4.6 *In the join order of m^{n}, any two elements have a greatest lower bound and a least upper bound. Thus m^{n} is a lattice under the join order. Similarly, m^{n} is a lattice also under the meet order.*

Proof. The supremum of f and g in the join order is $f \sqcup g$. The infimum of f and g in the join order is the supremum of all elements below both. Such a supremum exists because m^{n} is finite and there is at least one element below both, namely the element 0. Thus m^{n} is a lattice in the join order, and the result for the meet order follows from the dual isomorphism *. ∎

We note that the join of two elements of m^{n} in the join order is given by $\sqcup$, but it is not usually the case that the meet of two elements in the join order is given by $\sqcap$. Examples in Figure 10.1 show this. Similarly, the meet of two elements in the meet order is given by $\sqcap$, but their join is given by something other than $\sqcup$.

Definition 10.4.7 *In* m^{n}*, we let* $f \odot g$ *be the meet of* f *and* g *in the join order, and we let* $f \oplus g$ *be the join of* f *and* g *in the meet order.*

So the proof of Theorem 10.4.6 has shown that

$$f \odot g = \bigsqcup\{ h : h \sqsubseteq_{\sqcup} f, g\}$$
$$f \oplus g = \bigsqcap\{ h : f, g \sqsubseteq_{\sqcap} h\}$$

So $(\mathrm{m}^{\mathrm{n}}, \odot, \sqcup)$ is the algebraic version of the bounded lattice m^{n} under the join order, and $(\mathrm{m}^{\mathrm{n}}, \sqcap, \oplus)$ is the algebraic version of the lattice m^{n} under the meet order. The proof of Theorem 10.4.6 also yields the following.

Proposition 10.4.8 *If* S *is a subalgebra of* $(\mathrm{m}^{\mathrm{n}}, \sqcup, 0)$*, then* S *is a lattice under the join order, but not necessarily a sublattice of* $(\mathrm{m}^{\mathrm{n}}, \odot, \sqcup)$.

Two important subalgebras of $(\mathrm{m}^{\mathrm{n}}, \sqcup, 0)$ are the subalgebra N of normal elements and the subalgebra C of convex elements. The result above shows that both are lattices under the join order, and that joins in these lattices are given by $\sqcup$. To get them to be sublattices of the lattice m^{n} in the join order requires that meets in these lattices be given by $\odot$. For this, we need more.

Lemma 10.4.9 *In* m^{n}*, if* f *is normal, then any element* g *below* f *in the join order is normal.*

Proof. Suppose that f is normal and $f \sqcup g = f$. We need that g is normal.

$$f = f \sqcup g = (f \vee g) \wedge f^L \wedge g^L$$

Since f assumes the value m, so does g^L, whence g is normal. ∎

Theorem 10.4.10 *The normal functions of* m^{n} *are a sublattice of the lattice* m^{n} *under the join order, and a sublattice of the lattice* m^{n} *under the meet order.*

Proof. The supremum of two elements f and g in the join order is $f \sqcup g$. So by Proposition 10.3.5, the normal elements are closed under suprema in this poset. The infimum of f and g in this poset is the supremum of all their lower bounds in the join order. If f and g are normal, Lemma 10.4.9 shows these lower bounds are normal, hence their supremum is again normal. This shows the normal elements are a sublattice of the lattice $\mathrm{m^n}$ under the join order, and the proof for the meet order follows via the dual isomorphism *. ∎

A similar situation holds for the set C of convex elements of $\mathrm{m^n}$, but is a bit more delicate. For example, it is not true that elements below convex elements are convex, as illustrated by Figure 10.1, where the nonconvex element 212 is below the convex element 222. The proof of the following uses several results about convex functions in M that hold also in the finite setting.

Theorem 10.4.11 *The convex functions of* $\mathrm{m^n}$ *are a sublattice of the lattice* $\mathrm{m^n}$ *under the join order, and a sublattice of the lattice* $\mathrm{m^n}$ *under the meet order.*

Proof. We give the proof for the join order. By Proposition 10.3.10 the set of convex elements is closed under $\sqcup$, so the convex elements are closed under suprema in the join order. The infimum $\odot$ of two elements in the join order is the supremum of their common lower bounds in this order. Suppose f and g are convex and that h is a lower bound of f and g in the join order.

By Theorem 3.5.6, $\Gamma(h) = h^L \wedge h^R$ is a convex element that lies above h in the join order. Also by Theorem 3.5.6, the function Γ preserves $\sqcup$, so is order preserving with respect to the join order, and it fixes the convex elements f and g. So $\Gamma(h)$ is a convex lower bound of f, g that lies above h. Thus the infimum of f and g is the supremum of convex elements that are common lower bounds, hence is convex. ∎

We next consider the convex normal functions L. Since they are the intersection of the normal functions N and the convex functions C, they form a sublattice of $\mathrm{m^n}$ under both the join order and under the meet order. But more can be said. By Proposition 10.3.10, L is a distributive lattice under the operations $\sqcap$ and $\sqcup$. This immediately gives the following.

Proposition 10.4.12 *For convex normal functions f and g in* $\mathrm{m^n}$*,*

$$f \sqsubseteq_{\sqcap} g \quad \text{if and only if} \quad f \sqsubseteq_{\sqcup} g$$

The normal functions N are a sublattice of $(\mathrm{m^n}, \odot, \sqcup)$. Joins in N are given by $\sqcup$, but meets are given by $\odot$, and this is not equal to $\sqcap$ in N. To see this, in Figure 10.1 we have $212 \sqcap 122 = 222$, while in the join order 212 lies beneath 122, so $212 \odot 122 = 212$. Similar comments hold for C as is seen by considering the elements 111 and 222 in Figure 10.1. The situation for the convex normal functions L is different.

Theorem 10.4.13 *The convex normal functions* L *are a sublattice of* m^{n} *under the join order and under the meet order. In both cases, joins are given by* $\sqcup$ *and meets are given by* $\sqcap$*, and in both cases this sublattice is distributive.*

Proof. Let $f, g \in \mathrm{L}$. Then $f \odot g$ is their greatest lower bound in $(\mathrm{m}^{\mathrm{n}}, \sqsubseteq_{\sqcup})$, and hence also in $(\mathrm{L}, \sqsubseteq_{\sqcup})$. Since $\sqsubseteq_{\sqcup}$ agrees with $\sqsubseteq_{\sqcap}$ in L, we have that $f \odot g$ is their greatest lower bound in $(\mathrm{L}, \sqsubseteq_{\sqcap})$. But this greatest lower bound is $f \sqcap g$. So $f \odot g = f \sqcap g$. A similar argument shows that $f \oplus g = f \sqcup g$. ∎

We now turn our attention to the matter of describing meets $\odot$ in the lattice m^{n}. Of course, we know that

$$f \odot g = \bigsqcup \{h : h \sqsubseteq_{\sqcup} f, g\}$$

For even moderately large examples, it would be intractable to use this formula to compute $\odot$. We seek a better description. Ideal would be a description of $\odot$ in terms of the operations $\wedge, \vee, L, R$ much like Theorem 1.4.5. We show that this is not possible.

Proposition 10.4.14 *There is no equational expression that describes* $\odot$ *in terms of* $\wedge, \vee, L, R$*.*

Proof. Consider the elements $f = 221$ and $g = 333$ in 3^3 under the join order. Their meet $f \odot g$ can be computed to be 331. However, the closure of $\{f, g\}$ under the operations $\wedge, \vee, L, R$ is the set $\{221, 222, 333\}$ and this does not include 331. ∎

While a simple equation to describe $\odot$ is beyond us, we do give a polynomial time algorithm to compute $f \odot g$ without finding all the lower bounds of f and g in the join order. This algorithm would not be so friendly to compute by hand, but could be easily implemented on a computer.

Theorem 10.4.15 *There is a polynomial time algorithm to compute* $f \odot g$*.*

The proof is rather technical, and is found in complete detail in [51]. We do not reproduce the proof here, but do describe the algorithm.

Step 1 Find t and numbers a_i, b_i in $\{1, \dots, n\}$ for each $1 \le i \le t$ so that

1. $a_i \le b_i < a_{i+1}$ for each $1 \le i < t$.
2. $f = f^L$ and $g = g^L$ on each interval $[a_i, b_i]$.
3. The intervals $[a_i, b_i]$ are maximal with the property in item 2.
4. Each point where $f = f^L$ and $g = g^L$ belongs to some $[a_i, b_i]$.

Note that $f = f^L$ and $g = g^L$ at i for $i = 1$. Continue as long as $f = f^L$ and $g = g^L$. The last value before a break in this pattern is set to b_1. If we continue, there may or may not come another spot i where $f = f^L$ and $g = g^L$. If there is, then the first such spot is a_2, and we continue until the pattern again stops, with b_2 being the last time before the break, and so forth. If we set $a_{t+1} = n + 1$, the domain $\{1, \ldots, n\}$ is partitioned into intervals

$$\{1, \ldots, n\} = [a_1, b_1] \cup (b_1, a_2) \cup [a_2, b_2] \cup \cdots \cup [a_t, b_t] \cup (b_t, a_{t+1})$$

Definition 10.4.16 *For $f \in \mathrm{m}^{\mathrm{n}}$, let $\hat{f} \in \mathrm{m}^{\mathrm{n}}$ be the function*

$$\hat{f}(x) = \begin{cases} f(x) & \text{if } f(x) < f^L(x) \\ m & \text{if } f(x) = f^L(x) \end{cases}$$

Step 2 With this definition, we can now describe the meet $f \odot g$.

$$(f \odot g)(x) = \begin{cases} (f \vee g)(x) & \text{if } x \in [a_i, b_i) \text{ for some } i \\ \sup\{(f \vee g)(y) : b_i \leq y < a_{i+1}\} & \text{if } x = b_i \text{ for some } i \\ (\hat{f} \wedge \hat{g})(x) & \text{otherwise} \end{cases}$$

Example 10.4.17 Consider $f = 13355616$ and $g = 45627737$ in 7^8. We note that $f = f^L$ and $g = g^L$ at the spots $1, 2, 3, 5, 6, 8$. So we wind up with $t = 3$ intervals

$$[1, 3] \cup [5, 6] \cup [8, 8]$$

Then the first case of the definition of $f \odot g$ specifies its value to be that of $f \vee g$ at $x = 1, 2, 5$. The second case of this definition specifies the value of $f \odot g$ at $x = 3, 6, 8$, and the third case gives the value of $f \odot g$ at the other spots, $x = 4, 7$. It follows that $f \odot g = 45527717$.

10.5 The double order

In this section, we provide an alternate ordering of the elements of m^{n} and show that under this ordering, the elements N_k of height k form an involutive lattice, and that m^{n} under this order is formed in a simple way from the lattices N_k. This seems not to have direct application to fuzzy theory, but is of interest from a purely algebraic perspective.

Definition 10.5.1 *Suppose that R and S are relations on the same set X. Then $R \cap S$ is the relation on X where*

$$x\,(R \cap S)\,y \quad \text{if and only if} \quad x\,R\,y \text{ and } x\,S\,y$$

If a relation on X is viewed as a set of ordered pairs of elements of X as in the preliminaries of Section 9.1, then the relation $R \cap S$ is literally the intersection of the relations R and S.

Proposition 10.5.2 *If R and S are partial orderings on the same set X, then their intersection $R \cap S$ is also a partial ordering on X.*

Proof. Suppose $x \in X$. Since R is a partial order, it is reflexive, so $x\,R\,x$, and similarly $x\,S\,x$. Thus $x\,(R \cap S)\,x$. So $R \cap S$ is reflexive. The remainder of the proof is left as an exercise (Exercise 11). ■

Definition 10.5.3 *Let $\sqsubseteq$ be the intersection of the join order $\sqsubseteq_{\sqcup}$ on $\mathrm{m^n}$ and the meet order $\sqsubseteq_{\sqcap}$ on $\mathrm{m^n}$. We call $\sqsubseteq$ the **double order** on $\mathrm{m^n}$.*

From Theorem 10.4.2 we get the following.

Proposition 10.5.4 *$f \sqsubseteq g$ if and only if the following conditions hold.*

1. $f \wedge g^L \leq g \leq f^L$
2. $f^R \wedge g \leq f \leq g^R$

Theorem 10.3.6 shows that as a set, $\mathrm{m^n}$ is equal to the disjoint union of its subalgebras N_k of functions of height k. This is not overly helpful in describing the structure of $\mathrm{m^n}$ under the join or meet order since there may be non-trivial relationships between members of different subalgebras N_k. The following shows that the situation for the double order is much simpler.

Proposition 10.5.5 *If two elements f and g in $\mathrm{m^n}$ are comparable in the double order $\sqsubseteq$, then they have the same height.*

Proof. By Theorem 10.5.4, if $f \sqsubseteq g$, then $g \leq f^L$ and $f \leq g^R$. Since $f^{LR} = f^{RL}$ is the height of f, it follows that f and g have the same height. ■

So $\mathrm{m^n}$ is not only the disjoint union of the sets N_k where $k = 1, \ldots, m$, the double order of $\mathrm{m^n}$ is determined from the double orders of the N_k. Further, N_k is isomorphic to the normal functions of $\mathrm{k^n}$ as an algebra and as a poset in the double order. So the double order of $\mathrm{m^n}$ is completely determined by the double orders of the normal functions of $\mathrm{k^n}$ for $k = 1, \ldots, n$.

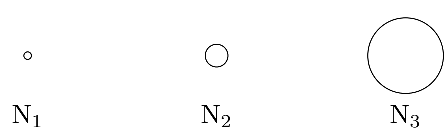


FIGURE 10.2: A sketch of the double order of 3^3

In Figure 10.2, a rough sketch is made of 3^3 under the double order. This poset is built of three unrelated pieces. The piece N_1 is just 111, the only function of height 1. The piece N_2 is all functions of height 2, and there are 7 of them, all strings of length 3 using the entries 1 and 2 except the 111. The final piece N_3 has 19 elements, for a total of 27 elements in 3^3. The diagram for N_3 is already complicated. Rather than show it, we show a somewhat simpler case below. The solid circles in this figure indicate the convex normal functions.

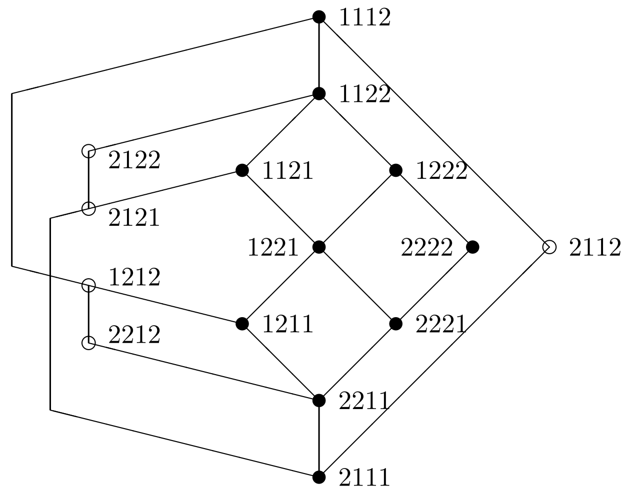


FIGURE 10.3: The normal functions in 2^4 under the double order

We next develop basic, and surprising, properties of the posets of normal functions of m^n under the double order. We need another notion.

Definition 10.5.6 *An **involution** on a poset P is a mapping $^*: P \to P$ that satisfies $x^{**} = x$ and $x \le y$ implies $y^* \le x^*$. An **involutive lattice** is a lattice with an involution.*

Proposition 10.5.7 *The map * on m^n is an involution.*

Proof. Since f^* reverses the order of f when viewed as a string, $f^{**} = f$. Corollary 10.2.6 provides $(f \sqcup g)^* = f^* \sqcap g^*$ and $(f \sqcap g)^* = f^* \sqcup g^*$. Therefore $f \sqsubseteq_\sqcap g$ implies $f \sqcap g = f$, giving $f^* \sqcup g^* = f^*$, so $g^* \sqsubseteq_\sqcup f^*$. Similarly $f \sqsubseteq_\sqcup g$ implies $g^* \sqsubseteq_\sqcap f^*$. It follows that $f \sqsubseteq g$ implies $g^* \sqsubseteq f^*$. ■

A much stronger result holds. Proposition 10.5.2 shows that the intersection of two partial orderings is a partial ordering. However, the intersection of two partial orderings that are lattice orderings is seldom a lattice ordering. (Exercises 12 and 13 discuss this in detail). So the following result is quite unexpected, and we have no firm understanding why it is true. Its proof is also demanding, and we refer the reader to [51].

Theorem 10.5.8 *The set* N *of normal functions in* $\mathrm{m^n}$ *is an involutive lattice under the double order.*

It would be nice to have a simple term description of meet in the double order involving only $\wedge, \vee, L, R$. There is none. In 3^4, the functions $f = (3,3,1,2)$ and $g = (3,3,2,3)$ have meet $(3,3,1,1)$ in the double order, and this cannot be expressed by applying these operations to f and g. For the subalgebra of convex normal functions, however, the situation is less complicated.

Proposition 10.5.9 *The convex normal functions are a sub-involutive lattice of the involutive lattice of normal functions of* $\mathrm{m^n}$ *under the double order. Further, these convex normal functions form a De Morgan algebra.*

Proof. Let f and g be convex normal functions. Note that Proposition 10.4.12 shows that the join order, meet order, and double order on the convex normal functions all agree. Further, meet and join in the lattice of convex normal functions are given by $\sqcap$ and $\sqcup$. We show that if h is a lower bound of f and g in the double order, then $h \sqsubseteq f \sqcap g$. This shows that meet in the lattice of convex normal functions agrees with the meet of convex normal functions in the lattice of normal functions under the double order. A similar proof establishes the corresponding result for joins.

Assume h is a lower bound of f and g in the double order. This implies that (a) $f \sqcap h = h$, (b) $g \sqcap h = h$, (c) $f \sqcup h = f$, and (d) $g \sqcup h = g$. From the first two items and the associativity of $\sqcap$, we obtain $(f \sqcap g) \sqcap h = h$, hence h lies beneath $f \sqcap g$ in the meet order. For the join order, we use the first item to obtain $(f \sqcap g) \sqcup h = (f \sqcap g) \sqcup (f \sqcap h)$. Since f is convex, Theorem 10.3.9 gives $(f \sqcap g) \sqcup (f \sqcap h) = f \sqcap (g \sqcup h)$. Using the fourth item, this becomes $f \sqcap g$. Thus $(f \sqcap g) \sqcup h = f \sqcap g$, showing that h lies beneath $f \sqcap g$ in the join order. Thus $h \sqsubseteq f \sqcap g$, showing that $f \sqcap g$ is the meet of f and g in the double order. ∎

Definition 10.5.10 *For elements f, g of a poset, f is a* ***cover*** *of g if $f > g$ and there does not exist an h such that $f > h > g$.*

A useful way to describe a finite poset is to give a description of the covers of each element. If one is to draw a picture of the partial ordering, this is among the best ways to describe the poset. Here, we give a simple algorithm for computing covers in the lattice of normal functions of $\mathrm{m^n}$ under the double order. Again, the proofs are involved, and are found in [51].

Theorem 10.5.11 *Suppose* N *is the normal functions of* $\mathrm{m^n}$ *under the double order and $f \in \mathrm{N}$ is given by the string $x_1 \cdots x_n$. Then $g \in \mathrm{N}$ is a cover of f if and only if it is obtained from f by one of the following rules.*

Rule 1 *For some x_i where $j < i \Rightarrow x_j < x_i$, change x_i to $x_i - 1$.*

Rule 2 *For some x_i where $i < j \Rightarrow x_i \geq x_j$, change x_i to $x_i + 1$.*

Example 10.5.12 As an illustration of the use of Theorem 10.5.11, consider the normal elements of 2^4 shown in Figure 10.3. The constant $0 = 2111$ is the least element of this lattice. It is not possible to apply Rule 1 to it since decreasing any element would not give a normal function in N. There are three ways to apply Rule 2 to 2111, at the second, third, and fourth spots. This gives the covers 2211, 2121, and 2112 of 2111. Consider now the element 2211. Rule 1 can be applied at the second spot to produce the cover 1211. Rule 2 can be applied at the third and fourth spots to produce its other two covers 2221 and 2212. Continuing in this way, it is simple to construct the lattice of Figure 10.3.

10.6 Varieties related to m^n

In this section, we describe the varieties generated by the algebras m^n, and show that except for some small values of m, n, these are the variety $\mathcal{V}(\mathrm{M})$ generated by M. The key step is in relating the algebras m^n of this chapter to the complex algebras 2^C of a chain C discussed in Chapter 7. We recall the definition.

Definition 10.6.1 *For a chain C with bounds 0 and 1 and involution $'$, the* ***complex algebra*** 2^C *is all subsets of C with constants $1_0 = \{0\}$, $1_1 = \{1\}$, and operations $\sqcap$, $\sqcup$, and * given by*

$$\begin{aligned} A \sqcap B &= \{a \wedge b : a \in A, b \in B\} \\ A \sqcup B &= \{a \vee b : a \in A, b \in B\} \\ A^* &= \{a' : a \in A\} \end{aligned}$$

Of course, when $m = 2$ and C is the finite chain $\mathrm{n} = \{1, \ldots, n\}$, there is confusion between the algebra m^n of this chapter, and the complex algebra 2^n. These algebras are not literally equal; the elements of m^n are functions $f : \mathrm{n} \to \mathrm{m}$ and the elements of the complex algebra 2^n are subsets of n. However, they are isomorphic algebras.

Proposition 10.6.2 *For $m = 2$ and any n, the algebra m^n of all functions from* n *to* m *with the convolution operations is isomorphic to the complex algebra 2^n of the chain* n.

Proof. It is well known that there is a bijection Φ from $\mathrm{m}^\mathrm{n} = \{f \mid f : \mathrm{n} \to 2\}$ to the power set $2^\mathrm{n} = \{A : A \subseteq \mathrm{n}\}$ given by

$$\Phi(f) = \{k \in \mathrm{n} : f(k) = 1\}$$

This map Φ is a homomorphism with respect to the pointwise meet and join operations $\wedge$ and $\vee$ of $\mathrm{m^n}$ and the operations of intersection and union of the power set. As in Definition 7.6.3, there are operations L and R on 2^{n} given by

$$A^L = \{k : a \leq k \text{ for some } a \in A\}$$
$$A^R = \{k : k \leq a \text{ for some } a \in A\}$$

It is easily seen that $\Phi(f^L) = \Phi(f)^L$ and $\Phi(f^R) = \Phi(f)^R$. Lemma 7.6.4 shows that $\sqcap$ and $\sqcup$ are defined from $\wedge, \vee, L, R$ in the same way in both $\mathrm{m^n}$ and 2^{n}, so Φ preserves $\sqcap$ and $\sqcup$. Finally, it is easy to see that Φ takes the constants 0 and 1 of $\mathrm{m^n}$ to the constants $1_0 = \{0\}$ and $1_1 = \{1\}$ of the complex algebra. ■

The results of Section 7.7 then give the following corollary. Here, as before, $\mathcal{V}(\mathrm{m^n})$ is the variety generated by the algebra $(\mathrm{m^n}, \sqcap, \sqcup, ^*, 0, 1)$ of full type, and $\mathcal{V}(\mathrm{m^n}, \sqcap, \sqcup)$ is the variety generated by its reduct $(\mathrm{m^n}, \sqcap, \sqcup)$.

Corollary 10.6.3 *For any $n \geq 5$, $\mathcal{V}(2^{\mathrm{n}})$ is equal to $\mathcal{V}(\mathrm{M})$, and for any $n \geq 3$, $\mathcal{V}(2^{\mathrm{n}}, \sqcap, \sqcup)$ is equal to $\mathcal{V}(\mathrm{M}, \sqcap, \sqcup)$.*

We now consider varieties generated by $\mathrm{m^n}$ for values of m other than 2. For $m = 1$ the algebra $\mathrm{m^n}$ has one element for each n. So we consider cases when $m \geq 2$.

Proposition 10.6.4 *Suppose that $m \geq 2$.*

1. *If $n \geq 5$, then $\mathrm{m^n}$ generates the same variety as M.*
2. *If $n \geq 3$, then $(\mathrm{m^n}, \sqcap, \sqcup)$ generates the same variety as $(\mathrm{M}, \sqcap, \sqcup)$.*

Proof. For $m, n \geq 2$, Proposition 3.9.3 shows that M has a subalgebra that is isomorphic to $\mathrm{m^n}$. This implies that for $m, n \geq 2$, $\mathcal{V}(\mathrm{m^n}) \subseteq \mathcal{V}(\mathrm{M})$, and $\mathcal{V}(\mathrm{m^n}, \sqcap, \sqcup) \subseteq \mathcal{V}(\mathrm{M}, \sqcap, \sqcup)$. For any m, n with $m \geq 2$, the algebra 2^{n} is isomorphic to a subalgebra of m^n, namely to the subalgebra of functions that take only values in $\{1, m\}$. So $\mathcal{V}(2^{\mathrm{n}}) \subseteq \mathcal{V}(\mathrm{m^n})$ and $\mathcal{V}(2^{\mathrm{n}}, \sqcap, \sqcup) \subseteq \mathcal{V}(\mathrm{m^n}, \sqcap, \sqcup)$. The result then follows from Corollary 10.6.3. ■

There remain some cases for smaller values of m or n that are not settled by these results. We have not placed these among known varieties, but this is likely not difficult.

10.7 The automorphism group of $\mathrm{m^n}$

In this section we describe the automorphism groups of the algebras $\mathrm{m^n}$. We recall the main result from Chapter 4.

Theorem 10.7.1 *The automorphisms of* $\mathrm{M} = \mathrm{Map}(\mathrm{I}, \mathrm{I})$ *are formed by taking automorphisms* $\alpha, \beta \in \mathrm{I}$ *and using these to map* $f \in \mathrm{M}$ *to* $g \in \mathrm{M}$ *where*

$$g = \alpha \circ f \circ \beta^{-1}$$

Thus, $\mathrm{Aut}(\mathrm{M})$ *is isomorphic to the product* $\mathrm{Aut}(\mathrm{I}) \times \mathrm{Aut}(\mathrm{I})$.

Since $\mathrm{m}^{\mathrm{n}} = \mathrm{Map}(\mathrm{n}, \mathrm{m})$, one might expect that a similar result holds in the finite case. Indeed, it does. But a finite chain has no automorphisms other than the trivial one, the identity map. In fact, we will show the following.

Theorem 10.7.2 *The automorphism group of* $(\mathrm{m}^{\mathrm{n}}, \sqcap, \sqcup)$ *is trivial.*

This implies that the automorphism group of m^{n} with all the operations is trivial as well. The proof is similar to the result for $\mathrm{Aut}(\mathrm{M})$ in Chapter 4 and is involved. We sketch an outline since it provides interesting information about the algebras m^{n}. As in Chapter 4, a key element is determining the irreducibles in the algebra.

Definition 10.7.3 *An element* f *of* $(\mathrm{m}^{\mathrm{n}}, \sqcap, \sqcup)$ *is*

1. ***join irreducible*** *if* $f = g \sqcup h$ *implies that* $f = g$ *or* $f = h$.
2. ***meet irreducible*** *if* $f = g \sqcap h$ *implies that* $f = g$ *or* $f = h$.
3. ***irreducible*** *if it is both join and meet irreducible.*

We use the following notation for functions in m^{n}. For $x \in \mathrm{m}$, $\underline{x}$ is the constant function with value x; that is, $\underline{x}(i) = x$ for all $i \in \mathrm{n}$. For $x \in \mathrm{m}$ and $i \in \mathrm{n}$, x_i is the point function whose i-th component is x and whose other components are 1. Finally, for any $f \in \mathrm{m}^{\mathrm{n}}$, f_i denotes the i-th component of f; that is, $f_i = f(i)$.

Theorem 10.7.4 *Let* $m, n \geq 2$. *The irreducible elements of* $(m^n, \sqcap, \sqcup)$ *are these:*

1. *The absorbing element* $\underline{1}$.
2. *The elements* m_i.
3. *The elements* $m_1 \vee x_n$ *and* $x_1 \vee m_n$.
4. *If* $n = 2$, *all normal elements and the absorbing element* $\underline{1}$.

Although the constants 0 and 1 are not operations in the algebra $(\mathrm{m}^{\mathrm{n}}, \sqcap, \sqcup)$, they are preserved by automorphisms of that algebra since they are the unique elements with $f \sqcup 0 = f$ and $f \sqcap 1 = f$ for all f. The following finite version of Proposition 4.5.1 follows.

Theorem 10.7.5 *If φ is an automorphism of* $(\mathrm{m}^{\mathrm{n}}, \sqcap, \sqcup)$*, the following hold:*

1. *If f is convex, then so is $\varphi(f)$.*
2. *If f is normal, then so is $\varphi(f)$.*

The path is then to show that any automorphism φ fixes the element $\underline{m}$ of m^{n}, and this immediately gives the following.

Proposition 10.7.6 *Let φ be an automorphism of* $(\mathrm{m}^{\mathrm{n}}, \sqcap, \sqcup)$ *and* $f \in \mathrm{M}^{\mathrm{n}}$.

1. $\varphi(f)^L = \varphi(f^L)$.
2. $\varphi(f)^R = \varphi(f^R)$.

With these tools, the proof that the automorphism group of $(\mathrm{m}^{\mathrm{n}}, \sqcap, \sqcup)$ is trivial proceeds along lines similar to those in Chapter 4. However, the proofs are somewhat involved.

10.8 Convex normal functions

The convex normal functions of m^{n} form a De Morgan algebra which we denote by $\mathrm{L} = \mathrm{L}(\mathrm{m}^{\mathrm{n}})$. One problem in investigating L is that the partial order given by the lattice operations $\sqcup$ and $\sqcap$ is not the coordinate-wise partial order on the n-tuples. In this section, we give other representations of L based fundamentally on the notion of straightening from Chapter 6.

Definition 10.8.1 *Let* $\mathrm{D}_1 = \mathrm{D}_1(\mathrm{m}^{\mathrm{n}})$ *be the algebra whose elements are all of the decreasing n-tuples of elements from* $\{1, 2, \ldots, 2m-1\}$ *that include m, and whose operations are given by pointwise $\wedge$ and $\vee$ on these n-tuples, with negation* $(a_1, a_2, ..., a_n)^* = (2m - a_n, 2m - a_{n-1}, ..., 2m - a_1)$ *and with constants* 0 *and* 1 *being the least and largest elements.*

In Section 6.2, we constructed $\mathrm{I}^{\dagger}$ by placing a dual copy of I on top of I and identifying the bottom of the dual copy of I with the top of I. This produced a chain that is isomorphic to $[0, 2]$. If we repeat this process with the m-element chain m in place of I, then we produce a chain with $2m - 1$ elements. So the algebra $\mathrm{D}_1(\mathrm{m}^{\mathrm{n}})$ is completely analogous to the algebra D_1 of Section 6.3.

Proposition 10.8.2 *The algebra* $\mathrm{D}_1 = \mathrm{D}_1(\mathrm{m}^{\mathrm{n}})$ *is a De Morgan algebra.*

Proof. It is clear that D_1 is a sublattice of the lattice of the distributive lattice of all functions from n to m with the pointwise meet and join. The constants 0 and 1 are its bounds. It is routine to see that * is order inverting and of period two. ■

Theorem 10.8.3 *For any m, n, the De Morgan algebras* $\mathrm{L}(\mathrm{m}^{\mathrm{n}})$ *and* $\mathrm{D}_1(\mathrm{m}^{\mathrm{n}})$ *are isomorphic.*

Proof. For $f = (a_1, a_2, \ldots, a_n) \in \mathrm{L}(\mathrm{m}^{\mathrm{n}})$, let i be the smallest index for which $a_i = m$. For $j < i$, replace a_j by $2m - a_j$. We get the n-tuple

$$f^{\dagger} = (2m - a_1, 2m - a_2, \ldots, 2m - a_{i-1}, a_i, a_{i+1}, \ldots, a_n)$$

Then $\varphi(f) = f^{\dagger}$ is an isomorphism from $\mathrm{L}(\mathrm{m}^{\mathrm{n}})$ onto $\mathrm{D}_1(\mathrm{m}^{\mathrm{n}})$. ∎

The situation is entirely analogous to Definition 6.2.4. The function $f^{\dagger}$ takes the mirror image of the increasing part of f in the "line" $y = m$ and leaves the remainder alone. Applying the isomorphism † is called **straightening**. The point is that the meet, join, and partial order of D_1 are componentwise, and therefore much easier to deal with.

There are several modifications to the construction of $\mathrm{D}_1(\mathrm{m}^{\mathrm{n}})$ possible in the finite setting.

Definition 10.8.4 *Given m, n, consider the following sets of functions.*

$\mathrm{D}_2(\mathrm{m}^{\mathrm{n}})$ *is all decreasing functions from* $\mathrm{n-1}$ *into* $\mathrm{2\,m-1}$

$\mathrm{D}_3(\mathrm{m}^{\mathrm{n}})$ *is all strictly decreasing functions from* $\mathrm{n-1}$ *into* $\mathrm{2\,m+n-3}$

Here we use $\mathrm{n-1}$ *for the chain* $\{1, \ldots, n-1\}$ *and so forth.*

Both of these sets have obvious operations $\wedge$ and $\vee$ of pointwise meet and join as well as constants 0 and 1 being their least and largest elements. They both have negations * that work the same as the negation in D_1. Specifically, for $f = (a_1, \ldots, a_{n-1})$

$$f^* = (k - a_{n-1}, \ldots, k - a_1)$$

where k is equal to $2m$ for $\mathrm{D}_2(\mathrm{m}^{\mathrm{n}})$ and $k = 2m + n - 2$ for $\mathrm{D}_3(\mathrm{m}^{\mathrm{n}})$.

Theorem 10.8.5 *The following De Morgan algebras are isomorphic.*

$$\mathrm{L}(\mathrm{m}^{\mathrm{n}}) \approx \mathrm{D}_1(\mathrm{m}^{\mathrm{n}}) \approx \mathrm{D}_2(\mathrm{m}^{\mathrm{n}}) \approx \mathrm{D}_3(\mathrm{m}^{\mathrm{n}})$$

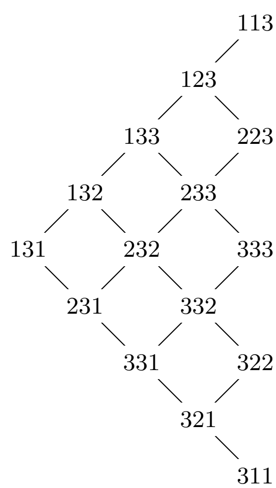
Proof. Each n-tuple in $\mathrm{D}_1(\mathrm{m}^{\mathrm{n}})$ must contain the entry m. Removing the first (or any) entry m leaves an element of $\mathrm{D}_2(\mathrm{m}^{n})$. Conversely, to each $n-1$ tuple in $\mathrm{D}_2(\mathrm{m}^{\mathrm{n}})$, it is possible to insert an entry m in an essentially unique way to produce a decreasing n-tuple that belongs to $\mathrm{D}_1(\mathrm{m}^{\mathrm{n}})$. This produces a bijection between these sets. It is easily seen that this bijection preserves $\wedge, \vee, ^*$ and the constants 0 and 1 (Exercise 10.8.5). So $\mathrm{D}_1(\mathrm{m}^{\mathrm{n}})$ and $\mathrm{D}_2(\mathrm{m}^{\mathrm{n}})$ are isomorphic.

Consider the mapping $\Phi : \mathrm{D}_2(\mathrm{m}^{\mathrm{n}}) \to \mathrm{D}_3(\mathrm{m}^{\mathrm{n}})$ given by

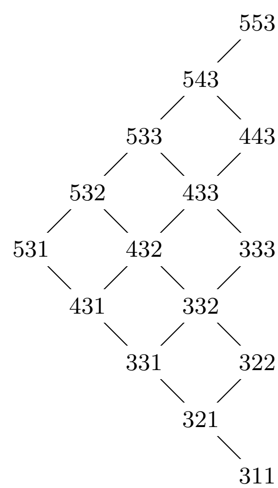
$$\Phi(a_1, a_2, \ldots, a_{n-1}) = (a_1 + (n-2), a_2 + (n-3), \ldots, a_{n-1})$$

Since the original $n-1$ tuple is decreasing, the result is strictly decreasing, and it is not difficult to see that every member of $\mathrm{D}_3(\mathrm{m}^\mathrm{n})$ arises this way. So Φ is a bijection. Verifying that Φ preserves the operations is an exercise (Exercise 17). We earlier proved that $\mathrm{L}(\mathrm{m}^\mathrm{n})$ is isomorphic to $\mathrm{D}_1(\mathrm{m}^\mathrm{n})$, so all four algebras are isomorphic. ■

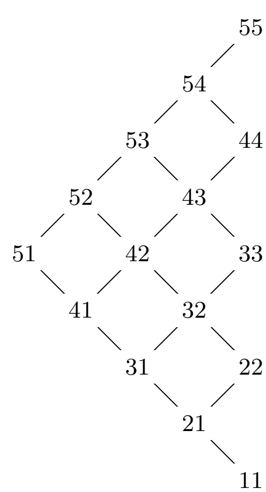
To illustrate, in the diagrams that follow, we show each representation for $m = n = 3$. In depicting elements, we write an element $(a_1, a_2, ..., a_n)$ simply as $a_1 a_2 \cdots a_n$. For example, $(2,3,2)$ is written as 232.



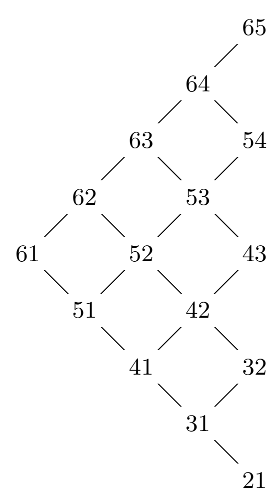
$\mathrm{L}(3^3)$



$\mathrm{D}_1(3^3)$



$\mathrm{D}_2(3^3)$



$\mathrm{D}_3(3^3)$

There is difficulty in depicting such lattices as those above for larger m and n both because of their size and the fact that they are not **planar**. We recall that a lattice is planar if it can be drawn in the plane without lines crossing.

Proposition 10.8.6 *The algebras* $\mathrm{D}_1(\mathrm{m}^\mathrm{n})$ *are not planar if* $m \geq 4$ *and* $n \geq 3$.

Proof. A finite distributive lattice is planar if and only if no element has 3 covers ([38], page 90, problem 45). For example, in $\mathrm{D}_1(4^3)$ the 3-tuple $(3,2,1)$ has covers $(4,2,1)$, $(3,3,1)$, and $(3,2,2)$. ■

To determine the size of $\mathrm{L}(\mathrm{m}^\mathrm{n})$, we use the representation $\mathrm{D}_3(\mathrm{m}^\mathrm{n})$, which is the set of strictly decreasing $(n-1)$-tuples from $\{1,2,\ldots,2m+n-3\}$. This is exactly the number of $(n-1)$-element subsets of a set with $2m+n-3$ elements. This is easy to determine, and is well known. This provides the following.

Theorem 10.8.7 $|\,\mathrm{L}(\mathrm{m}^\mathrm{n})\,| = \dfrac{(2m+n-3)!}{(2m-2)!(n-1)!}$.

We now consider certain Kleene subalgebras $\mathrm{K\,L}(\mathrm{m}^\mathrm{n})$ of $\mathrm{L}(\mathrm{m}^\mathrm{n})$ and the cardinalities of these algebras. We recall the **floor function** $[x]$ that returns the largest integer less than or equal to x.

Definition 10.8.8 $\mathrm{K\,L}(\mathrm{m}^\mathrm{n})$ *consists of the elements of* $\mathrm{L}(\mathrm{m}^\mathrm{n})$ *whose support is either in the first* $\left[\frac{n}{2}\right]+1$ *entries or in the last* $\left[\frac{n}{2}\right]+1$ *entries.*

This definition is similar to that of the subalgebra K of M. The proof of the following result is similar to that for K given in Theorem 3.6.8.

Theorem 10.8.9 $\mathrm{K\,L}(\mathrm{m}^\mathrm{n})$ *is a Kleene subalgebra of* $\mathrm{L}(\mathrm{m}^\mathrm{n})$.

First, we determine the relationship between the sizes of $\mathrm{L}(\mathrm{m}^\mathrm{n})$ and $\mathrm{K\,L}(\mathrm{m}^\mathrm{n})$.

Theorem 10.8.10 *Let* $\underline{n} = \left[\frac{n}{2}\right]+1$. *Then*

$$|\,\mathrm{K\,L}(\mathrm{m}^\mathrm{n})\,| = \begin{cases} 2\,|\,\mathrm{L}(\mathrm{m}^{\underline{\mathrm{n}}})\,| - 1 & \text{if } n \text{ is odd} \\ 2\,|\,\mathrm{L}(\mathrm{m}^{\underline{\mathrm{n}}})\,| - (2m-1) & \text{if } n \text{ is even} \end{cases}$$

Proof. For $n=1$, both $\mathrm{K\,L}(\mathrm{m}^\mathrm{n})$ and $\mathrm{L}(\mathrm{m}^\mathrm{n})$ have one element, and for $n=2$ they both have $2m-1$ elements. Assume $n \geq 2$. Suppose n is odd. The elements of $\mathrm{K\,L}(\mathrm{m}^\mathrm{n})$ begin with $n-\underline{n}$ entries equal to 1 or end with $n-\underline{n}$ entries equal to 1. If they so begin, then the other entries form $\underline{n}$-tuples in one-to-one correspondence with $\mathrm{L}(\mathrm{m}^{\underline{\mathrm{n}}})$. Similarly, if elements end with $n-\underline{n}$ entries equal to 1, then the other entries form $\underline{n}$-tuples also in one-to-one correspondence with $\mathrm{L}(\mathrm{m}^{\underline{\mathrm{n}}})$. For n odd, one n-tuple is counted twice. Similarly, if n is even, $2m-1$ n-tuples get counted twice. ■

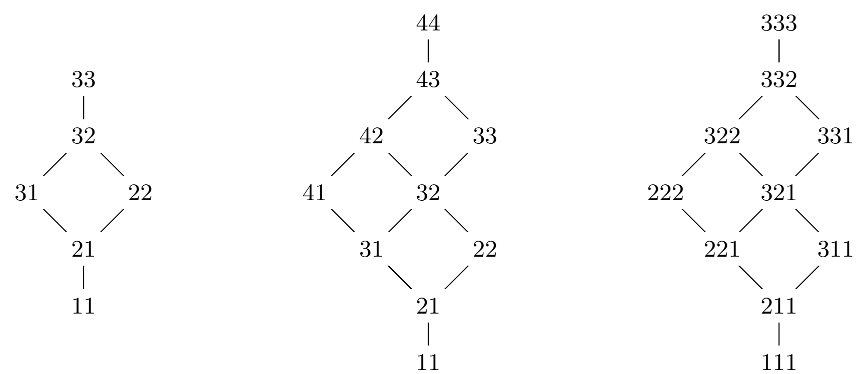
Note that an explicit formula for $|\,\mathrm{K\,L}(\mathrm{m}^\mathrm{n})\,|$ can then be obtained from the formula for the size of in terms of $|\,\mathrm{L}(\mathrm{m}^\mathrm{n})\,|$.

10.9 The De Morgan algebras $\mathrm{H}(\mathrm{m}^{\mathrm{n}})$

In the algebra $\mathrm{D}_2(\mathrm{m}^{\mathrm{n}})$, the tuples can be of any positive integer length, but entries must come from a set with an odd number of elements, namely $\{1, 2, \ldots, 2m - 1\}$. This suggests considering a more general class, namely the one in the following definition.

Definition 10.9.1 *For positive integers m and n, let $\mathrm{H}(\mathrm{m}^{\mathrm{n}})$ be the algebra of all decreasing n-tuples from $\{1, 2, \ldots, m\}$, with pointwise meet and join, negation $(a_1, a_2, \ldots, a_n)^* = (m+1-a_n, m+1-a_{n-1}, \ldots, m+1-a_1)$, and the obvious constants.*

It is easy to see that $\mathrm{H}(\mathrm{m}^{\mathrm{n}})$ is a De Morgan algebra and that $\mathrm{D}_2(\mathrm{m}^{\mathrm{n}})$ and $\mathrm{H}((2\mathrm{m}-1)^{\mathrm{n}-1})$ are the same. Three examples are shown below.



$\mathrm{H}(3^2)$ $\mathrm{H}(4^2)$ $\mathrm{H}(3^3)$

Note that $\mathrm{H}(3^2)$ has a non-trivial automorphism, and $\mathrm{H}(4^2)$ and $\mathrm{H}(3^3)$ are isomorphic and have no non-trivial automorphisms. These are special instances of a general phenomenon as we will see later.

Theorem 10.9.2 $|\mathrm{H}(\mathrm{m}^{\mathrm{n}})| = \dfrac{((m-1)+n)!}{(m-1)!\,n!}$.

Proof. The proof is left as Exercise 18. ■

Note that this implies $|\mathrm{H}(\mathrm{m}^{\mathrm{n}})| = |\mathrm{H}((\mathrm{n}+1)^{\mathrm{m}-1})|$. Actually, these De Morgan algebras are isomorphic as we will see later.

The De Morgan algebra $\mathrm{H}(\mathrm{m}^{\mathrm{n}})$ is in particular a finite distributive lattice. As discussed in the preliminaries, its set of join irreducible elements is a poset $J(\mathrm{H}(\mathrm{m}^{\mathrm{n}}))$ under the induced order, and this poset determines the lattice $\mathrm{H}(\mathrm{m}^{\mathrm{n}})$. We determine now the poset of join irreducible elements of $\mathrm{H}(\mathrm{m}^{\mathrm{n}})$.

Definition 10.9.3 *For $1 \le i \le n-1$, an n-tuple in* $\mathrm{H}(\mathrm{m}^{\mathrm{n}})$ *has a **jump** at i if its $i+1$ entry is strictly less than its i-th entry. It has a **jump** at n if the n-th entry is at least 2.*

For example, the 5-tuple $(5,5,5,3,1)$ has jumps at 3 and 4, the 6-tuple $(8,7,2,2,2,2)$ has jumps at 1 and 2 and 6, and the 6-tuple $(5,5,5,5,1,1)$ has a jump at 4. The only n-tuple with no jumps is $(1,1,\ldots,1)$, the zero of the lattice $\mathrm{H}(\mathrm{m}^{\mathrm{n}})$. The following is easy.

Theorem 10.9.4 *The join irreducibles $J(\mathrm{H}(\mathrm{m}^{\mathrm{n}}))$ of the distributive lattice $\mathrm{H}(\mathrm{m}^{\mathrm{n}})$ are those n-tuples with exactly one jump.*

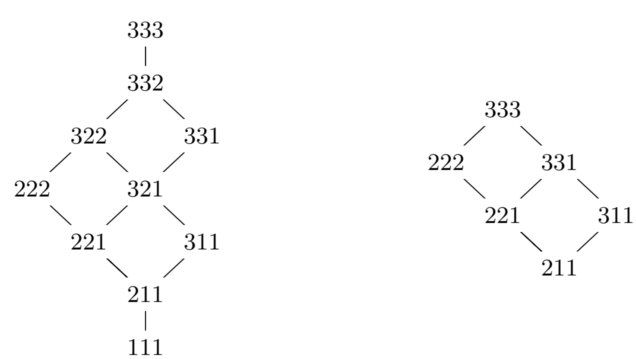
Since the only element with no jumps is the n-tuple $(1,1,\ldots,1)$, the join irreducible elements of $\mathrm{H}(\mathrm{m}^{\mathrm{n}})$ are of the form $(a,a,\ldots,a,1,1,\ldots,1)$, with $a > 1$ and at least one a in the tuple. Thus with each non-zero join irreducible, there is associated a pair of integers, the integer a and the index of the last a. For example, we have the following associations.

$$(5,5,1,1,1) \to (5,2)$$
$$(5,5,5,5,5) \to (5,5)$$

This association gives a map from $J(\mathrm{H}(\mathrm{m}^{\mathrm{n}}))$ to the poset $(\mathrm{m}-1)\times\mathrm{n}$. (Here, we are associating the poset $\{2,3,\ldots,m\}$ with the poset $\mathrm{m}-1$.) This is rather obviously a one-to-one mapping of the join irreducibles of $\mathrm{H}(\mathrm{m}^{\mathrm{n}})$ onto the poset $(\mathrm{m}-1)\times\mathrm{n}$, and preserves component-wise order. Thus we have the following.

Theorem 10.9.5 *The poset of join irreducibles $J(\mathrm{H}(\mathrm{m}^{\mathrm{n}}))$ of $\mathrm{H}(\mathrm{m}^{\mathrm{n}})$ is isomorphic to the poset $(\mathrm{m}-1)\times\mathrm{n}$. (Note that the poset $(\mathrm{m}-1)\times\mathrm{n}$ is actually a bounded distributive lattice.)*

The example below shows $\mathrm{H}(3^3)$ and its poset of join irreducibles $J(\mathrm{H}(3^3))$, which in this case is the product of a 2-element chain and a 3-element chain.



$\mathrm{H}(3^3)$ $\qquad$ $J(\mathrm{H}(3^3))$

It is interesting to recall ([38], page 85) that the **length of maximal chains** (length is one less than the number of elements in the chain) in a finite distributive lattice is the same as the size of the set of join irreducible elements. This gives the following, which can also be calculated directly.

Proposition 10.9.6 *Maximal chains in* $\mathrm{H}(\mathrm{m}^{\mathrm{n}})$ *have length* $(m-1)\times n$.

Because of the categorical equivalence of finite distributive lattices and finite posets, the lattice $\mathrm{H}(\mathrm{m}^{\mathrm{n}})$ is isomorphic to the lattice $\mathrm{H}(\mathrm{p}^{\mathrm{q}})$ if and only if the posets $(\mathrm{m}-1)\times\mathrm{n}$ and $(\mathrm{p}-1)\times\mathrm{q}$ are isomorphic. Further, it is clear that the poset $(\mathrm{m}-1)\times\mathrm{n}$ has only the trivial automorphism unless $m-1=n$, in which case it has exactly two automorphisms. Thus the lattice $\mathrm{H}(\mathrm{m}^{\mathrm{n}})$ has only the trivial automorphism unless $\mathrm{m}-1=\mathrm{n}$, in which case it has exactly two automorphisms, its poset of non-zero join irreducibles being the poset $\mathrm{n}\times\mathrm{n}$. Thus we get the following corollaries.

Corollary 10.9.7 *The lattices* $\mathrm{H}(\mathrm{m}^{\mathrm{n}})$ *and* $\mathrm{H}(\mathrm{p}^{\mathrm{q}})$ *are isomorphic if and only if* $m=p$ *and* $n=q$, *or* $m-1=q$ *and* $p-1=n$.

Corollary 10.9.8 *The automorphism group* $\mathrm{Aut}(\mathrm{H}(\mathrm{m}^{\mathrm{n}}))$ *of the lattice* $\mathrm{H}(\mathrm{m}^{\mathrm{n}})$ *has only one element unless* $m-1=n$, *in which case it has exactly two elements.*

Since $\mathrm{L}(\mathrm{m}^{\mathrm{n}})\approx\mathrm{H}((2\,\mathrm{m}-1)^{\mathrm{n}-1})$, the join irreducibles of $\mathrm{L}(\mathrm{m}^{\mathrm{n}})$ are isomorphic to the poset $(2\,\mathrm{m}-2)\times(\mathrm{n}-1)$. Thus we get the following corollary.

Corollary 10.9.9 *The automorphism group* $\mathrm{Aut}(\mathrm{L}(\mathrm{m}^{\mathrm{n}}))$ *of the lattice* $\mathrm{L}(\mathrm{m}^{\mathrm{n}})$ *has only one element unless* $2m-1=n$, *in which case it has exactly two elements.*

Actually the following holds. We omit the proof, which is found in [114].

Corollary 10.9.10 *The lattice automorphisms of* $\mathrm{H}(\mathrm{m}^{\mathrm{n}})$ *are De Morgan automorphisms of* $\mathrm{H}(\mathrm{m}^{\mathrm{n}})$.

To conclude, we remark that there are Kleene subalgebras of $\mathrm{H}(\mathrm{m}^{\mathrm{n}})$ corresponding to the Kleene subalgebras $\mathrm{K\,L}(\mathrm{m}^{\mathrm{n}})$ of $\mathrm{L}(\mathrm{m}^{\mathrm{n}})$ in Theorem 10.8.9. We omit discussion of these subalgebras here, but refer to [114] for details.

10.10 Summary

This chapter has studied finite analogs m^{n} of the algebra M. They inherit many properties of M. They are De Morgan Birkhoff systems that, except in small cases, generate the same variety as M.

These algebras, like M, have two orders. The meet order $\sqsubseteq_\sqcap$ is given by the meet semilattice operation $\sqcap$, and the join order $\sqsubseteq_\sqcup$ is given by the join semilattice operation $\sqcup$. Unlike the case of M, both orders give lattice orders. Algorithmic descriptions of the joins and meets in these orders were given.

As with M, the intersection of the join and meet orders of m^{n} is a partial order. Unlike M, this double order is a lattice order on the subalgebra of normal functions. There is no intuitive reason why this should be true. Algorithmic descriptions for join and meet in these lattices were given. This provides an interesting class of non-distributive finite involutive lattices.

The irreducible elements of the algebras m^{n} were described, and this was used to show that the automorphism groups of these algebras are trivial.

The algebras m^{n} have subalgebras of convex functions, of normal functions, and of convex normal functions $\mathrm{L}(\mathrm{m}^{\mathrm{n}})$, among others. Several isomorphic realizations of the algebras $\mathrm{L}(\mathrm{m}^{\mathrm{n}})$ were given based on the notion of straightening. These descriptions were used to develop formulas for their cardinalities and heights.

Generalizations $\mathrm{H}(\mathrm{m}^{\mathrm{n}})$ of the algebras $\mathrm{L}(\mathrm{m}^{\mathrm{n}})$ were considered. These were algebras of decreasing functions from the chain n to the chain m. Their posets of join irreducibles were shown to be the lattices $(\mathrm{m}-1)\times\mathrm{n}$. This was used to determine the automorphism groups of the lattices $\mathrm{H}(\mathrm{m}^{\mathrm{n}})$, and when they are isomorphic.

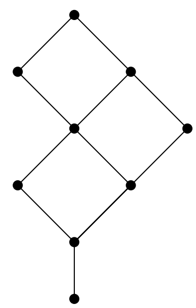
10.11 Exercises

1. Let X be a set and $\mathcal{P}(X)$ be its power set. What are the join irreducibles in the lattice $\mathcal{P}(X)$?

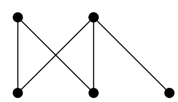
2. Any chain is a lattice. What elements of a chain are join irreducible?

3. The plane $\mathbb{R}\times\mathbb{R}$ is a lattice where $(x_1, y_1) \leq (x_2, y_2)$ if $x_1 \leq x_2$ and $y_1 \leq y_2$. (Show this.) Prove that there are no join irreducible elements in this lattice.

4. The following figure depicts a distributive lattice L. Sketch and label its poset $J(L)$ of join irreducible elements, and the distributive lattice $D(J(L))$ of downsets of this poset.



5. A poset P is shown below. Sketch and label its distributive lattice of downsets $D(P)$, and the poset $J(D(P))$ of its join irreducibles.



6. Prove Proposition 10.1.9. Here, the main difficulty is in showing that $J(f)(j)$ is join irreducible in L. It is useful to first prove that $J(f)(j)$ is the smallest element of L that is mapped above j.

7. Let X be any finite set, and consider X as a poset where no two distinct elements are comparable. Prove that the set of downsets of this poset X is the lattice $\mathcal{P}(X)$, the power set of X. This leads to the result that the category of finite Boolean algebras and the homomorphisms between them is dually isomorphic to the category of finite sets and the functions between them.

8. Show that for a natural number n, the algebra n is a Kleene algebra whose underlying lattice is a chain.

9. Draw a diagram to indicate the join and meet orders of 3^2.

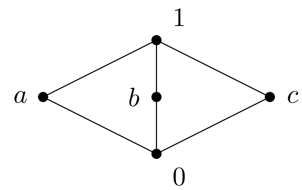
10. Use the algorithm of Section 10.4 to compute $f \odot g$ in 4^6 for $f = 312143$ and $g = 413232$.

11. Prove the remainder of Proposition 10.5.2, namely, that the intersection of two partial orderings on a set is a partial ordering on the set.

12. Prove that for any partial order $\leq$ on a finite set X, there is a linear order (a chain) $\sqsubseteq$ on X so that if $x \leq y$, then $x \sqsubseteq y$.

13. Use the result of the previous exercise to prove that for any partial ordering $\leq$ on a finite set X, there is a family of linear orders on X whose intersection is $\leq$. The fewest number of such linear orders whose intersection is $\leq$ is called the **order dimension** of the poset $(X, \leq)$.

14. Find the order dimension of the following poset.



15. Fill in details for the proof of Proposition 10.5.5.

16. Show that the mapping † of Theorem 10.8.3 is a one-to-one mapping from $\mathrm{L}(\mathrm{m}^{\mathrm{n}})$ onto $\mathrm{D}_1(\mathrm{m}^{\mathrm{n}})$.

17. Prove that the mappings discussed in Theorem 10.8.5 preserve the operations $\wedge, \vee, *, 0, 1$ of the algebras involved.

18. Prove Theorem 10.9.2.

19. Prove Theorem 10.9.4.

Appendix A

Properties of the Operations on $\mathbb{M}$

The following properties hold in $\mathbb{M}$.

1. $f \sqcup g = (f \wedge g^L) \vee (f^L \wedge g) = (f \vee g) \wedge (f^L \wedge g^L)$
2. $f \sqcap g = (f \wedge g^R) \vee (f^R \wedge g) = (f \vee g) \wedge (f^R \wedge g^R)$

3. $f \leq f^L$; $f \leq f^R$
4. $f \leq g$ implies $f^L \leq g^L$ and $f^R \leq g^R$
5. $f^{LL} = f^L$; $f^{RR} = f^R$
6. $f^{LR} = f^{RL}$ and this is a constant function with value $\sup f$

7. $f^{**} = f$
8. $f^{L*} = f^{*R}$; $f^{R*} = f^{*L}$
9. $(f \wedge g)^* = f^* \wedge g^*$; $(f \vee g)^* = f^* \vee g^*$
10. $(f \vee g)^L = f^L \vee g^L$; $(f \vee g)^R = f^R \vee g^R$

11. $f^L \sqcup g^L = f^L \sqcup g = f \sqcup g^L = f^L \wedge g^L$
12. $f^R \sqcap g^R = f^R \sqcap g = f \sqcap g^R = f^R \wedge g^R$

13. $(f \sqcup g)^L = f^L \sqcup g^L$
14. $(f \sqcup g)^R = f^R \sqcup g^R$
15. $(f \sqcap g)^R = f^R \sqcap g^R$
16. $(f \sqcap g)^L = f^L \sqcap g^L$

17. $f \sqcup f = f$; $f \sqcap f = f$
18. $f \sqcup g = g \sqcup f$; $f \sqcap g = g \sqcap f$

19. $f \sqcup (g \sqcup h) = (f \sqcup g) \sqcup h$; $f \sqcap (g \sqcap h) = (f \sqcap g) \sqcap h$

20. $f \sqcup (f \sqcap g) = f \sqcap (f \sqcup g)$

21. $1_0 \sqcup f = f$; $1_1 \sqcap f = f$

22. $f^{**} = f$

23. $(f \sqcup g)^* = f^* \sqcap g^*$; $(f \sqcap g)^* = f^* \sqcup g^*$

24. $f \sqsubseteq_{\sqcap} g$ if and only if $f^R \wedge g \leq f \leq g^R$

25. $f \sqsubseteq_{\sqcup} g$ if and only if $f \wedge g^L \leq g \leq f^L$

26. $f \sqcup (g \vee h) = (f \sqcup g) \vee (f \sqcup h)$

27. $f \sqcap (g \vee h) = (f \sqcap g) \vee (f \sqcap h)$

28. $f_1 \sqcup \cdots \sqcup f_n = (f_1 \vee \cdots \vee f_n) \wedge f_1^L \wedge \cdots \wedge f_n^L$

29. $f_1 \sqcap \cdots \sqcap f_n = (f_1 \vee \cdots \vee f_n) \wedge f_1^R \wedge \cdots \wedge f_n^R$

Bibliography

[1] S. Abramsky and B. Coecke. A categorical semantics of quantum protocols. In *19th IEEE Symposium on Logic in Computer Science (LICS 2004),* 14–17 July 2004, Turku, Finland, Proceedings, pages 415–425, 2004.

[2] J. Baez and M. Stay. *Physics, Topology, Logic and Computation: A Rosetta Stone*. Springer, 2011.

[3] R. Balbes and P. Dwinger. *Distributive Lattices*. University of Missouri Press, 1974.

[4] M. Barr. Fuzzy set theory and topos theory. *Canad. Math. Bull.*, 29(4):501–508, 1986.

[5] G. Bezhanishvili. Locally finite varieties. *Algebra Universalis*, 46(4):531–548, 2001.

[6] G. Bezhanishvili, M. Gehrke, J. Harding, C. Walker, and E. Walker. Varieties of algebras in fuzzy set theory. In *Logical, Algebraic, Analytic, and Probabilistic Aspects of Triangular Norms*, pages 321–344. Elsevier B. V., Amsterdam, 2005.

[7] G. Birkhoff. *Lattice Theory*, volume 25 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, R.I., third edition, 1979.

[8] G. Bruns. Free ortholattices. *Canad. J. Math.*, 28(5):977–985, 1976.

[9] J. A. Brzozowski. De Morgan bisemilattices. In *30th IEEE International Symposium on Multiple-Valued Logic* (ISMVL 2000) Portland, OR, pages 173–178. IEEE Computer Soc., Los Alamitos, CA, 2000.

[10] S. N. Burris and H. P. Sankappanavar. *A Course in Universal Algebra*. Number 78 in Graduate Texts in Mathematics. Springer, 1981.

[11] O. Castillo. *Type-2 Fuzzy Logic in Intelligent Control Applications*, volume 272 of *Studies in Fuzziness and Soft Computing*. Springer-Verlag, Berlin, 2012.

[12] O. Castillo and P. Melin. *Type-2 Fuzzy Logic: Theory and Applications*. Studies in Computational Intelligence. Springer, 2008.

[13] O. Castillo and P. Melin. *Recent Advances in Interval Type-2 Fuzzy Systems*. SpringerBriefs in Computational Intelligence. Springer, 2012.

[14] C.-H. Chiu and W.-J. Wang. A simple computation of min and max operations for fuzzy numbers. *Fuzzy Sets and Systems*, 126(2):273–276, 2002.

[15] B. Davey and H. Priestley. *Introduction to Lattices and Order*. Cambridge Mathematical Textbooks. Cambridge University Press, 2002.

[16] B. De Baets and R. Mesiar. Triangular norms on product lattices. *Fuzzy Sets and Systems*, 104(1):61–75, 1999.

[17] G. Deschrijver. A representation of t-norms in interval-valued L-fuzzy set theory. *Fuzzy Sets and Systems*, 159(13):1597–1618, 2008.

[18] D. Dubois and H. Prade. Operations in a fuzzy-valued logic. *Information & Control*, 42(2):224–240, 1979.

[19] D. Dubois and H. Prade. Twofold fuzzy sets: An approach to the representation of sets with fuzzy boundaries based on possibility and necessity measures. *Fuzzy Mathematics*, 3(4), 1983.

[20] D. Dubois and H. Prade. *Fuzzy Sets and Systems*. Academic Press, New York, 1988.

[21] D. Dubois and H. Prade. *Possibility Theory*. Plenum, New York, 1988.

[22] H. Ebbinghaus, J. Flum, and W. Thomas. *Mathematical Logic (2. ed.)*. Undergraduate Texts in Mathematics. Springer, 1994.

[23] M. Emoto and M. Mukaidono. Necessary and sufficient conditions for fuzzy truth values to form a De Morgan algebra. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, 7(4):309–318, 1999.

[24] M. Emoto and M. Mukaidono. Necessary and sufficient conditions for fuzzy truth values to satisfy the distributive laws. In Liu, Chen, Ying, and Cai, editors, *Proceedings of the International Conference on Fuzzy Information Processing*, pages 159–164, 2003.

[25] Z. Ésik. Free De Morgan bisemigroups and bisemilattices. *Algebra Colloq.*, 10(1):23–32, 2003.

[26] N. Galatos, P. Jipsen, T. Kowalski, and H. Ono. *Residuated Lattices: An Algebraic Glimpse at Substructural Logics*, volume 151 of *Studies in Logic and the Foundations of Mathematics*. Elsevier B. V., Amsterdam, 2007.

[27] M. Gehrke and J. Harding. Bounded lattice expansions. *J. Algebra*, 238(1):345–371, 2001.

[28] M. Gehrke, C. Walker, and E. Walker. De Morgan systems on the unit interval. *International Journal of Intelligent Systems*, 11:733–750, 1996.

[29] M. Gehrke, C. Walker, and E. Walker. Some comments on interval-valued fuzzy sets. *International Journal of Intelligent Systems*, 11:751–759, 1996.

[30] M. Gehrke, C. Walker, and E. Walker. A mathematical setting for fuzzy logics. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, 5:223–238, 1997.

[31] M. Gehrke, C. Walker, and E. Walker. Algebraic aspects of fuzzy sets and fuzzy logic. In B.K. Papadopoulos and A. Syropoulos, editors, *Proceedings of the Workshop on Current Trends and Developments in Fuzzy Logic,* Thessaloniki, Greece, October 16–20, 1998, pages 101–170, 1999.

[32] G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. W. Mislove, and D. S. Scott. *A Compendium of Continuous Lattices.* Springer-Verlag, Berlin-New York, 1980.

[33] G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. W. Mislove, and D. S. Scott. *Continuous Lattices and Domains*, volume 93 of *Encyclopedia of Mathematics and its Applications.* Cambridge University Press, Cambridge, 2003.

[34] J. A. Goguen. L-fuzzy sets. *J. Math. Anal. Appl.*, 18:145–174, 1967.

[35] J. A. Goguen. Categories of V-sets. *Bull. Amer. Math. Soc.*, 75:622–624, 1969.

[36] Jonathan S. Golan. *Semirings and Their Applications.* Kluwer Academic Publishers, 1999.

[37] I. Grattan-Guiness. Fuzzy membership mapped onto interval and many-valued quantities. *Z. Math. Logik, Grundladen Math.*, 22:149–160, 1975.

[38] G. Grätzer. *General Lattice Theory.* Birkhäuser Verlag, Basel, 2003.

[39] G. Grätzer. *Universal Algebra.* Springer, New York, second edition, 2008.

[40] P. Hájek. *Metamathematics of Fuzzy Logic*, volume 4 of *Trends in Logic—Studia Logica Library.* Kluwer Academic Publishers, Dordrecht, 1998.

[41] J. Harding. A link between quantum logic and categorical quantum mechanics. *Intern. J. of Theoret. Phys.*, 48(3):769–802, 2009.

[42] J. Harding and A. Romanowska. Varieties of Birkhoff systems, Part I. Manuscript submitted for publication, 2014.

[43] J. Harding and A. Romanowska. Varieties of Birkhoff systems part II. Manuscript submitted for publication, 2014.

[44] J. Harding, C. Walker, and E. Walker. Convex normal functions revisited. *Fuzzy Sets and Systems*, 161(9):1343–1349, 2010.

[45] J. Harding, C. Walker, and E. Walker. Lattices of convex normal functions. *Fuzzy Sets and Systems*, 159(9):1061–1071, 2010.

[46] J. Harding, C. Walker, and E. Walker. The variety generated by the truth value algebra of type-2 fuzzy sets. *Fuzzy Sets and Systems*, 161(5):735–749, 2010.

[47] J. Harding, C. Walker, and E. Walker. Projective bichains. *Algebra Universalis*, 67(4):347–374, 2012.

[48] J. Harding, C. Walker, and E. Walker. Partial orders on the truth value algebra of finite type-2 fuzzy sets. In *Joint IFSA World Congress and NAFIPS Annual Meeting, IFSA/NAFIPS 2013,* Edmonton, Alberta, Canada, June 24–28, 2013, pages 163–168, 2013.

[49] J. Harding, C. Walker, and E. Walker. Type-2 fuzzy sets and bichains. In A. Sadeghian, J. Mendel, and H. Tahayori, editors, *Advances in Type-2 Fuzzy Sets and Systems*, volume 301 of *Studies in Fuzziness and Soft Computing*, pages 97–112. Springer New York, 2013.

[50] J. Harding, C. Walker, and E. Walker. Categories with fuzzy sets and relations. *Fuzzy Sets and Systems*, 256:149–165, 2014.

[51] J. Harding, C. Walker, and E. Walker. Partial orders on fuzzy truth value algebras. *Internat. J. Uncertain. Fuzziness Knowledge-Based Systems*, 23(2):193–219, 2015.

[52] P. Hernández, S. Cubillo, and C. Torres-Blanc. About t-norms on type-2 fuzzy sets. In *Proceedings of the 8th Conference of the European Society for Fuzzy Logic and Technology, EUSFLAT-13,* Milano, Italy, September 11-13, 2013.

[53] D. Hobby and R. McKenzie. *The structure of finite algebras*, volume 76 of *Contemporary Mathematics*. American Mathematical Society, Providence, RI, 1988.

[54] U. Höhle and L. Stout. Foundations of fuzzy sets. *Fuzzy Sets and Systems*, 40(2):257–296, 1991.

[55] B. Q. Hu and C. K. Kwong. On type-2 fuzzy sets and their t-norm operations. *Inform. Sci.*, 255:58–81, 2014.

[56] K.U. Jahn. Intervall-wertige mengen. *Math. Nach.*, 68:115–132, 1975.

[57] P. Jipsen and H. Rose. *Varieties of Lattices*, volume 1533 of *Lecture Notes in Mathematics.* Springer Verlag, 1992.

[58] R. John. Type-2 fuzzy sets: An appraisal of theory and applications. *Int. J. Uncertainty, Fuzziness Knowledge-Based Systems*, 6(6):563–576, 1998.

[59] B. Jónsson and A. Tarski. Boolean algebras with operators. I. *Amer. J. Math.*, 73:891–939, 1951.

[60] B. Jónsson and A. Tarski. Boolean algebras with operators. II. *Amer. J. Math.*, 74:127–162, 1952.

[61] A. Joyal and R. Street. The geometry of tensor calculus, I. *Advances in Mathematics*, 88(1):55–112, 1991.

[62] N. Karnik and J. Mendel. Operations on type-2 fuzzy sets. *Fuzzy Sets and Systems*, 122:327–348, 2001.

[63] M. Kawaguchi and M. Miyakoshi. Extended triangular norms in type-2 fuzzy logic. In *CD-ROM Proceedings of EUFIT'99,* Aachen, Germany, 1999.

[64] M. Kawaguchi and M. Miyakoshi. Extended t-norms as logical connectives of fuzzy truth values. *Mult.-Valued Log.*, 8(1):53–69, 2002.

[65] M. Kawaguchi and M. Miyakoshi. Generalized extended t-norms as t-norms of type 2. In *ISMVL 2009, 39th International Symposium on Multiple-Valued Logic,* 21–23 May 2009, Naha, Okinawaw, Japan, pages 292–297, 2009.

[66] J. Kelley. *General Topology.* Springer-Verlag, New York-Berlin, 1975.

[67] H. Kikuchi and N. Takagi. De Morgan bisemilattice of fuzzy truth value. In *Proceedings of the 32nd International Symposium on Multiple-Valued Logic*, ISMVL '02, pages 180–184, Washington, DC, USA, 2002. IEEE Computer Society.

[68] E. P. Klement, R. Mesiar, and E. Pap. *Triangular Norms.* Kluwer Academic Publishers, Dordrecht, The Netherlands, 2000.

[69] V. Kreinovich. From processing interval-valued fuzzy data to general type-2: Towards fast algorithms. In *2011 IEEE Symposium on Advances in Type-2 Fuzzy Logic Systems, T2FUZZ 2011,* Paris, France, April 12–13, 2011.

[70] C. Ling. Representations of associative functions. *Publ. Math. Debrecen*, 12:189–212, 1965.

[71] Y. M. Liu. Some properties of convex fuzzy sets. *J. Math. Anal. Appl.*, 111(1):119–129, 1985.

[72] R. Lowen. Convex fuzzy sets. *Fuzzy Sets and Systems*, 3(3):291–310, 1980.

[73] S. Mac Lane. *Categories for the Working Mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1998.

[74] M. Mares. *Computation over Fuzzy Quantities (9.1)*. CRC-Press, Boca Raton, 1994.

[75] W. McCune. `http://www.cs.unm.edu/~mccune/prover9/`. Prover9 and Mace4, 2005–2010.

[76] R. McKenzie and A. Romanowska. Varieties of $\cdot$-distributive bisemilattices. In *Contributions to General Algebra (Proc. Klagenfurt Conf., Klagenfurt, 1978)*, pages 213–218. Heyn, Klagenfurt, 1979.

[77] J. Mendel. *Uncertain Rule-Based Fuzzy Logic Systems*. Prentice Hall PTR, Upper Saddle River, NJ, 2001.

[78] J. Mendel, H. Hagras, W. Tan, W. Melek, and H. Ying. *Introduction to Type-2 Fuzzy Logic Control: Theory and Applications*. IEEE Press Series on Computational Intelligence. Wiley-IEEE Press, 2014.

[79] J. Mendel and R. John. Type-2 fuzzy sets made simple. *IEEE Transactions on Fuzzy Systems*, 10(2):117–127, 2002.

[80] K. Menger. Statistical metrics. *Proc. Nat. Acad. Sci.*, 28:535–537, 1942.

[81] M. Mizumoto and K. Tanaka. Some properties of fuzzy sets of type-2. *Information and Control*, 31:312–340, 1976.

[82] M. Mizumoto and K. Tanaka. Fuzzy sets and type-2 under algebraic product and algebraic sum. *Fuzzy Sets and Systems*, 5:277–290, 1981.

[83] M. Mukaidono. Algebraic structures of truth values in fuzzy logic. In *Fuzzy Logic and Fuzzy Control*, volume 833 of *Lecture Notes in Artificial Intelligence*, pages 15–21. Springer-Verlag, 1994.

[84] H. Nguyen. A note on the extension principle for fuzzy sets. *J. Math. Anal. Appl.*, 64(2):369–380, 1978.

[85] H. Nguyen, N. R. Prasad, C. Walker, and E. Walker. *A First Course in Fuzzy and Neural Control*. Chapman & Hall/CRC, Boca Raton, FL, 2003.

[86] H. Nguyen and E. Walker. *A first course in fuzzy logic*. Chapman & Hall/CRC, Boca Raton, FL, third edition, 2006.

[87] J. Nieminen. On the algebraic structure of fuzzy sets of type 2. *Kybernetika*, 13:261–273, 1997.

[88] R. Padmanabhan. Regular identities in lattices. *Trans. Amer. Math. Soc.*, 158:179–188, 1971.

[89] E. Palmeira, B. Bedregal, R. Mesiar, and J. Fernandez. A new way to extend t-norms, t-conorms and negations. *Fuzzy Sets and Systems*, 240:1–21, 2014.

[90] D. Papert Strauss. Topological lattices. *Proc. London Math. Soc.*, 18(3):217–230, 1968.

[91] J. Płonka. On distributive quasilattices. *Fund. Math.*, 60:197–200, 1967.

[92] A. Romanowska. On bisemilattices with one distributive law. *Algebra Universalis*, 10:36–47, 1980.

[93] K.I. Rosenthal. *Quantales and Their Applications*. Pitman Research Notes in Mathematics Series. Longman Scientific & Technical, 1990.

[94] W. Rudin. *Principles of Mathematical Analysis*. McGraw-Hill Book Co., New York-Auckland-Düsseldorf, third edition, 1976. International Series in Pure and Applied Mathematics.

[95] R. Sambuc. *Fonctions ϕ-floues, Application l'aide au diagnostic en pathologie thyroidienne*. PhD thesis, Univ. Marseille, France, 1975.

[96] B. Schweizer and A. Sklar. Statistical metric spaces. *Pacific J. Math.*, 10:313–334, 1960.

[97] B. Schweizer and A. Sklar. Associative functions and statistical triangle inequalities. *Publ. Math.*, 8:169–186, 1961.

[98] B. Schweizer and A. Sklar. *Probabilistic Metric Spaces*. Elsevier, Amsterdam, 1983.

[99] P. Selinger. Dagger compact closed categories and completely positive maps. *Electron. Notes Theor. Comput. Sci.*, 170:139–163, 2007.

[100] P. Selinger. A survey of graphical languages for monoidal categories. In B. Coecke, editor, *New Structures for Physics*, volume 813 of *Lecture Notes in Physics*, pages 289–355. Springer, 2011.

[101] J. Starczewski. Extended triangular norms. *Inform. Sci.*, 179(6):742–757, 2009.

[102] J. Starczewski. *Advanced Concepts in Fuzzy Logic and Systems with Membership Uncertainty*, volume 284 of *Studies in Fuzziness and Soft Computing*. Springer, 2013.

[103] A. Tepavčević. On special elements of bisemilattices. *Novi Sad J. Math.*, 27(1):83–92, 1997.

[104] C. Walker. Categories of fuzzy sets. *Soft Comput.*, 8(4):299–304, 2004.

[105] C. Walker and E. Walker. Algebraic properties of type-2 fuzzy sets. In H. P. Nguyen, H. T. Nguyen, C. H. Nguyen, and P. Santiprabhob, editors, *Proceedings of the Third International Conference on Intelligent Technologies and Third Vietnam-Japan Symposium on Fuzzy Systems and Applications,* Hanoi, Vietnam, December 3–5, 2002.

[106] C. Walker and E. Walker. Algebraic structures on fuzzy sets of type-2. In Liu, Chen, Ying, and Cai, editors, *Proceedings of International Conference on Fuzzy Information Processing: Theories and Applications,* Beijing, China, March 1–4, pages 97–100, 2003.

[107] C. Walker and E. Walker. Groups, t-norms, and families of De Morgan systems. In *Topological and Algebraic Structures in Fuzzy Sets*, volume 20 of *Trends Log. Stud. Log. Libr.*, pages 455–467. Kluwer Acad. Publ., Dordrecht, 2003.

[108] C. Walker and E. Walker. The algebra of fuzzy truth values. *Fuzzy Sets and Systems*, 149:309–347, 2005.

[109] C. Walker and E. Walker. Automorphisms of powers of linearly ordered sets. *Internat. J. Uncertain. Fuzziness Knowledge-Based Systems*, 14(1):77–85, 2006.

[110] C. Walker and E. Walker. Automorphisms of the algebra of fuzzy truth values. *Internat. J. Uncertain. Fuzziness Knowledge-Based Systems*, 14(6):711–732, 2006.

[111] C. Walker and E. Walker. Automorphisms of the algebra of fuzzy truth values. II. *Internat. J. Uncertain. Fuzziness Knowledge-Based Systems*, 16(5):627–643, 2008.

[112] C. Walker and E. Walker. Sets with type-2 operations. *Int. J. Approx. Reasoning*, 50(1):63–71, 2009.

[113] C. Walker and E. Walker. Some general comments on fuzzy sets of type-2. *Int. J. Intell. Syst.*, 24(1):62–75, 2009.

[114] C. Walker and E. Walker. A family of finite De Morgan and Kleene algebras. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, 20(5):631–654, 2012.

[115] C. Walker and E. Walker. Type-2 operations on finite chains. *Fuzzy Sets and Systems*, 236:33–49, 2014.

[116] M. Winter. *Goguen Categories: A Categorical Approach to L-fuzzy Relations*, volume 25 of *Trends in Logic—Studia Logica Library*. Springer, Dordrecht, 2007.

[117] L. Zadeh. Fuzzy sets. *Information and Control*, 8(3):338–353, 1965.

[118] L. Zadeh. The concept of a linguistic variable and its application to approximate reasoning I. *Information Sciences*, 8(3):199–249, 1975.

[119] L. Zadeh. The concept of a linguistic variable and its application to approximate reasoning II. *Information Sciences*, 8:301–357, 1975.

[120] L. Zadeh. The concept of a linguistic variable and its application to approximate reasoning III. *Information Sciences*, 9(1):43–80, 1975.

[121] L. Zadeh. Yes, no and relatively. *Chemtech*, pages 340–344 and 406–410, June and July 1987.

[122] H. J. Zimmerman. *Fuzzy Set Theory and Its Applications, Second Edition.* Kluwer Academic Publishers, Boston/Dordrecht/London, 1991.

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